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Quadratic excesses of covers with triples of λK_n

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Abstract

A cover (with triples) of λK_n is an ordered pair (V, B) where V is an n-element set and B is a set of 3-element subsets of V called blocks such that each 2-element subset of V appears in at least λ blocks. Define $E(B) = \{\{x, y\}, \{x, z\}, \{y, z\} \mid \{x, y, z\} \in B\}$. The excess (sometimes called the padding) of a cover is defined to be $E(B) - E(\lambda K_n)$, although it is often convenient to think of the excess as the graph on n vertices with edge set induced by $E(B) - E(\lambda K_n)$. A quadratic graph is a graph in which every vertex has degree 2 or 0. In 1987, Colbourn and Rosa characterized which quadratic graphs are the excess of some cover of K_n. In this paper, these results are extended by characterizing which quadratic graphs are the excess of some cover of λK_n for $\lambda \geq 2$.

1 Introduction

A λ-fold triple system is an ordered pair (V, B) where V is an n-element set and B is a set of 3-element subsets of V called blocks such that each 2-element subset of V appears in λ blocks of B. The blocks of B are also called triples. A K_3-decomposition of a graph G is a partition of $E(G)$ into sets, each of which induces a graph isomorphic to K_3. It is straightforward to see that a λ-fold triple system where $|V| = n$ is equivalent to a K_3-decomposition of λK_n (the multigraph on n vertices in which each pair of vertices is joined by λ edges). Necessary and sufficient conditions for the existence of λ-fold triple systems are well known (see Theorem 1). If there does not exist a K_3-decomposition of λK_n, it is natural to ask what is the closest similar structure. Packings and covers provide two natural notions of closeness to a K_3-decomposition.

For the purposes of this paper, a packing of λK_n on the vertex set V is a K_3-decomposition of a subgraph H of λK_n and a cover of λK_n on the vertex set V is a K_3-decomposition of $\lambda K_n + P$ where P is a set of edges with underlying vertex set V. In the case of a packing, define $L = E(\lambda K_n) - E(H)$. L and P are called the leave of the packing and the excess (or padding) of the cover respectively, although it is often convenient to think of the leave or excess as the subgraph on n vertices with the edge set equal to the leave or excess (so there may be isolated vertices). It is natural to ask which types of graphs can be leaves or excesses of packings and covers of λK_n. Of some interest in this context are quadratic graphs, which are graphs in which each vertex has degree 2 or 0. In [3, 4], Colbourn and Rosa found necessary and sufficient conditions for the existence of a quadratic leave or excess of λK_n in the case when $\lambda = 1$. Recently, in joint work with Rodger, the author found a characterization of the quadratic graphs which are leaves of λK_n, finding necessary and sufficient conditions in [2]. The obvious follow-up to this is to find necessary and sufficient conditions for a quadratic graph to be an excess of λK_n; these conditions are the main result of the paper (see Theorem 3).

Before proceeding to our main result, it is worth noting that other work on excesses of covers has been considered; for instance, the possible excesses for a minimum cover of K_n with 4-cycles and with 5-cycles were found by Bialostocki and Schönheim, and by Milici, respectively; (see [1] and [6]).

2 Results

In this section, the main result of the paper is given, namely, necessary and sufficient conditions for a quadratic graph to be the excess of a cover of λK_n (naturally we assume that $\lambda \geq 1$ in this paper). We start however by stating a very well known theorem on maximum packings and minimum covers of λK_n (packings and covers for which the leave and excess have as few edges as possible).

Theorem 1. *[5, 7] Let $\lambda \geq 1$ and $n \neq 2$. Let P (or L) be any multigraph with the least number of edges in which all vertices have degree congruent to $\lambda(n-1) \pmod 2$ and with $|E(P)| + \frac{\lambda n(n-1)}{2} \equiv 0 \pmod 3$ (or $\frac{\lambda n(n-1)}{2} - |E(L)| \equiv 0 \pmod 3$ respectively). Then there exists a K_3-decomposition of $\lambda K_n \cup E(P)$ (or $\lambda K_n - E(L)$ respectively).*

The next theorem provides necessary and sufficient conditions for a quadratic graph to be the excess of a cover of K_n.

