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Packing four copies of a tree into a complete graph

Sean P. Haler   Hong Wang

Department of Mathematics
University of Idaho
Moscow, ID 83844-1103
U.S.A.
seanhaler@gmail.com   hwang@uidaho.edu

Abstract

A graph G of order n is k-placeable if there exist k edge-disjoint copies of G in the complete graph K_n. Previous work characterized all trees that are k-placeable for $k \leq 3$. This work extends those results by giving a complete characterization of all 4-placeable trees.

1 Introduction

Only finite simple graphs are considered here and standard terminology and notation from [1] is used unless otherwise indicated. For any graph G, let $V(G)$ and $E(G)$ denote the vertex and edge sets of G, respectively. The *degree* of a vertex $v \in V(G)$, denoted $\mathrm{d}_G(v)$ (or $\mathrm{d}(v)$ when the context is clear) is the number of edges incident with v. Furthermore, a vertex of degree 1 is called an *end vertex* and the maximum (minimum) degree of G is denoted $\Delta(G)$ ($\delta(G)$). Denote by K_n the complete graph of order n and P_n the path of order n and length $n-1$.

For graphs G and H, an *embedding* of G into H is an injective function $\phi\colon V(G) \to V(H)$ such that $\phi(a)\phi(b) \in E(H)$ whenever $ab \in E(G)$. It is notationally convenient to write $\phi\colon G \to H$ as opposed to $\phi\colon V(G) \to V(H)$ and to write $\phi(ab)$ for the edge $\phi(a)\phi(b)$. Furthermore, when $V' \subseteq V(G)$ or $E' \subseteq E(G)$ let $\phi(V') = \{\phi(v) : v \in V'\}$ and $\phi(E') = \{\phi(ab) : ab \in E'\}$. A *packing* of k graphs $G_1, G_2, ..., G_k$ into H is a k-tuple $\Phi = (\phi_1, \phi_2, ..., \phi_k)$ such that, for $i = 1, 2, \ldots, k$, ϕ_i is an embedding of G_i into H and the k sets $\phi_i(E(G_i))$ are mutually disjoint. If G is a graph of order n, a packing where $G = G_1 = G_2 = \cdots = G_k$ and $H = K_n$ is a *k-placement* of G.

A *tree* T is a connected acyclic graph. Besides the trees in Figure 1 and Figure 2 (which will be frequently referenced) several other trees of order $n \geq 8$ are important. A *star* S_n is a tree of order n where every edge is incident with a single vertex (e.g. $S_8 \cong T_1$). Denote by S_n^k the tree of order n obtained by replacing a single edge of S_{n-k+1} with a path of length k (e.g. $S_8^2 \cong T_2$, $S_8^3 \cong T_5$, and $S_8^4 \cong T_{12}$). Let $S_n^{2,2}$ be the tree of order n obtained by replacing two edges of S_{n-2} with paths of length 2 (e.g. $S_8^{2,2} \cong T_4$). Similarly let S_n^{2+} be the tree of order n obtained from S_{n-1}^2 by joining a new end vertex to the vertex of degree 2 (e.g. $S_8^{2+} \cong T_3$). Finally, define the tree Y_n obtained from S_{n-2}^2 by joining two end vertices to the end vertex of the length 2 path (e.g. $Y_8 \cong T_{11}$).

Finally, let W be the set of trees consisting of T_9, T_{13}, and all trees Y_n and S_n^4 where $n \geq 8$. The main result of this work is Theorem 1.1, which characterizes all trees that are 4-placeable.

Theorem 1.1. *A tree T of order $n \geq 8$ has a 4-placement if and only if $\Delta(T) \leq n-4$ and $T \notin W$.*

It is generally accepted that H. J. Straight first observed that each non-star tree of order n has a 2-placement [4, 11]. This result was first generalized in [4] and led to a great amount of work on packings of two graphs [2, 3, 5, 7, 8, 13]. The main inspiration for this work comes from H. Wang and N. Sauer who proved an analogous result for $k = 3$ in [9]. A good deal of work on packings of

