AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **59(3)** (2014), Pages 342–351

Broadcast domination in graph products of paths

K.W. Soh* K.M. Koh

Department of Mathematics
National University of Singapore
Lower Kent Ridge Road, 119260
Singapore

skw45610@gmail.com matkohkm@nus.edu.sg

Abstract

Let $\gamma_b(G)$ denote the broadcast domination number for a graph G. In [*Discrete Applied Math.* 154 (2006), 59–75], Dunbar et al. determined the value of $\gamma_b(G)$, where G is the Cartesian product of two paths. In this paper, we evaluate the value of $\gamma_b(G)$, whenever G is the strong product, the direct product and the lexicographic product of two paths.

1 Introduction

A radio station wishes to broadcast from towers of varying capacity so that the broadcast is heard by all intended recipients. Larger capacity towers can broadcast further, but will incur a higher associated cost (or transmission power, say in watts). To minimize this cost, the radio station has to place appropriate towers at carefully selected locations. The problem of choosing the locations and appropriate capacities of the towers can be rephrased as a broadcast domination problem. First introduced by Erwin [2] using the term *cost domination* in 2002, it is one of the many variations of domination. Observe that if we allow only capacity towers of distance 1 to be built in the broadcast scenario, we return to the domination problem. An overview of domination and its variants can be found in [3].

As in [2], we can model the problem by a graph whose vertices are the sections of the region and where an edge between two vertices indicates that the two sections are close to each other. Let G be a connected simple graph with vertex set $V(G)$. The *order* of G is the number of vertices in G. The *distance* between two vertices $u, v \in V(G)$, which we denote as $d(u, v)$, is the length of a shortest $u - v$ path in G. The *eccentricity* of a vertex $v \in V(G)$ is $e(v) = \max\{d(v, u) \mid u \in V(G)\}$. The *radius* and *diameter* of G are defined as $\text{rad}(G) = \min\{e(v) \mid v \in V(G)\}$ and $\text{diam}(G) = \max\{e(v) \mid v \in V(G)\}$ respectively. A vertex $v \in V(G)$ is a *central vertex* if $e(v) = \text{rad}(G)$.

* Corresponding author.

A function $f : V(G) \to \{0, 1, ..., \text{diam}(G)\}$ is a *broadcast* if $f(v) \le e(v)$ for every vertex $v \in V(G)$. We define $V_f^+ = \{v \mid f(v) > 0\}$, and say that every vertex in V_f^+ is a *broadcast vertex*. We define the *broadcast neighborhood* of a broadcast vertex v as $N_f[v] = \{u \mid d(u, v) \le f(v)\}$, and say that each vertex $u \in N_f[v]$ *hears* the broadcast f from vertex v. For a set $S \subseteq V_f^+$, we write $N_f[S] = \bigcup_{v \in S} N_f[v]$. If $N_f[V_f^+] = V(G)$, then f is a *dominating broadcast*. We say that vertex u is *f-dominated* by S if $u \in N_f[S]$. The *cost* of a broadcast f is denoted as $\sigma(f) = \sum_{v \in V_f^+} f(v)$. A dominating broadcast f with minimum cost is called a *γ_b-broadcast* of G, and the *broadcast domination number* for graph G, denoted by $\gamma_b(G)$, is defined as $\gamma_b(G) = \sigma(f)$, where f is a *γ_b-broadcast* of G.

Let $G = G_1 \star G_2$ be a graph product of two graphs G_1 and G_2, which has vertex set $V(G) = V(G_1) \times V(G_2) = \{(x_1, x_2) \mid x_i \in V(G_i) \text{ for } i = 1, 2\}$. There are four standard graph products, namely, the Cartesian, the strong, the direct, and the lexicographic product. Their respective graph products are denoted by $\square$, $\boxtimes$, $\times$, and $\bullet$. Two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if

1. $\star = \square$, and either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$,
2. $\star = \boxtimes$, and either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$, or $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$,
3. $\star = \times$, $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$, or
4. $\star = \bullet$, and either $u_1v_1 \in E(G_1)$, or $u_1 = v_1$ and $u_2v_2 \in E(G_2)$.

The problem of determining the broadcast domination number for the Cartesian product of two graphs was first studied in [1]. Let P_n denote a path of order n. In that paper, the authors found a closed formula for the special case where the two graphs are paths, i.e., a grid graph $P_m \square P_n$. We present their result as follows.

Theorem 1. *[1] For $m \ge n \ge 2$, $\gamma_b(P_m \square P_n) = \text{rad}(P_m \square P_n) = \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$.*

It is natural to consider the other three graph products. The remainder of this paper is organized as follows. In Section 2, we state some known results on broadcast domination, and give a property that is common to the strong, and the direct product of two paths. These results are used in Sections 3 to 5 to enumerate the broadcast domination number for the strong product, the direct product, and the lexicographic product of two paths respectively.