Theorem 2. *[3] Let Q be a quadratic graph on n vertices. Then Q is the excess of a cover of K_n if and only if*

1. *n is odd, and*
2. *$|E(Q)| + |E(K_n)| \equiv 0 \pmod 3$.*

We are now ready to proceed to the main theorem.

Theorem 3. *Let Q be a quadratic graph on n vertices. Then Q is the excess of a cover of λK_n if and only if*

1. $\lambda(n-1)$ *is even,*

2. $|E(Q)| + |E(\lambda K_n)| \equiv 0 \pmod 3$*, and*

3. $n \neq 2$.

Proof. The necessity of Conditions (1) and (2) follows since each vertex in each triple has even degree and each triple contains 3 edges respectively. The necessity of Condition (3) is clear since $\lambda K_2 + E(Q)$ contains no copies of K_3.

To prove the sufficiency, suppose that $(1-3)$ hold. We consider several cases in turn.

Case 1: $n \equiv 1, 3 \pmod 6$

By Theorem 1, let (V, B_1) be a K_3-decomposition of $(\lambda - 1)K_n$ (with $L = \emptyset$). By Condition (2) and Theorem 2, let (V, B_2) be a cover of K_n with excess Q. Then $(V, B_1 \cup B_2)$ is the required cover.

Case 2: $n \equiv 5 \pmod 6$

Let $\epsilon \in \{1, 2, 3\}$ with $\epsilon \equiv \lambda \pmod 3$. Note that $|E(\epsilon K_n)| \equiv |E(\lambda K_n)|$ (mod 3). Since $n \equiv 5 \pmod 6$, by Theorem 1, there exists a K_3-decomposition (V, B_1) of $(\lambda - \epsilon)K_n$.

If $\epsilon = 1$ then by Condition (2) and Theorem 2, let (V, B_2) be a cover of K_n with excess Q.

Suppose $\epsilon = 2$. By Condition (2), $|E(Q)| \equiv 1 \pmod 3$. Since $n \equiv 2 \pmod 3$, some vertex in V has degree 0 in Q, say x. Let $c = (c_0, c_1, ..., c_i)$ be a cycle in Q and let $c' = (c_0, x, c_1, ..., c_i)$. Form Q' from Q by replacing c with c'. Q' is quadratic on the vertex set V and $|E(Q')| \equiv 2 \pmod 3$, so by Theorem 2, let (V, B_2') be a cover of K_n with excess Q'. By Theorem 1, let (V, B_2'') be a maximum packing of K_n with leave the 4-cycle (c_0, x, c_1, d) where d is any vertex in V other than x, c_0, and c_1 (this exists since in this case $n \equiv 5 \pmod 6$ so $n \geq 5$). Let $B_2 = B_2' \cup B_2'' \cup \{\{c_0, c_1, d\}\}$.

If $\epsilon = 3$ then by Condition 2 $|E(Q)| \equiv 0 \pmod 3$. Since $n \equiv 2 \pmod 3$, some vertex in V has degree 0 in Q, say x. Let $c = (c_0, c_1, ..., c_i) \in Q$ and let $c' = (c_0, x, c_1, ..., c_i)$. Form Q' from Q by replacing c with c'. Q' is quadratic on the vertex set V and $|E(Q')| \equiv 1 \pmod 3$, so by the previous case in this proof when $\epsilon = 2$, let (V, B_2') be a cover of $2K_n$ with excess Q'. By Theorem 1, let (V, B_2'') be a maximum packing of K_n with leave the 4-cycle (c_0, x, c_1, d) where d is any vertex in V other than x, c_0, and c_1. Let $B_2 = B_2' \cup B_2'' \cup \{\{c_0, c_1, d\}\}$.