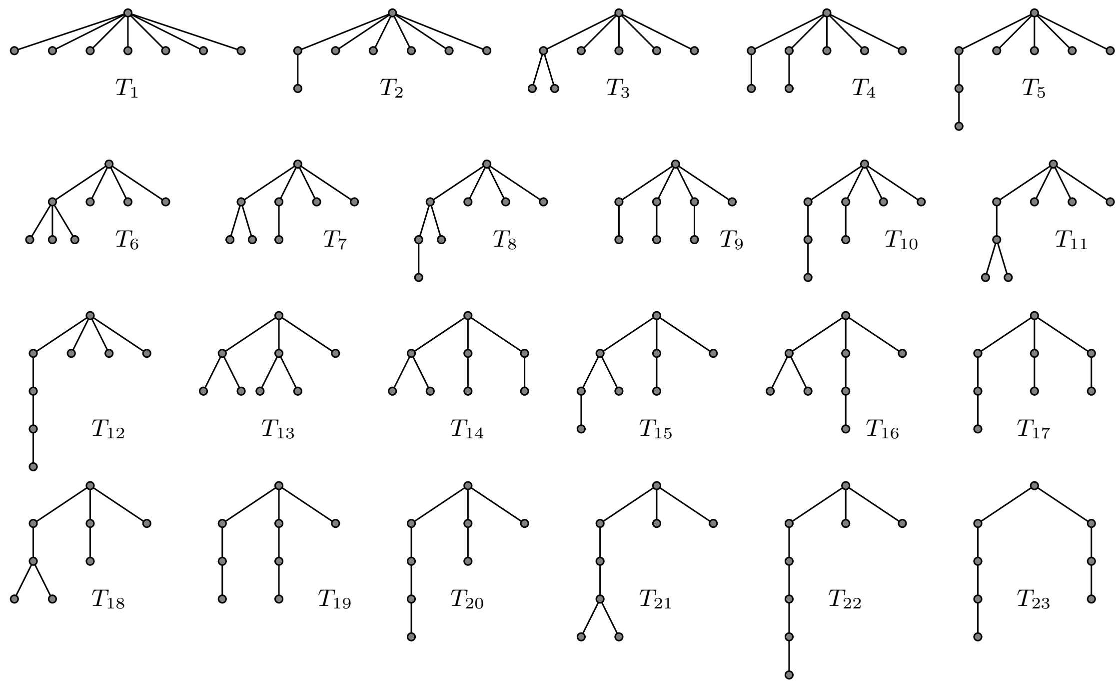

Figure 1: The 23 trees of order 8.

3 graphs has also been done [6, 10, 12, 13]. There have been some results for arbitrary k [14], but the amount of work is rare by comparison. We present the following conjecture for arbitrary k.

Conjecture 1.2. *Let $k \geq 1$ be an integer and let T be a tree of order n with $n > 2k$. If $\Delta(T) < n-k$ then there is a k-placement of T.*

The proof of Theorem 1.1 is based mainly on the induction argument of Lemma 2.5. Several other supporting lemmas are given in Section 2. A "base case" for Lemma 2.5 involving trees of order $8, 9, 10$, and 11 is addressed separately in Section 3. A special case where Lemma 2.5 cannot be used is addressed in Section 4. Finally, the proof of Theorem 1.1 is given in Section 5.

2 Preliminaries

Let G be a graph, $V' \subset V(G)$, and $E' \subset E(G)$. A vertex adjacent to an end vertex is a *node*. Let $G - E'$ be the graph with vertex set $V(G)$ and edge set $E(G) \setminus E'$. Denote by $G - V'$ the subgraph of G induced by $V(G) \setminus V'$ and if $V' = \{x\}$ then the notation of $G - \{x\}$ is relaxed to $G - x$. If V' consists entirely of end vertices of G then $G - V'$ is called a *shrub* of G. For example, P_2 is a shrub of P_2, P_3, and P_4 but not P_5. The *neighborhood* of a vertex x in G, denoted here as $N_G(x)$ is the set of vertices adjacent to x in G and $N_G(V') = \bigcup\{N_G(x) : x \in V'\}$ ($N(x)$ or $N(V')$ when G is clear).

Let Φ be a k-placement of G. A vertex v of G is *k-placed* by Φ if for each $i, j \in \{1, 2, \ldots k\}$ with $i \neq j$, $\phi_i(v) \neq \phi_j(v)$. Moreover if every vertex of G is k-placed then Φ is *dispersed*. An edge ab is *k-placed* by Φ if the set of edges $\{\phi_i(ab) : i = 1, 2, ..., k\}$ are independent.