2 Background

A *packing broadcast* is a broadcast such that every vertex in G hears from at most one broadcast vertex. An *efficient broadcast* is a dominating broadcast such that every vertex hears from one broadcast vertex only (possibly including itself). It is clear that every graph G has an efficient broadcast, since we can broadcast from a central vertex with cost equal to rad(G). We also have the following stronger result.

Observation 2. *[1] Every graph G has a γ_b-broadcast which is efficient.*

Observation 3. *[2] For every positive integer n, $\gamma_b(P_n) = \lceil \frac{n}{3} \rceil$.*

For any graph product, the vertices of $P_m \star P_n$ can be placed in a grid consisting of m columns and n rows. Let f be a broadcast of $P_m \star P_n$. We say that a column (respectively row) is *covered* by $U \subseteq V_f^+$ if at least one vertex in that column (respectively row) is f-dominated by U.

Observation 4. *Let G be either the graph $P_m \boxtimes P_n$ or a connected component of $P_m \times P_n$, where $m, n \geq 1$, and let f be a broadcast on G. Then for any $U \subseteq V_f^+$, the number of rows (or columns) covered by U is at most $\sum_{u \in U} \big(2f(u) + 1\big)$, which is less than or equal to $3 \cdot \sum_{u \in U} f(u)$.*

3 Strong product of two paths

In this section, we evaluate the broadcast domination number for the graph $P_m \boxtimes P_n$, whose vertices are labeled as (i, j) or $v_{(i,j)}$, where $i \leq m$ and $j \leq n$ are the column and row numbers respectively. Let f be a broadcast on graph $P_m \boxtimes P_n$ with $m \geq n \geq 1$. Observe that for any $x \in V_f^+$, the induced subgraph $\big[N_f[x]\big]$ is also a strong product of two paths.

Lemma 5. *Consider the graph $P_m \boxtimes P_n$, where $m \geq n \geq 1$. Then there exists a γ_b-broadcast f such that $f(v) \geq \lceil \frac{n-1}{2} \rceil$ for all $v \in V_f^+$.*

Proof. Let $W_f = \big\{v \in V_f^+ \mid 1 \leq f(v) \leq \lceil \frac{n-1}{2} \rceil - 1\big\}$. Suppose on the contrary that W_f is nonempty. Consider vertex $w \in W_f$ with the largest $f(w)$. By Observation 4, since the number of rows covered by $\{w\}$ is at most $\big(2 \cdot f(w) + 1\big) \leq \big(2 \cdot \lceil \frac{n-1}{2} \rceil - 1\big) < n$, it follows that not all vertices in $V(P_m \boxtimes P_n)$ are f-dominated by $\{w\}$. Hence there exists a nonempty set $S \subseteq V_f^+ \backslash \{w\}$ such that for any vertex $s \in S$, the induced subgraph $\big[N_f[s] \cup N_f[w]\big]$ is connected.

Let $t \in S$ be the vertex satisfying $f(t) = \max_{s \in S} f(s)$. We define $\alpha = 2 \cdot \big(f(w) + f(t) + 1\big)$, and consider any subgraph $A = P_\alpha \boxtimes P_{\min\{\alpha,n\}}$ with $N_f[w] \cup N_f[t] \subseteq V(A)$. If vertex t is in $V_f^+ \backslash W_f$, i.e., $f(t) \geq \lceil \frac{n-1}{2} \rceil$, we have $\alpha > 2 \cdot \big(f(t) + 1\big) > n$. Then for the subgraph $A = P_\alpha \boxtimes P_n$ (see Figure 1), not all vertices in A are f-dominated by $\{w, t\}$. Since f is a γ_b-broadcast, there must be another broadcast vertex in A, so that $\sum_{v \in V(A)} f(v) \geq f(t) + f(w) + 1 \geq \lceil \frac{\alpha-1}{2} \rceil$.

Otherwise, vertex $t \in W_f$, i.e., $f(t) \leq f(w)$. By the maximality of $f(t)$, there exists another broadcast vertex in the subgraph $A = P_\alpha \boxtimes P_{\min\{\alpha,n\}}$ other than vertices w and t (see Figure 2). Hence we have $\sum_{v \in V(A)} f(v) \geq f(w) + f(t) + 1 \geq \lceil \frac{\alpha-1}{2} \rceil$.