Then in each subcase ($\epsilon = 1, 2$, and 3), $(V, B_1 \cup B_2)$ is the required cover.

Case 3: $n \equiv 4 \pmod 6$

Since $n \equiv 4 \pmod 6$, by Condition (1), λ is even. By Theorem 1, let (V, B_1) be a K_3-decomposition of $(\lambda - 2)K_n$.

By Condition (2), $|E(Q)| \equiv 0 \pmod 3$. Hence, Q also satisfies the conditions for an excess for $v = n - 1$ and $\lambda = 1$. So let $x \in V$ be an isolated vertex in Q (in this case $|V(Q)| \equiv 1 \pmod 3$ and $|E(Q)| \equiv 0 \pmod 3$ so such an x exists) and let $(V \setminus \{x\}, B_2)$ be a cover of K_{n-1} with excess Q. By Theorem 1, let (V, B_3) be a maximum packing of K_n with leave Q' consisting of a $K_{1,3}$ and $\frac{n-4}{2}$ independent edges, where the vertex set of the $K_{1,3}$ is $\{w, x, y, z\} \subset V$ with y being the vertex of degree 3. Then $(V, B_1 \cup B_2 \cup B_3 \cup \{\{x, a_i, b_i\} \mid \{a_i, b_i\}$ is an independent edge in $Q'\} \cup \{\{x, y, z\}, \{x, y, w\}\})$ is the required decomposition.

Case 4: $\lambda = 2$ and $n \equiv 0, 2 \pmod 6$

First suppose Q consists entirely of 2-cycles and isolated vertices. Hence $|E(Q)|$ is even, and by Condition (2), $|E(Q)| \equiv 0$ or $1 \pmod 3$ when $n \equiv 0$ or $2 \pmod 6$ respectively (recall $\lambda = 2$ in this case). Hence $|E(Q)| \equiv 0$ or $4 \pmod 6$ when $n \equiv 0$ or $2 \pmod 6$ respectively and thus the number of isolated vertices in Q is equivalent to 0 or 4 (mod 6) when $n \equiv 0$ or $2 \pmod 6$ respectively. If Q contains no isolated vertices (so $n \equiv 0 \pmod 6$), then by Theorem 1, for each $k \in \{1, 2\}$ let (V, B_k) be a minimum cover of K_n with the same 1-factor excess. Then $(V, B_1 \cup B_2)$ is the required cover. Otherwise Q contains at least 4 isolated vertices, and this case is handled below by using the observation in the next paragraph.

Note that if Q is a quadratic graph in which there are three isolated vertices, say a, b, and c, in Q, then Q is a quadratic excess if and only if $Q \cup \{\{a, b\}, \{a, c\}, \{b, c\}\}$ is a quadratic excess. If Q contains at least 3 isolated vertices, then add a cycle of length 3 on three of the isolated vertices to Q.

In light of the last two paragraphs, to complete the proof of Case 4 it now suffices to consider the situation where Q has a cycle $c = (v_0, v_1, ..., v_x)$ of length $x + 1 \geq 3$. Form Q' from Q by replacing c with $c' = (v_1, v_2, ..., v_x)$. Note that $|E(Q')| \equiv 2 \pmod 3$ and $0 \pmod 3$ when $n \equiv 0$ and $2 \pmod 6$ respectively. Further note that $v_0 \notin V(Q')$ and that Q' satisfies the conditions for an excess when $\lambda = 1$ and $n' = n - 1 \equiv 5 \pmod 6$ and for an excess when $\lambda = 1$ and $n' = n - 1 \equiv 1 \pmod 6$. So by Theorem 2, let $(V \setminus \{v_0\}, B_1)$ be a cover of K_{n-1} with excess Q'.