Lemma 2.1. *Let V be a set of end vertices in a graph G of order n. If $G - V$ has a 4-placement with each vertex in $N_G(V)$ 4-placed, then G has a 4-placement.*

Proof: Suppose $|V| = r$ and let $V = \{v_1, v_2, \ldots, v_r\}$. Let $H \cong K_n$ and let $X \subset V(H)$ where $X = \{x_1, x_2, \ldots, x_r\}$. Let $N_G(V) = \{u_1, u_2, \ldots, u_r\}$ where $u_iv_i \in E(G)$ for $i = 1, 2, \ldots, r$ and note

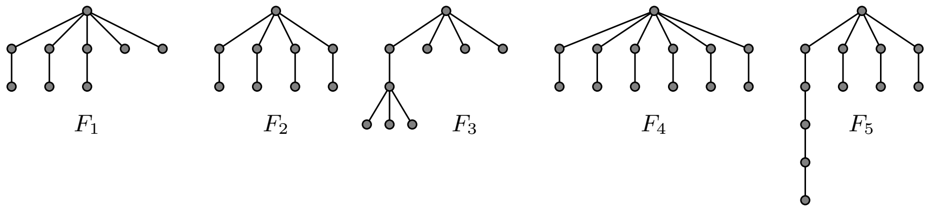

Figure 2: Special trees.

that the u_i's may not be distinct. By assumption there is a 4-placement $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ of $G-V$ into $H-X$ such that each vertex in $N_G(V)$ is 4-placed. For $j = 1, 2, 3, 4$, define $\gamma_j : G \to H$ so that $\gamma_j|_{G-V} = \phi_j$ and $\gamma_j(v_i) = x_i$ for each $i \in \{1, 2, \ldots, r\}$. It is straightforward that $\Gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is a 4-placement of G. □

Lemma 2.2. *Let G be a graph of order n with $ab \in E(G)$. Let G' be the graph with $V(G') = V(G) \cup \{w\}$ (for some $w \notin V(G)$) and $E(G') = E(G) - ab + aw + bw$. If Φ is 4-placement of G such that ab is 4-placed, then G' has a 4-placement.*

Proof: Let $H' \cong K_{n+1}$ and let $x \in V(H')$. Let $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ be a 4-placement of G into $H' - x$ that 4-places ab. For $i = 1, 2, 3, 4$, define $\gamma_i : G' \to H'$ by $\gamma_i|_G = \phi_i$ and $\gamma_i(w) = x$. Let $\Gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$.

Suppose to contradict that Γ is not a 4-placement of G'. Then there are two edges e and f of G' such that $\gamma_i(e) = \gamma_j(f)$ for some distinct $i, j \in \{1, 2, 3, 4\}$. Clearly $\gamma_i(e)$ and $\gamma_j(f)$ are not in $H'-x$, since then $\phi_i(e) = \phi_j(f)$. Thus $\gamma_i(e)$ and $\gamma_j(f)$ are incident with x. Thus $e = rw$ and $f = sw$ where $r, s \in \{a, b\}$. Since $\gamma_i(e) = \gamma_j(f)$ then $\gamma_i(r) = \gamma_j(s)$. But then $\phi_i(r) = \phi_j(s)$ contradicting the assumption that ab is 4-placed by Φ. Thus Γ is 4-placement of G'. □

In Lemma 2.2 vertices and edges that are 4-placed by Φ are also 4-placed by Γ, with the exception of the ab edge. Thus Lemma 2.2 can be applied once to each 4-placed edge to produce new 4-placements of larger graphs. This is done in Section 4.

The following well-known observation is given here for completeness.

Lemma 2.3. *There exists a dispersed 4-placement of P_n if $n \geq 8$.*

Proof: Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and let a be an end vertex of $T = P_n$. Suppose first that $n = 2t$ for a positive integer t. For $i = 1, 2, 3, 4$, define the path $P^i = v_i v_{i+1} v_{i-1} \cdots v_{i-t+1} v_{i+t}$, where the subscripts of the v_j's are taken modulo n in $\{1, 2, ..., n\}$. It is easy to see the set of P^1, P^2, P^3, P^4 are edge disjoint paths of order n in K_n. For $i = 1, 2, 3, 4$, define $\phi_i(T) = P^i$ with $\phi_i(a) = v_i$. Thus $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ is a dispersed 4-placement of T (see the 4-placement of T_{23} in Figure 3).

