

On Boole’s formula for factorials

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Abstract

We present a simple new proof and a new generalization of Boole’s formula

$$n! = \sum_{j=1}^n (-1)^{n-j} \binom{n}{j} j^n \quad (n \in \mathbf{N}).$$

1 Introduction

The elegant formula

$$n! = \sum_{j=1}^n (-1)^{n-j} \binom{n}{j} j^n \quad (n \in \mathbf{N}) \tag{1.1}$$

is given in Boole’s classical book “Calculus of Finite Differences” [3, p.5, p.19]. In 2005, Anglani and Barile [2] used methods from real analysis and combinatorics to provide two proofs. An interesting extension of Boole’s identity was published in 2008 by Pohoata [3]. He applied Lagrange’s interpolating polynomial theorem to establish

$$a_0 \cdot b^n \cdot n! = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} P(a + jb) \quad (n \in \mathbf{N}),$$

where a and b are real numbers with $b \neq 0$ and P is a real polynomial of degree n with leading coefficient a_0 . The special case $a = 0, b = 1, P(x) = x^n$ leads to (1.1).

The aim of this note is twofold. In Section 2, we present a simple new proof of (1.1), and in Section 3, we offer a new generalization of (1.1).

2 A new proof

Here, we apply the method of induction to obtain a short and elementary proof of Boole’s identity. We need the following well-known formulas:

$$j \binom{n+1}{j} = (n+1) \binom{n}{j-1}, \tag{2.1}$$

$$\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}, \tag{2.2}$$

$$\sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} j^\nu = 0 \quad (\nu = 0, 1, \dots, n). \tag{2.3}$$

Using the formula

$$\sum_{j=0}^{n+2} (-1)^{n+2-j} \binom{n+2}{j} j^\nu = (n+2) \sum_{k=0}^{\nu-1} \binom{\nu-1}{k} \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} j^k$$

$(\nu = 1, 2, \dots, n+1)$

it follows by induction on n that (2.3) is valid; see also [1, chapter 2.4].

Proof of (1.1). If $n = 1$, then both sides of (1.1) are equal to 1. Next, we assume that (1.1) holds. Applying (2.1), (2.2), (2.3) with $\nu = n$, and the induction hypothesis we obtain

$$\begin{aligned} \sum_{j=1}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} j^{n+1} &= (n+1) \sum_{j=1}^{n+1} (-1)^{n+1-j} \binom{n}{j-1} j^n \\ &= (n+1) \sum_{j=1}^{n+1} (-1)^{n+1-j} \left[\binom{n+1}{j} - \binom{n}{j} \right] j^n = (n+1) \left[0 + \sum_{j=1}^n (-1)^{n-j} \binom{n}{j} j^n \right] \\ &= (n+1) \cdot n! = (n+1)!. \end{aligned}$$

This reveals that (1.1) is valid if we replace n by $n + 1$.

3 A new generalization

We prove the following

Theorem. Let $(f(j))_{j=0}^\infty$ be a sequence of complex numbers with $f(0) = 1$. Then,

$$\sum_{j=0}^n (-1)^{n-j} \binom{n}{j} \sum_{\substack{c_1+\dots+c_j=m \\ c_1, \dots, c_j \geq 0}} f(c_1) \cdots f(c_j) = \sum_{\substack{d_1+\dots+d_n=m \\ d_1, \dots, d_n \geq 1}} f(d_1) \cdots f(d_n) \quad (m, n \in \mathbf{N}). \tag{3.1}$$

Proof. We define the formal power series

$$F(X) = \sum_{j=0}^\infty f(j)X^j.$$

On the one hand we have

$$\begin{aligned} (F(X) - 1)^n &= \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} F(X)^j \\ &= \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} \sum_{c_1, \dots, c_j=0}^\infty f(c_1) \cdots f(c_j) X^{c_1+\dots+c_j} \\ &= \sum_{m=0}^\infty X^m \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} \sum_{\substack{c_1+\dots+c_j=m \\ c_1, \dots, c_j \geq 0}} f(c_1) \cdots f(c_j). \end{aligned}$$

On the other hand we obtain

$$(F(X) - 1)^n = \left(\sum_{k=1}^\infty f(k)X^k \right)^n = \sum_{m=n}^\infty X^m \sum_{\substack{d_1+\dots+d_n=m \\ d_1, \dots, d_n \geq 1}} f(d_1) \cdots f(d_n).$$

Comparing the coefficients of X^m gives (3.1).

Remark

The special case $m = n$ leads to

$$\sum_{j=0}^n (-1)^{n-j} \binom{n}{j} \sum_{\substack{c_1+\dots+c_j=n \\ c_1, \dots, c_j \geq 0}} f(c_1) \cdots f(c_j) = f(1)^n. \tag{3.2}$$

Since

$$\sum_{\substack{c_1+\dots+c_j=n \\ c_1, \dots, c_j \geq 0}} \frac{1}{c_1! \cdots c_j!} = \frac{j^n}{n!},$$

we conclude that formula (3.2) with $f(c) = 1/c!$ implies (1.1).

References

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