By Theorem 1, let (V, B_2) be a maximum packing of K_n with leave a 1-factor F named to contain the edges $\{v_1, v_x\}$ and $\{v_0, d\}$ where d is a vertex for which $\{v_1, v_x, d\} \in B_1$. Then $(V, (B_1 \cup B_2 \setminus \{\{v_1, v_x, d\}\}) \cup \{\{v_0, a_i, b_i\} \mid \{a_i, b_i\} \in F \setminus \{\{v_1, v_x\}, \{v_0, d\}\}\} \cup \{\{v_0, v_1, d\}, \{v_0, v_1, v_x\}, \{v_0, v_x, d\}\})$ is the required cover.

Case 5: $\lambda > 2$ and $n \equiv 0 \pmod 6$

By Condition (1), λ is even. Hence, by Theorem 1, let (V, B_1) be a K_3-decomposition of $(\lambda - 2)K_n$. By Condition (2), $|E(Q)| \equiv 0 \pmod 3$. Hence, by Case 4, let (V, B_2) be a cover of $2K_n$ with excess Q. Then $(V, B_1 \cup B_2)$ is the required cover.

Case 6: $\lambda > 2$ and $n \equiv 2 \pmod 6$

By Condition 1, λ is even. Let $\epsilon \in \{2, 4, 6\}$ with $\epsilon \equiv \lambda \pmod 6$. Note that $|E(\lambda K_n)| \equiv |E(\epsilon K_n)| \pmod 3$ and $\epsilon \equiv \lambda \pmod 2$ so Q satisfies the necessary conditions for an excess of λK_n precisely when it satisfies the necessary conditions for an excess of ϵK_n. By Condition (3), $n \geq 8$, so by Theorem 1, let (V, B_1) be a K_3-decomposition of $(\lambda - \epsilon)K_n$.

If $\epsilon = 2$, by Case 4, let (V, B_2) be a cover of $2K_n$ with excess Q.

If $\epsilon = 4$ then by Condition 2 $|E(Q)| \equiv 2 \pmod 3$. First suppose Q consists only of 2-cycles and isolated vertices. Since $|E(Q)| \equiv 4 \equiv n \pmod 6$, the number of isolated vertices in Q must be a multiple of 6. If Q has no isolated vertices, then let Q' consist of two of the 2-cycles and Q'' consist of the remaining $\frac{n}{2} - 2 \geq 2$ 2-cycles ($n \geq 8$ in this case). Note that $|E(Q')| \equiv |E(Q'')| \equiv 1 \pmod 3$ so by Case 4, let (V, B_2') be a cover of $2K_n$ with excess Q' and (V, B_2'') be a cover of $2K_n$ with excess Q''. Let $B_2 = B_2' \cup B_2''$. Otherwise Q has at least 6 isolated vertices; add a 3-cycle on three of the isolated vertices to Q.

It remains to consider the case where Q has a cycle $c = (v_0, v_1, ..., v_x)$ of length $x+1 \geq 3$. Form Q' from Q by replacing c with $c' = (v_1, v_2, ..., v_x)$. Note $|E(Q')| \equiv 1 \pmod 3$ so by Case 4, let (V, B_2') be a cover of $2K_n$ with excess Q'. By Theorem 1, let (V, B_2'') be a packing of $2K_n$ with leave $\{\{c_1, c_x\}, \{c_1, c_x\}\}$. Let $B_2 = B_2' \cup B_2'' \cup \{\{c_0, c_1, c_x\}\}$.

If $\epsilon = 6$ then $|E(Q)| \equiv 0 \pmod 3$. Since $n \equiv 2 \pmod 6$, there are at least two vertices, say v_0 and v_1 such that $v_0, v_1 \notin V(Q)$. Form Q' from Q by adding the 2-cycle (v_0, v_1). Note $|E(Q')| \equiv 2 \pmod 3$, so by the previous case where $\epsilon = 4$, let (V, B_2') be a cover of $4K_n$ with excess Q'. By Theorem 1, let (V, B_2'') be a packing of $2K_n$ with leave $\{\{v_0, v_1\}, \{v_0, v_1\}\}$. Let $B_2 = B_2' \cup B_2''$.

In each case, $(V, B_1 \cup B_2)$ is the required cover, so the result is proved. □

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