The case when $n = 2t - 1$ is similar and is therefore omitted. □

Before presenting the main induction lemma a technical result is needed. Define a subset V of $V(G)$ as *nondeficient* if $|N(S)| \geq |S|$ for every subset S of V. The proof of Lemma 2.4 uses Hall's Theorem which states (paraphrased) that in a bipartite graph, one partite set B can be matched into the other partite set A if and only if B is nondeficient (see Theorems 1.2.3 and 2.1.1 of [1]).

Lemma 2.4. *Let $H = K_{4,m}$ where $m \geq 4$ and let A and B be the partite sets of H with sizes 4 and m, respectively. If B_1, B_2, B_3, B_4 are arbitrary subsets of B each with order 4, then there exist disjoint matchings M_1, M_2, M_3, M_4 such that M_i matches B_i into A, for $i = 1, 2, 3, 4$.*

Proof: Let $A = \{a_1, a_2, a_3, a_4\}$ and let $z = |B^*|$ where $B^* = \bigcap_{i=1}^4 B_i = \{b_1, b_2, \ldots, b_z\}$. Suppose first that $z \geq 3$. For $i = 1, 2, 3, 4$, let $M_i' = \{a_i b_1, a_{i+1} b_2, a_{i+2} b_3\}$ where the subscripts are taken modulo 4 in $\{1, 2, 3, 4\}$. In this case, each M_i' can easily be extended to satisfy the lemma. Suppose next that $z = 2$. For $i = 1, 2, 3, 4$, let $M_i'' = \{a_ib_1, a_{i+1}b_2\}$ where the subscripts are taken modulo 4 in $\{1, 2, 3, 4\}$. Again, each M_i'' can be extended, in turn, to satisfy the lemma.

Thus suppose $z \leq 1$ and assume to contradict that B_1, B_2, B_3, B_4 cannot be matched into A by disjoint matchings. Let c be the maximum number of the B_i's that can be matched into A and note that trivially $1 \leq c < 4$. Assume without loss of generality that M_i is a matching of B_i into A for all $i = 1, 2, \ldots, c$ such that the M_i's are disjoint. Let $C = \bigcup_{i=1}^{c} M_i$ and $D = H - C$. Since c is maximal by Hall's Theorem B_{c+1} is not nondeficient in D. That is, there exists $S \subset B_{c+1}$ such that $|N_D(S)| < |S|$. Let $R = N_D(S)$. Note all the edges from S to $A \setminus R$ are in C so $c \geq \min\{|S|, |A \setminus R|\}$. Thus $1 \leq |R| < |S| \leq 3$. If $|R| = 1$, then $|A \setminus R| = 3$ implying $c = 3$. But then $S \subset B^*$ and $|S| \geq 2$, contradicting $z \leq 1$. Therefore $|R| \neq 1$, implying $|R| = 2$, $|S| = 3$, and $c = 3$.

Let $B_4 = \{s_1, s_2, s_3, \overline{s}\}$ and $A = \{r_1, r_2, \overline{r_1}, \overline{r_2}\}$ where $S = \{s_1, s_2, s_3\}$ and $R = \{r_1, r_2\}$. Without loss of generality, $M_1 \supset \{s_2\overline{r_1}, s_3\overline{r_2}\}$, $M_2 \supset \{s_1\overline{r_2}, s_3\overline{r_1}\}$, and $M_3 \supset \{s_1\overline{r_1}, s_2\overline{r_2}\}$. If $s_i \in B_i$ for some $i = 1, 2, 3$, then $s_i \in B^*$. It may be assumed without loss of generality that $s_1 \notin B_1$ and $s_2 \notin B_2$. There exists $p \in B_2 \setminus S$ such that $pr_1 \in M_2$. Let $M_2' = (M_2 \setminus \{pr_1, s_1\overline{r_2}\}) \cup \{p\overline{r_2}, s_1r_1\}$ and note that M_1, M_2', and M_3 are mutually disjoint. Since $s_2 \notin B_2$, then there exists a matching M^* of $\{s_2, s_3\}$ into $\{r_1, r_2\}$ in D. Let $M_4 = M^* \cup \{s_1\overline{r_2}, \overline{sr_1}\}$. Then M_1, M_2', M_3, and M_4 are mutually disjoint and $c = 4$. □