On the other hand, we can define a dominating broadcast g as follows:

$$g(v) = \begin{cases} \lceil \frac{\alpha-1}{2} \rceil & \text{if } v = a \text{ is a central vertex in } A \\ 0 & \text{if } v \in V(A) \backslash \{a\} \\ f(v) & \text{if } v \in V(P_m \boxtimes P_n) \backslash V(A). \end{cases}$$

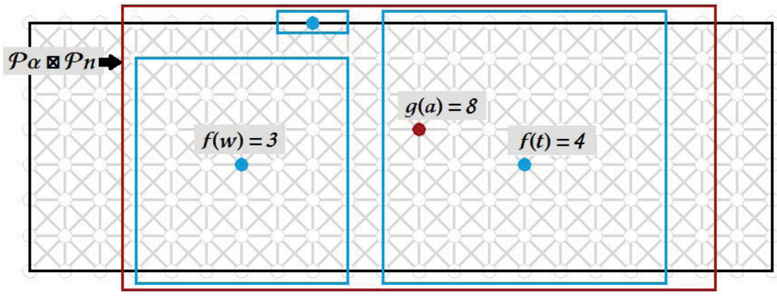

Figure 1: An illustration of vertex $t \in V_f^+ \backslash W_f$ when $\alpha > n = 8$.

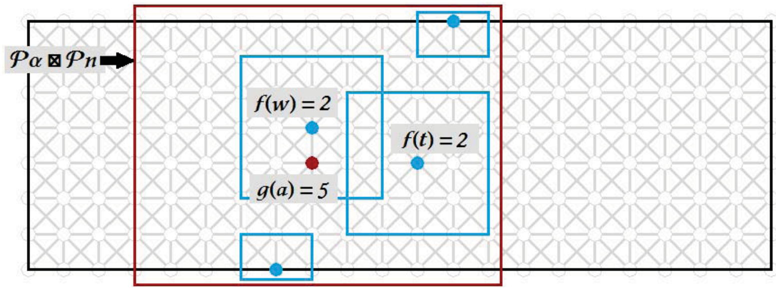

Figure 2: An illustration of vertex $t \in W_f$ when $\alpha > n = 8$.

To see why g is dominating, we observe that vertex $a \in V(A)$ covers $\big(2 \cdot g(a)+1\big) \geq \alpha$ rows (or columns). Therefore if $\alpha \leq n$, we have $V(A) = V(P_\alpha \boxtimes P_\alpha) \subseteq N_g[a]$. Otherwise $\alpha > n$, and $V(A) = V(P_\alpha \boxtimes P_n) \subseteq N_g[a]$. In fact, since f is a γ_b-broadcast and $\sigma(f) = \sigma(g)$, g is therefore a γ_b-broadcast.

Finally, we consider the set $W_g = \big\{v \in V_g^+ \mid 1 \leq g(v) \leq \lceil \frac{n-1}{2} \rceil - 1\big\}$. For $\alpha \leq n$, $|W_g| < |W_f|$. This is because vertex t is in W_f, which implies that $W_g \subset \big(W_f \cup \{a\}\big) \backslash \{w, t\}$. The same inequality holds for $\alpha > n$ since $g(a) = \lceil \frac{\alpha-1}{2} \rceil > \lceil \frac{n-1}{2} \rceil$, i.e., vertex a is not in W_g. This implies that $W_g \subset \big(W_f \cup \{a\}\big) \backslash \{w\}$. By applying the above procedure repeatedly for a finite number of times, we obtain a γ_b-broadcast (say h) such that $h(v) \geq \lceil \frac{n-1}{2} \rceil$ for all $v \in V_h^+$. □

Lemma 6. *For any graph $P_m \boxtimes P_n$ with $m \geq n \geq 1$, there exists a γ_b-broadcast f such that f is efficient and $f(v) \geq \lceil \frac{n-1}{2} \rceil$ for all $v \in V_f^+$.*

Proof. By Observation 2, there exists a γ_b-broadcast f which is efficient. If there is a vertex $v \in V_f^+$ with $f(v) < \lceil \frac{n-1}{2} \rceil$, we apply the steps described in the proof for Lemma 5, so that the result follows. □

Lemma 7. *For $m \geq n \geq 2$, $\gamma_b(P_m \boxtimes P_n) \geq \left\lceil \frac{m - \lfloor \frac{m}{p} \rfloor}{2} \right\rceil$, where $p = \big(2 \cdot \lceil \frac{n-1}{2} \rceil + 1\big)$ is the smallest odd integer greater than or equal to n.*

Proof. By Lemma 6, the graph $P_m \boxtimes P_n$ has an efficient γ_b-broadcast $f : V \to \big\{0, \lceil \frac{n-1}{2} \rceil, \lceil \frac{n-1}{2} \rceil + 1, \ldots, \text{diam}(P_m \boxtimes P_n)\big\}$. We define a broadcast g as follows:

$$g(v) = \begin{cases} \lceil \frac{n-1}{2} \rceil & \text{if } f(v) \geq \lceil \frac{n-1}{2} \rceil \\ 0 & \text{otherwise.} \end{cases}$$

