Some Infinite Classes of Williamson Matrices and Weighing Matrices

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ABSTRACT

Williamson type matrices A, B, C, D will be called nice if $AB^T + CD^T = 0$, perfect if $AB^T + CD^T = AC^T + BD^T = 0$, special if $AB^T + CD^T = AC^T + BD^T = AD^T + BC^T = 0$.
Type 1 (1 , -1)-matrices A, B, C, D of order n will be called tight Williamson-like matrices if $AA^T + BB^T + CC^T + DD^T = 4nI_n$ and $AB^T + BA^T + CD^T + DC^T = 0$.
Write $N = 3^{2r} \cdot p_1^{4r_1} \ldots p_n^{4r_n}$, where $p_j \equiv 3(mod\ 4)$, $p_j > 3$, $j = 1, \ldots, n$ and $r, r_1, \ldots, r_n$ are non–negative integers. In this paper we prove:

(i) if there exist special Williamson type matrices of order n then there exist two disjoint amicable $W(2n, n)$, whose sum and difference are (1 ,-1)-matrices, and four disjoint and amicable $W(4n, n)$, whose sum is a (1 , -1)-matrix;

(ii) there exists an Hadamard matrix of order $4mn$, where m is the order of tight Williamson-like matrices and n is the order of nice Williamson type matrices.

Definition 1 Williamson type matrices A, B, C, D will be called *nice* if $AB^T + CD^T = 0$, *perfect* if $AB^T + CD^T = AC^T + BD^T = 0$, *special* if $AB^T + CD^T = AC^T + BD^T = AD^T + BC^T = 0$ (see Definition 4, [2]).

Definition 2 Type 1 (1 , -1)-matrices A, B, C, D of order n will be called *tight Williamson-like matrices* if $AA^T + BB^T + CC^T + DD^T = 4nI_n$ and $AB^T + BA^T + CD^T + DC^T = 0$ (see Definition 5, [2]).

Notation 1 Write $N = 3^{2r} \cdot p_1^{4r_1} \ldots p_n^{4r_n}$, where $p_j \equiv 3(mod\ 4)$, $p_j > 3$, $j = 1, \ldots, n$ and $r, r_1, \ldots, r_n$ are non–negative integers.

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Theorem 1 *If there exist special Williamson type matrices of order n then there exist two disjoint amicable $W(2n,n)$, whose sum and difference are (1 , -1)-matrices.*

Proof. Let A_1, A_2, A_3, A_4 be the Williamson type matrices of order n. Set

$$P = \frac{1}{2}\begin{bmatrix} A_1 + A_2 & A_3 + A_4 \\ A_3 + A_4 & A_1 + A_2 \end{bmatrix},$$

$$Q = \frac{1}{2}\begin{bmatrix} A_1 - A_2 & A_3 - A_4 \\ A_3 - A_4 & A_1 - A_2 \end{bmatrix}.$$

Then P and Q are the required two $W(2n,n)$. □

Remark. $W(2n,n)$, n odd, exist only if n is the sum of two squares (see Corollary 2.11 [1]).

Corollary 1 *There exist two disjoint and amicable $W(2N, N)$, whose sum and difference are (1,-1)-matrices.*

Proof. From Theorem 5 [3] there exist special Williamson type matrices of order N. □

Theorem 2 *If there exist special Williamson type matrices of order n then there exist four disjoint and amicable $W(4n,n)$, whose sum is a (1 , -1)-matrix.*

Proof. Set $E = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}$, $F = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}$, $G = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}$, $H = \begin{bmatrix} 0 & Q \\ Q & 0 \end{bmatrix}$, where P, Q were given in the proof of Theorem 1. Then E, F, G, H are the required weighing matrices. □

Corollary 2 *There exist four disjoint and amicable $W(4N, N)$, whose sum is a (1 , -1)-matrix.*

It is known that:

(i) if there exist nice Williamson type matrices of order m and special Williamson type matrices of order n then there exist nice Williamson type matrices of order mn (see Theorem 5 [2]);

(ii) if there exist tight Williamson-like matrices of order m and special Williamson type matrices of order n then there exist tight Williamson type matrices of order mn (see Theorem 1 [4]).

Theorem 3 *If there exist tight Williamson-like matrices of order m and nice Williamson type matrices of order n then there exists an Hadamard matrix of order $4mn$.*

Proof. Let A_1, A_2, A_3, A_4 be the tight Williamson-like matrices of order m on a additive abelian group $G = \{g_1, \cdots, g_m\}$ and B_1, B_2, B_3, B_4 be nice Williamson type matrices of order n. Set

$$C_1 = \frac{1}{2}(A_1 + A_2) \times B_1 + \frac{1}{2}(A_1 - A_2) \times B_2,$$

$$C_2 = \frac{1}{2}(A_1 + A_2) \times B_3 + \frac{1}{2}(A_1 - A_2) \times B_4,$$

$$C_3 = \frac{1}{2}(A_3 + A_4) \times B_1 + \frac{1}{2}(A_3 - A_4) \times B_2,$$

$$C_4 = \frac{1}{2}(A_3 + A_4) \times B_3 + \frac{1}{2}(A_3 - A_4) \times B_4.$$

We have

$$\sum_{i=1}^{4} C_j C_j^T = \frac{1}{4}(\sum_{j=1}^{4} A_j A_j^T) \times (\sum_{j=1}^{4} B_j B_j^T) = 4mnI_{mn}.$$

Let $R_1 = (r_{ij})$ be the permutation matrix of order m, defined on G by

$$r_{ij} = \begin{cases} 1 & \text{if } g_i + g_j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Write $R = R_1 \times I_n$ and set

$$H = \begin{bmatrix} C_1 & C_2R & C_3R & C_4R \\ -C_2R & C_1 & -\tilde{C}_4R & \tilde{C}_3R \\ -C_3R & \tilde{C}_4R & C_1 & -\tilde{C}_2R \\ -C_4R & -\tilde{C}_3R & \tilde{C}_2R & C_1 \end{bmatrix}$$

where

$$\tilde{C}_2 = \frac{1}{2}(A_1 + A_2)^T \times B_3 + \frac{1}{2}(A_1 - A_2)^T \times B_4,$$

$$\tilde{C}_3 = \frac{1}{2}(A_3 + A_4)^T \times B_1 + \frac{1}{2}(A_3 - A_4)^T \times B_2,$$

$$\tilde{C}_4 = \frac{1}{2}(A_3 + A_4)^T \times B_3 + \frac{1}{2}(A_3 - A_4)^T \times B_4.$$

We see that $\tilde{C}_i\tilde{C}_i^T = C_iC_i^T$, $i = 2, 3, 4$, and $C_iC_j^T = \tilde{C}_j\tilde{C}_i^T$, $C_i\tilde{C}_j^T = C_j\tilde{C}_i^T$, for $i \neq j$. Finally, it is easy to check that

$$HH^T = 4mnI_{mn}.$$

□

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References

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