Lemma 2.5. *Let T be a tree of order $n \geq 12$. Suppose that there are 4 end vertices v_1, v_2, v_3, v_4 of G adjacent to distinct nodes u_1, u_2, u_3, u_4, respectively. If there is a 4-placement of $G' = G - \{v_1, v_2, v_3, v_4\}$ then there is a 4-placement of G.*

Proof: Let $H \cong K_n$ and let $A \subset V(H)$ with $A = \{a_1, a_2, a_3, a_4\}$. By assumption there exists a 4-placement $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ of G' into $H - A$. For $i = 1, 2, 3, 4$, let $B_i = \{\phi_i(u_j) : 1 \leq j \leq 4\}$ and let $B = \bigcup_{i=1}^{4} B_i$. Let D be the complete bipartite subgraph of H with partite sets A and B. By Lemma 2.4, there exist disjoint matchings M_1, M_2, M_3, and M_4 such that M_i matches B_i into A within the subgraph D. It is straightforward that each ϕ_i can be extended to $\gamma_i : G \to H$ using M_i. Furthermore, since the M_i's are disjoint $\Gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is a 4-placement of G. □

This section concludes with a lemma showing the necessity condition of Theorem 1.1. The phrase *degree considerations* will refer to the fact that in a k-placement Φ of a tree T with order n, the sum of the degrees of vertices placed by Φ on a single vertex cannot exceed $n - 1$. Also, a k-placement of a tree is *tight* if all edges of K_n are required, i.e. when $n = 2k$.

Lemma 2.6. *Let T be a tree of order $n \geq 8$. T has no 4-placement if $\Delta(T) > n - 4$ or if $T \in W$.*

Proof: Any tree with $\Delta(T) > n - 4$ has no 4-placement by degree considerations. Similarly, any 4-placement of T_{13} must place two vertices of degree three on a single vertex which is not possible by degree considerations. Thus let $T \in W \setminus \{T_{13}\}$ and suppose to contradict that $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ is a 4-placement of T. Let a be the vertex of T with degree $n - 4$ and let $A = \{v_i : v_i = \phi_i(a), i = 1, 2, 3, 4\}$. By degree considerations the set of elements in A are distinct, and moreover, any vertex other than a that is placed on an element of A must be an end vertex.

Case 1: Let $T = T_9$. Let b be the end vertex adjacent to a. Note that $\{\phi_i(ab) : i = 1, 2, 3, 4\}$ are the only edges placed by Φ in the subgraph induced by A, a contradiction since Φ must be tight.

Case 2: Let $T = S_n^4$. Let c be the end vertex not adjacent to a and let $z_1, z_2, ..., z_{n-5}$ be the other end vertices of T. Note that, for each embedding, at least 2 of the z_i's must be placed in A. This means that Φ must place at least 8 distinct edges in the subgraph induced by A, a contradiction.

Case 3: Let $T = Y_n$. Let x_1 and x_2 be the end vertices not adjacent to a and $y_1, y_2, \ldots, y_{n-5}$ be the other end vertices of T. Furthermore, for $i = 1, 2, 3, 4$, let $r_i = |A \cap \{\phi_i(y_j) : j = 1, 2, \ldots, n-5\}|$ and note that since each ϕ_i must place three end vertices in A so that $r_i \geq 1$. Assume without loss of generality that $r_1 \geq r_2 \geq r_3 \geq r_4$. Finally, let c be the node adjacent to x_1 and for $i = 1, 2, 3, 4$ let $\phi_i(c) = w_i$.

Case 3a: Suppose $r_1 = 1$. It may be assumed that $\phi_1(y_1) = v_2$ and $\phi_2(y_1) = v_3$. It must be the case that $\phi_1(\{x_1, x_2\}) = \{v_3, v_4\}$ and $\phi_2(\{x_1, x_2\}) = \{v_1, v_4\}$. Thus $w_1 \neq w_2$. But then $\phi_1(N_T(a)) \cap \{v_1, v_3, v_4, w_1, w_2\} = \emptyset$, a contradiction since $d(a) = n - 4$.

Case 3b: Suppose $r_1 = 3$. It may be assumed that $\phi_1(\{y_1, y_2, y_3\}) = \{v_2, v_3, v_4\}$, $\phi_2(y_1) = v_3$, $\phi_3(y_1) = v_4$, and $\phi_4(y_1) = v_2$. Thus $\phi_2(\{x_1, x_2\}) = \{v_1, v_4\}$ and $\phi_3(\{x_1, x_2\}) = \{v_1, v_2\}$. Thus $w_2 \neq w_3$ and so $\phi_2(N_T(a)) \cap \{v_1, v_2, v_4, w_2, w_3\} = \emptyset$, a contradiction since $d(a) = n - 4$.