Since g is a packing broadcast (but not necessarily dominating), we have $\sum_{v \in V_g^+} \big(2 \cdot g(v) + 1\big) \leq m$, so that $|V_f^+| = |V_g^+| \leq \lfloor \frac{m}{2 \cdot \lceil \frac{n-1}{2} \rceil + 1} \rfloor = \lfloor \frac{m}{p} \rfloor$. Finally, as f is a dominating broadcast, we have $\sum_{v \in V_f^+} \big(2 \cdot f(v) + 1\big) \geq m$, or equivalently, $\gamma_b(P_m \boxtimes P_n) = \sum_{v \in V_f^+} f(v) \geq \lceil \frac{m - |V_f^+|}{2} \rceil \geq \lceil \frac{m - \lfloor \frac{m}{p} \rfloor}{2} \rceil$. □

In the following theorem, we give the broadcast domination number for the graph $P_m \boxtimes P_n$. Roughly speaking, the proof involves a construction of a low cost broadcast, whose broadcast vertices lie in the middle of the columns, and are separated by a distance that is roughly n. Except for one of the broadcast vertices, which has a larger broadcast capacity in order to broadcast to vertices in the m^{th} column of the graph, all the other broadcast vertices have the same broadcast capacity.

Theorem 8. *For the graph $P_m \boxtimes P_n$ with $m \geq n \geq 1$,*

$$\gamma_b(P_m \boxtimes P_n) = \left\lceil \frac{1}{2} \cdot \Big(m - \lfloor \frac{m}{\max\{p, 3\}} \rfloor \Big) \right\rceil,$$

where $p = \big(2 \cdot \lceil \frac{n-1}{2} \rceil + 1\big)$.

Proof. For $n = 1$, we have from Observation 3 that $\gamma_b(P_m \boxtimes P_1) = \gamma_b(P_m) = \lceil \frac{m}{3} \rceil = \lceil \frac{m - \lfloor \frac{m}{3} \rfloor}{2} \rceil$. For $n \geq 2$, we define a broadcast f as follows:

$$f(v_{(i,j)}) = \begin{cases} \lceil \frac{n-1}{2} \rceil & \text{if } i = \big(\lceil \frac{n+1}{2} \rceil + k \cdot p\big) \text{ and } j = \lceil \frac{n}{2} \rceil, \\ & \quad \text{where } k = 0, 1, ..., \lfloor \frac{m}{p} \rfloor - 2 \\ \left\lceil \frac{m - p \cdot \lfloor \frac{m}{p} \rfloor + p - 1}{2} \right\rceil & \text{if } i = m - \left\lceil \frac{m - p \cdot \lfloor \frac{m}{p} \rfloor + p - 1}{2} \right\rceil \text{ and } j = \lceil \frac{n}{2} \rceil \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that f is a dominating broadcast. Also since $\lceil \frac{n-1}{2} \rceil = \frac{p-1}{2}$, $\sigma(f) = \big(\lfloor \frac{m}{p} \rfloor - 1\big) \cdot \lceil \frac{n-1}{2} \rceil + \lceil \frac{m - p \cdot \lfloor \frac{m}{p} \rfloor + p - 1}{2} \rceil \leq \frac{p \cdot \lfloor \frac{m}{p} \rfloor - p - \lfloor \frac{m}{p} \rfloor + 1}{2} + \frac{m - p \cdot \lfloor \frac{m}{p} \rfloor + p - 1}{2} + \frac{1}{2} \leq \lfloor \frac{m - \lfloor \frac{m}{p} \rfloor}{2} + \frac{1}{2} \rfloor = \lceil \frac{m - \lfloor \frac{m}{p} \rfloor}{2} \rceil$. Then by Lemma 7, we get the required result. □

4 Direct product of two paths

In this section, we determine the broadcast domination number for graph $P_m \times P_n$, whose vertices are labeled as (i, j) or $v_{(i,j)}$, where $i \leq m$ and $j \leq n$ are the column and row numbers respectively. Observe that $P_m \times P_n$ consists of two isomorphic connected components if either m or n is even. Otherwise it has two non-isomorphic connected components [4]. We call the component of $P_m \times P_n$ containing $v_{(1,1)}$ the *even component*, and the other component the *odd component*.

Lemma 9. *For any component of graph $P_m \times P_n$ with $m \geq n \geq 1$, there exists a γ_b-broadcast f such that*
(i) f is efficient,
(ii) $f(v) \geq \lceil \frac{n-1}{2} \rceil$ for all $v \in V_f^+$,
(iii) $f(v_{(i,j)}) \geq \lceil \frac{n}{2} \rceil$ for all $v_{(i,j)} \in V_f^+$, where $j \neq \lceil \frac{n-1}{2} \rceil$, and
(iv) each column of the graph has at most one broadcast vertex.

Proof. The proofs for (i) and (ii) are similar to that of Lemmas 6 and 5 respectively, and are omitted. For (iii), since $\lceil \frac{n-1}{2} \rceil = \lceil \frac{n}{2} \rceil$ when n is even, it remains to show that (iii) holds for odd values of n. Suppose on the contrary that (iii) is false for some odd value of n. Then we can find a vertex $v_{(a,b)}$ with $b \neq \lceil \frac{n-1}{2} \rceil$ such that $f(v_{(a,b)}) \leq \frac{n-1}{2}$. Note that (ii) implies $f(v_{(a,b)}) = \frac{n-1}{2}$, so there is at least one vertex among all the vertices in columns $a-1$, a and $a+1$ that do not hear the broadcast from $v_{(a,b)}$. Since f is dominating, there exists a broadcast vertex u near the a^{th} column. By (ii), we have $f(u) \geq \lceil \frac{n-1}{2} \rceil$. However this contradicts our assumption in (i) that f is efficient. Hence (iii) must be true (see Figure 3). Finally, (iv) follows from (ii) and (iii). □

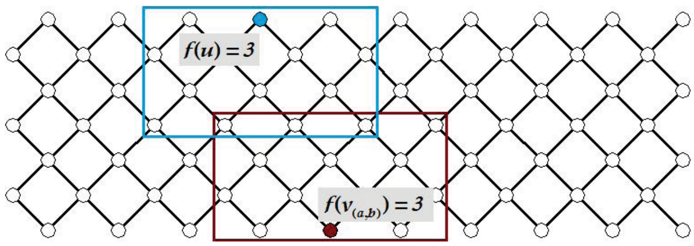

Figure 3: Illustration of proof for Lemma 9(iii) when $n = 7$ is odd.

We will now proceed to consider each of the components.

Even component with n odd:

For $m \geq n \geq 2$ with n odd, let graph A be the even component of $P_m \times P_n$. Since $v_{(1,1)}$ is in the even component, the number of vertices in the first row is $\lceil \frac{m}{2} \rceil$. Based on this fact, we have the following lemma.

Lemma 10. $\gamma_b(A) \geq \lceil \frac{m-k}{2} \rceil$, *where* $k = \left\lfloor \frac{2 \cdot \lceil \frac{m}{2} \rceil}{n+1} \right\rfloor$.

Proof. By Lemma 9(i)-(ii), graph A has an efficient γ_b-broadcast $f : V \to \{0, \lceil \frac{n-1}{2} \rceil, \lceil \frac{n-1}{2} \rceil + 1, ..., \text{diam}(A)\}$. Let $S = \{\, a \mid v_{(a,b)} \in V_f^+ \}$ be the column numbers that has one broadcast vertex. We define a broadcast g as follows:

$$g(v_{(i,j)}) = \begin{cases} \lceil \frac{n-1}{2} \rceil & \text{if } i = a \text{ or } a+1, \text{ and } j = \lceil \frac{n+1}{2} \rceil, \text{ where } a \in S \\ 0 & \text{otherwise.} \end{cases}$$

Note that vertices $(a, \lceil \frac{n+1}{2} \rceil)$ and $(a+1, \lceil \frac{n+1}{2} \rceil)$ cannot exist simultaneously in $V(A)$. Therefore $|V_f^+| = |V_g^+|$, and it is clear that g is a packing broadcast, but not necessarily a dominating broadcast.

Let $V_1(A)$ be the set of vertices in the first row of graph A. Since we have $|N_g(v) \cap V_1(A)| = \frac{n+1}{2}$ for each $v \in V_g^+$, it follows from the definition of packing broadcast that $|V_f^+| = |V_g^+| \leq \left\lfloor \frac{\lceil \frac{m}{2} \rceil}{\frac{n+1}{2}} \right\rfloor = k$. Finally, as f is a dominating broadcast, we have $\sum_{v \in V_f^+} \big(2 \cdot f(v) + 1\big) \geq m$, or equivalently, $\gamma_b(A) = \sum_{v \in V_f^+} f(v) \geq \left\lceil \frac{m - |V_f^+|}{2} \right\rceil \geq \left\lceil \frac{m-k}{2} \right\rceil$. □

Theorem 11. *$\gamma_b(A) = \left\lceil \frac{m-k}{2} \right\rceil$, where $k = \left\lfloor \frac{2 \cdot \lceil \frac{m}{2} \rceil}{n+1} \right\rfloor$.*