Case 3c: Suppose $r = 2$. It may be assumed that $\phi_1(\{y_1, y_2\}) = \{v_2, v_3\}$. It may further be assumed that $\phi_2(x_1) = \phi_3(x_1) = v_1$ and in particular $w_2 \neq w_3$. If Φ places no edge on v_2v_3, then $\phi_3(x_2) = v_2$, a contradiction since then $\phi_2(N_T(a)) \cap \{v_1, v_2, v_3, w_2, w_3\} = \emptyset$. Thus assume that $\phi_2(y_1) = v_3$. Note that $v_1v_4, v_1w_2, v_1w_3 \notin \phi_1(E(T))$. Thus $w_1 \in \{w_2, w_3\}$ and $\phi_1(\{x_1, x_2\}) \subset \{v_4, w_2, w_3\}$, so it must be the case that $w_2w_3 \in \phi_1(E(T))$. Similarly, $v_2v_1, v_2w_2, v_2w_3 \notin \phi_2(E(T))$, and thus $\phi_2(x_2) = w_3$, a contradiction since $w_2w_3 \in \phi_1(E(T))$. □

3 Small Order Trees

This section provides 4-placements for each tree that meets the criteria of Theorem 1.1 and has order $8, 9, 10$, or 11 as well as F_4 and F_5. It is convenient to label the vertices T_t as $a_t, b_t, c_t, d_t, e_t, f_t, g_t$, and h_t starting from the top (as pictured in Figure 1) and proceeding left to right, then top to bottom. Under this scheme, for example, $E(T_7) = \{a_7b_7, a_7c_7, a_7d_7, a_7e_7, b_7f_7, b_7g_7, c_7h_7\}$. Furthermore, let $\mathbb{T} = \{T_6, T_7, T_8, T_{10}, T_{14}, T_{15}, T_{16}, T_{17}, T_{18}, T_{20}, T_{21}, T_{23}\}$.

Lemma 3.1. *The following statements are true:*
- ***a)*** *Each tree $T \in \mathbb{T}$ has a dispersed 4-placement.*
- ***b)*** *T_{19} has a 4-placement where each vertex is 4-placed except b_{19}.*
- ***c)*** *T_{22} has a 4-placement where each vertex is 4-placed except f_{22}.*
- ***d)*** *F_1, F_2, F_4, and F_5 have dispersed 4-placements.*
- ***e)*** *F_3 has a 4-placement such that each vertex of degree 4 is 4-placed.*

Proof: Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Four embeddings for each of the trees in **a** through **d** are shown in Figure 3. Each embedding assumes the v_i's are placed on a circle with the subscripts strictly increasing as the angle increases from 0 to 2π. Occasionally, all the images of a particular vertex are colored to distinguish it from other vertices it may be mapped to in an automorphism. For example, the images of b_6 are colored red, the images of c_6 are colored green, etc. It is straightforward to verify that these embeddings produce the 4-placements required. The only vertices not 4-placed are b_{19} (the images of which are colored red) and f_{22} (the images of which are also colored red).

A 4-placement of F_3 satisfying **e** can be obtained from the 4-placement of T_6 and applying Lemma 2.2 to the a_6b_6 edge. □

Corollary 3.2. *Let T be a tree of order $n \in \{9, 10, 11\}$ not in W and let U be a shrub of T with order 8. If $\Delta(U) \leq 4$ then there is a 4-placement of T.*

Proof: First, it may be assumed by Lemmas 2.1 and 3.1 that $U \notin \mathbb{T}$ and furthermore that T contains no shrub in $\mathbb{T} \cup \{F_1, F_2\}$. This leaves six possibilities for U. Let $V = V(T) \setminus V(U)$ and let $N = N_T(V)$.

Case 1: Suppose $U = T_{19}$. By Lemmas 2.1 and 3.1 it may be assumed $b_{19} \in N$. If $d_{19} \in N$, then T_{17} is a shrub of T and if not T_{20} is a shrub of T, both contradictions.