Proof. The proof is similar to that of Theorem 8. We define a broadcast f as follows:

$$f(v_{(i,j)}) = \begin{cases} \left\lceil \frac{n+(-1)^q}{2} \right\rceil & \text{if } i = (2q-1) \cdot \left\lceil \frac{n+1}{2} \right\rceil \text{ and } j = \left\lceil \frac{n+1}{2} \right\rceil, \\ & \quad \text{where } q = 1, 2, ..., k-1 \\ \left\lceil \frac{m-k}{2} \right\rceil - \left\lfloor \frac{n(k-1)}{2} \right\rfloor & \text{if } i = m - \left\lceil \frac{m-k}{2} \right\rceil + \left\lfloor \frac{n(k-1)}{2} \right\rfloor, \\ & \quad \text{and } j = \left\lceil \frac{n+1}{2} \right\rceil - 1 \text{ or } \left\lceil \frac{n+1}{2} \right\rceil \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that all the broadcast vertices belong to $V(A)$, and that f is dominating. Also since n is odd, we have $\left\lfloor \frac{k-1}{2} \right\rfloor \cdot \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{k-1}{2} \right\rceil \cdot \left\lceil \frac{n-1}{2} \right\rceil = \frac{n(k-1)}{2}$ when k is odd. Similarly we have $\left\lfloor \frac{k-1}{2} \right\rfloor \cdot \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{k-1}{2} \right\rceil \cdot \left\lceil \frac{n-1}{2} \right\rceil = \frac{k-2}{2} \cdot \frac{n+1}{2} + \frac{k}{2} \cdot \frac{n-1}{2} = \frac{n(k-1)-1}{2}$ when k is even.

Therefore $\sigma(f) = \left\lceil \frac{m-k}{2} \right\rceil - \left\lfloor \frac{n(k-1)}{2} \right\rfloor + \left\lfloor \frac{k-1}{2} \right\rfloor \cdot \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{k-1}{2} \right\rceil \cdot \left\lceil \frac{n-1}{2} \right\rceil \leq \left\lceil \frac{m-k}{2} \right\rceil$. Then by Lemma 10, we get the desired result. □

Odd component with n odd:

For $m \geq n \geq 2$ with n odd, let graph B be the odd component of $P_m \times P_n$. Since $v_{(1,1)}$ is not in the odd component, the number of vertices in the first row is $\left\lfloor \frac{m}{2} \right\rfloor$. Based on this fact, we have the following lemma.

Lemma 12. *$\gamma_b(B) \geq \left\lceil \frac{m-k}{2} \right\rceil$, where $k = \left\lfloor \frac{m}{n+1} \right\rfloor$.*

Proof. The proof is similar to that of Lemma 10, and thus omitted. □

Theorem 13. *$\gamma_b(B) = \left\lceil \frac{m-k}{2} \right\rceil$, where $k = \left\lfloor \frac{m}{n+1} \right\rfloor$.*

Proof. The proof is similar to that of Theorem 11. We define a broadcast f as follows:

$$f(v_{(i,j)}) = \begin{cases} \left\lceil \frac{n+(-1)^{q+1}}{2} \right\rceil & \text{if } (i,j) = \left((2q-1)\left\lceil \frac{n+1}{2} \right\rceil + 1, \left\lceil \frac{n+1}{2} \right\rceil\right) \\ & \quad \text{where } q = 1, 2, ..., k-1 \\ \left\lceil \frac{m-k}{2} \right\rceil - \left\lceil \frac{n(k-1)}{2} \right\rceil & \text{if } i = m - \left\lceil \frac{m-k}{2} \right\rceil + \left\lceil \frac{n(k-1)}{2} \right\rceil, \\ & \quad \text{and } j = \left\lceil \frac{n+1}{2} \right\rceil - 1 \text{ or } \left\lceil \frac{n+1}{2} \right\rceil \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that all the broadcast vertices belong to $V(B)$, and that f is dominating. Also since n is odd, by considering the cases when k is odd and even separately, $\sigma(f) = \lceil \frac{m-k}{2} \rceil - \lceil \frac{n(k-1)}{2} \rceil + \lceil \frac{k-1}{2} \rceil \cdot \lceil \frac{n+1}{2} \rceil + \lfloor \frac{k-1}{2} \rfloor \cdot \lceil \frac{n-1}{2} \rceil \leq \lceil \frac{m-k}{2} \rceil$. Then by Lemma 12, we get the desired result. □

Even or odd component with n even:

When n is even, the even component is isomorphic to the odd component. Hence we consider only the even component. Coincidentally, the following proofs are almost identical to that of the strong product of two paths. For $m \geq n \geq 2$ with n even, let graph C be the even component of $P_m \times P_n$.

Lemma 14. *$\gamma_b(C) \geq \lceil \frac{m-k}{2} \rceil$, where $k = \lfloor \frac{m}{n+1} \rfloor$.*

Proof. The proof in Lemma 7 applies here. Notice that $p = \left(2 \cdot \lceil \frac{n-1}{2} \rceil + 1\right)$ can be simplified to $p = n + 1$. □

Theorem 15. *$\gamma_b(C) = \lceil \frac{m-k}{2} \rceil$, where $k = \lfloor \frac{m}{n+1} \rfloor$.*

Proof. Let $p = \left(2 \cdot \lceil \frac{n-1}{2} \rceil + 1\right) = n + 1$. We define a broadcast f as follows:

$$f(v_{(i,j)}) = \begin{cases} \lceil \frac{n-1}{2} \rceil & \text{if } i = \left(\lceil \frac{n+1}{2} \rceil + r \cdot p\right) \text{ and } j = \lceil \frac{n+(-1)^r}{2} \rceil, \\ & \quad \text{where } r = 0, 1, ..., \lfloor \frac{m}{p} \rfloor - 2 \\ \lceil \frac{m - p \cdot \lfloor \frac{m}{p} \rfloor + p - 1}{2} \rceil & \text{if } i = m - \lceil \frac{m - p \cdot \lfloor \frac{m}{p} \rfloor + p - 1}{2} \rceil, \\ & \quad \text{and } j = \lceil \frac{n-1}{2} \rceil \text{ or } \lceil \frac{n+1}{2} \rceil \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that all the broadcast vertices belong to $V(C)$, and that f is a dominating broadcast with cost $\sigma(f) \leq \lceil \frac{m-k}{2} \rceil$. Therefore by Lemma 14, we have the desired result. □

We now give a unified formula for the broadcast domination number of graph $P_m \times P_n$.

Corollary 16. *For the graph $P_m \times P_n$ with $m \geq n \geq 1$,*

$$\gamma_b(P_m \times P_n) = \begin{cases} m & \textit{if } n = 1 \\ n \cdot \left(\frac{m+1}{n+1}\right) & \textit{if } n \geq 2,\ m, n \textit{ both odd with } \frac{m+1}{n+1} \textit{ integer} \\ 2 \cdot \left\lceil \frac{m - \lfloor \frac{m}{n+1} \rfloor}{2} \right\rceil & \textit{otherwise.} \end{cases}$$

Proof. Use Observation 3, Theorems 11, 13, and 15, and then simplify to get the required result. □

5 Lexicographic product of two paths

In this section, we determine the broadcast domination number for graph $P_m \bullet P_n$, whose vertices are labeled as (i, j) or $v_{(i,j)}$, where $i \leq m$ and $j \leq n$ are the column and row numbers respectively. Note that generally $G_1 \bullet G_2$ is not isomorphic to $G_2 \bullet G_1$. Let f be a broadcast on graph $P_m \bullet P_n$. Observe that for any $x \in V_f^+$ with $f(x) \geq 2$, the induced subgraph $\left[N_f[x]\right]$ is also a lexicographic product of two paths. Furthermore, if vertex v is f-dominated by $\{x\}$, then all vertices in the same column as vertex v are f-dominated by $\{x\}$.

The following observation is similar to that of Observation 4.

Observation 17. *Let f be a broadcast on $P_m \bullet P_n$, where $m, n \geq 1$. Then for any $U \subseteq V_f^+$, the number of columns covered by U is at most $\sum_{u\in U}\big(2f(u)+1\big)$, which is less than or equal to $3 \cdot \sum_{u\in U} f(u)$.*

In the following lemma, we consider the graph $P_m \bullet P_n$ with either $m \leq 2$ or $n \leq 3$.

Lemma 18. *Consider the graph $P_m \bullet P_n$.*
(i) If either $m = 1$ or $n \leq 3$, then $\gamma_b(P_m \bullet P_n) = \left\lceil \frac{\max\{m,n\}}{3} \right\rceil$.
(ii) If $n \geq 4$, then $\gamma_b(P_2 \bullet P_n) = 2$.

Proof. We will only prove (i), as the proof for (ii) is obvious. If either $m = 1$ or $n = 1$, then $\gamma_b(P_m \bullet P_n) = \gamma_b(P_{\max\{m,n\}}) = \left\lceil \frac{\max\{m,n\}}{3} \right\rceil$ by Observation 3. Otherwise, $n \in \{2, 3\}$. Let f be a γ_b-broadcast on $P_m \bullet P_n$. By Observation 17, $3 \cdot \sum_{u\in V_f^+} f(u) \geq m$. Hence $\gamma_b(P_m \bullet P_n) = \sum_{u\in V_f^+} f(u) \geq \left\lceil \frac{m}{3} \right\rceil$. On the other hand, it can be verified that the following broadcast on $P_m \bullet P_n$ is dominating.