Case 2: Suppose $U = T_{22}$. By Lemmas 2.1 and 3.1 it may be assumed that $f_{22} \in N$. If $N = \{c_{22}, d_{22}, f_{22}\}$ then T_{21} is a shrub of T and if not then T_{20} is a shrub of T. Again, these are both contradictions.

Case 3: Suppose $U = T_9$. If $a_9 \in N$ (or $e_9 \in N$) then F_1 (F_2) is a shrub of T, a contradiction. Thus suppose $N \cap \{a_9, e_9\} = \emptyset$. If $\{b_9, c_9, d_9\} \cap N \neq \emptyset$ then T_{14} is a shrub of T, a contradiction. However, if $\{f_9, g_9, h_9\} \cap N \neq \emptyset$ then T_{17} is a shrub of T, also a contradiction.

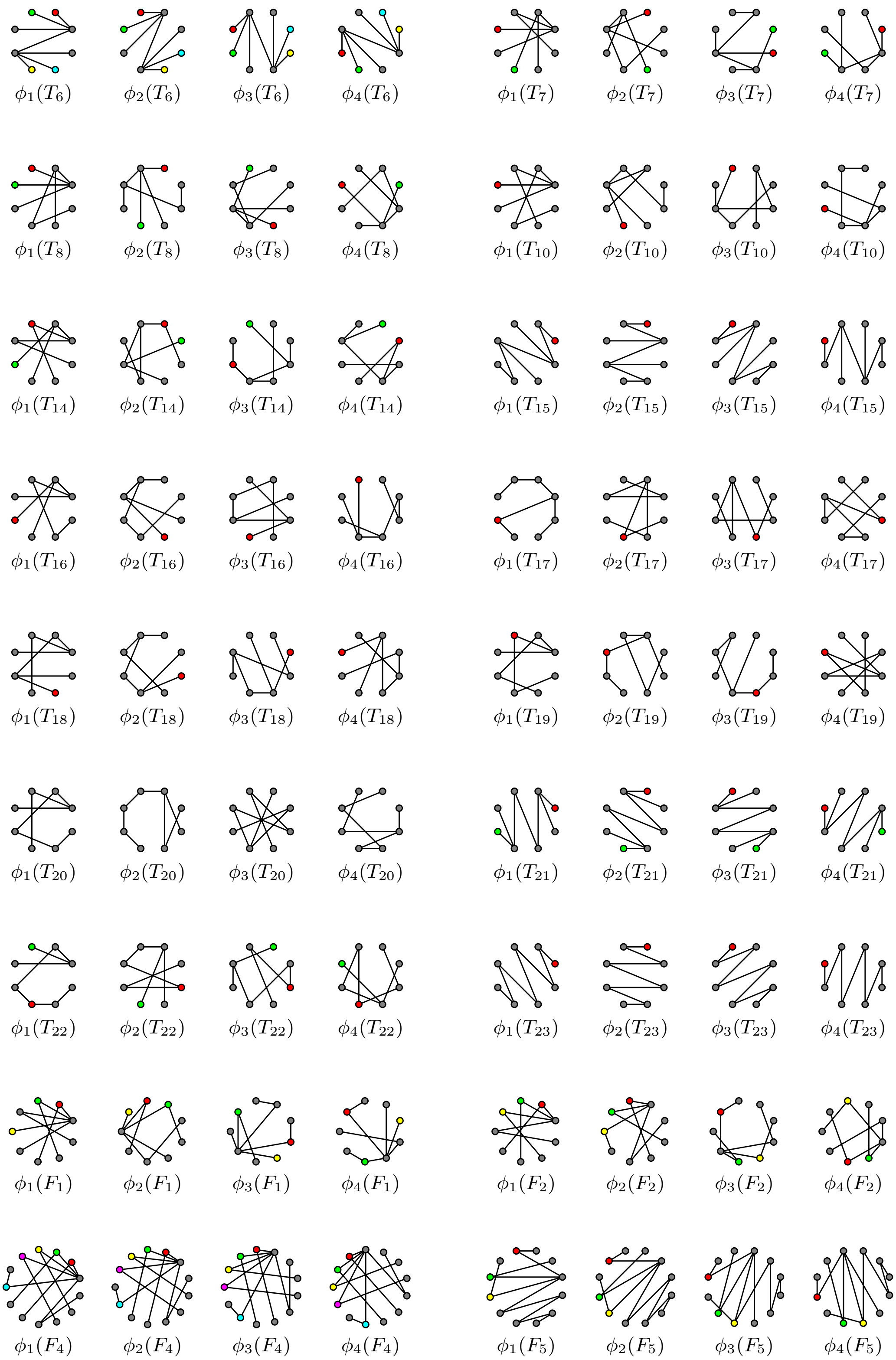

Figure 3: 4-placements for certain trees of small order. Similarly colored vertices in a packing are images of single vertex. These colors are used to make distinctions in trees with symmetry.