$$g(v_{(p,q)}) = \begin{cases} 1 & \text{if } (p,q) = (3k-1, 2), \text{ where } k = 1, 2, ..., \left\lfloor \frac{m}{3} \right\rfloor \\ \left\lceil \frac{m}{3} - \left\lfloor \frac{m}{3} \right\rfloor \right\rceil & \text{if } (p,q) = (m, 2) \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore when $m \equiv 0 \pmod 3$, $\gamma_b(P_m \bullet P_n) \leq \sum_{v\in V_g^+} g(v) = \frac{m}{3} = \left\lceil \frac{m}{3} \right\rceil$. Similarly when $m \not\equiv 0 \pmod 3$, $\gamma_b(P_m \bullet P_n) \leq \sum_{v\in V_g^+} g(v) \leq \left\lfloor \frac{m}{3} \right\rfloor + 1 = \left\lceil \frac{m}{3} \right\rceil$. Hence g is a γ_b-broadcast with cost equal to $\left\lceil \frac{m}{3} \right\rceil$. □

For the graph $P_m \bullet P_n$ with $m \geq 3$ and $n \geq 4$, we shall examine vertices in the bottom and top rows.

Observation 19. *Let f be a broadcast on $P_m \bullet P_n$, where $m \geq 3$ and $n \geq 4$. Let $B = \{v_{(i,1)} \mid 1 \leq i \leq m\}$ and $T = \{v_{(i,n)} \mid 1 \leq i \leq m\}$. Then for any $U \subseteq V_f^+$, the number of f-dominated vertices in $B \cup T$ is at most $5 \sum_{u\in U} f(u)$.*

Proof. Consider any vertex $v \in V_f^+$. If $f(v) = 1$, then $|N_f[v] \cap (B \cup T)| \leq 5$ as at most three vertices in $B \cup T$ hear the broadcast f from vertex v. For $f(v) = k \geq 2$, since $N_f[v]$ covers at most $(2k+1)$ columns, we have $|N_f[v] \cap (B \cup T)| \leq 2(2k+1) \leq 5k$. □

Lemma 20. *For $m \geq 3$ and $n \geq 4$, $\gamma_b(P_m \bullet P_n) = \left\lceil \frac{2m}{5} \right\rceil$.*

Proof. Let f be a γ_b-broadcast, and let B and T be as defined earlier. Then all vertices in $B \cup T$ hear the broadcast f. Hence by Observation 19, $2m = |B \cup T| \leq 5 \sum_{v\in V_f^+} f(v) = 5\gamma_b(P_m \bullet P_n)$. It follows that $\gamma_b(P_m \bullet P_n) \geq \left\lceil \frac{2m}{5} \right\rceil$.

On the other hand, the broadcast defined by

$$\begin{array}{llll} f(v_{(5i-2,1)}) = 2 & i = 1, 2, ..., \lceil \frac{m}{5} \rceil & \text{for } m \equiv 0, 3, 4 & \pmod 5 \\ f(v_{(4,1)}) = 3, f(v_{(5i,1)}) = 2 & i = 2, 3, ..., \lceil \frac{m}{5} \rceil & \text{for } m \equiv 1, 2 & \pmod 5 \end{array}$$

is a dominating broadcast of cost $\lceil \frac{2m}{5} \rceil$. The result follows. □

Theorem 21. *For the graph $P_m \bullet P_n$,*

$$\gamma_b(P_m \bullet P_n) = \begin{cases} \max\{\lceil \frac{m}{3} \rceil, \lceil \frac{n}{3} \rceil\} & \text{if } m = 1 \text{ or } n \in \{1, 2, 3\} \\ \max\{\lceil \frac{2m}{5} \rceil, 2\} & \text{if } m \geq 2 \text{ and } n \geq 4. \end{cases}$$

Proof. Result follows from Lemmas 18 and 20. □

To end this paper, we would like to present here some open problems for future research. The value of $\gamma_b(G)$, where G is the Cartesian product of two cycles, has been enumerated in [5].

Problem 1. Evaluate the value of $\gamma_b(G)$, where G is the strong product, the direct product or the lexicographic product of two cycles.

Problem 2. Evaluate the value of $\gamma_b(G)$, where G is the Cartesian product, the strong product, the direct product or the lexicographic product of a path and a cycle.

Acknowledgments

We would like to express our sincere thanks to the referees for their useful comments and suggestions.

References

[1] J. Dunbar, D. Erwin, T. Haynes, S. M. Hedetniemi and S. T. Hedetniemi, Broadcasts in graphs, *Discrete Applied Math.* **154** (2006), 59–75.

[2] D. Erwin, Dominating broadcasts in graphs, *Bull. Inst. Combin. Appl.* **42** (2004), 89–105.

[3] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[4] P. K. Jha, S. Klavžar and B. Zmazek, Isomorphic components of Kronecker product of bipartite graphs, *Discuss. Math. Graph Theory* **17** (1997), 301–309.

[5] K. W. Soh and K. M. Koh, Broadcast domination in tori, (submitted).

(Received 17 June 2012; revised 22 Feb 2013, 24 Dec 2013)