Case 4: Suppose $U = T_{12}$. If $h_{12} \in N$ then T_{22} is a shrub of T and this is handled by Case 2. Thus assume $h_{12} \notin N$. Note that $\{c_{12}, d_{12}, e_{12}\} \cap N = \emptyset$ since otherwise T_{10} is a shrub of T. Similarly, if b_{12}, f_{12}, or g_{12} are in N then T_8, T_{18}, or T_{21} are shrubs of T, respectively, all contradictions. But then $N = \{a\}$ and $T = S_n^4$, a contradiction. Thus T must have a 4-placement.

Case 5: Suppose $U = T_{11}$. Since T_{17} is not a shrub of T, then g_{11} and h_{11} cannot both be in N. If exactly one of g_{11} or h_{11} is in N, then T_{12} is a shrub of T and this reduces to Case 4. Thus it can be assumed that $\{g_{11}, h_{11}\} \cap N = \emptyset$. Similarly, $\{c_{11}, d_{11}, e_{11}\} \cap N = \emptyset$ since otherwise T_{10} is a shrub of T. Furthermore, $b_{11} \notin N$, since then T_8 would be a shrub of T. Thus $N \subset \{a_{11}, f_{11}\}$. Note that $f_{11} \in N$ since otherwise $N \subset \{a_{11}\}$ and then $T = Y_n$, a contradiction. Therefore F_3 is a shrub of T and Lemma 2.1 and Lemma 3.1 **e** provide a 4-placement of T.

Case 6: Suppose $U = T_{13}$. Note that a_{13} and d_{13} are not in N since then T_7 or T_{14} would be a shrub of T, respectively. If $\{e_{13}, f_{13}, g_{13}, h_{13}\} \cap N \neq \emptyset$ then T_{18} is a shrub of T, a contradiction. Thus $N \subset \{b_{13}, c_{13}\}$ and so T_8 is a shrub of T, a contradiction.

This completes the proof. □

Lemma 3.3. *Let T be a tree of order $n \in \{9, 10, 11\}$. If $\Delta(T) \leq n - 4$ and $T \notin W$, then there is a 4-placement of T.*

Proof: Suppose the Lemma is false and let T be a counterexample. By Corollary 3.2 T does not contain a shrub U of order 8 with $\Delta(U) \leq 4$. Let u be a vertex of T with maximum degree. By Lemma 2.3 it may be assumed that $T \neq P_{11}$, and so T contains shrubs of order 8; therefore $d(u) > 4$. If $n = 9$, then there exists an end vertex in $N(u)$ and deleting this end vertex creates a shrub of order 8 with maximum degree 4, a contradiction.

Suppose $n = 10$. If $d(u) = 6$, then there exists two end vertices in $N(u)$ and removing them gives a shrub of order 8 and maximum degree 4, a contradiction. Thus $d(u) = 5$. There exists an end vertex $v_1 \in N(u)$. If $\Delta(T - v_1) = 4$ then removing any additional end vertex of T produces a shrub of order 8 and maximum degree at most 4, a contradiction. Thus $\Delta(T - v_1) = 5$ and T contains two vertices of degree 5 and is thus uniquely determined. But then T_6 is a shrub of T, a contradiction.

Therefore $n = 11$. If $d(u) = 7$, then there exists three end vertices in $N(u)$ and removing them gives a shrub of maximum degree 4, a contradiction. If $d(u) = 6$, there are end vertices v_2 and v_3 in $N(u)$. If $\Delta(T - \{v_2, v_3\}) \geq 5$ then T_6 is a shrub of T, a contradiction. Thus $T - \{v_2, v_3\}$ has maximum degree less than 4 and removing any other end vertex produces a shrub of order 8 and maximum degree at most 4, a contradiction. Thus $d(u) = 5$. If $N(u)$ contains no end vertex then T is uniquely determined and contains F_1 as a shrub. But by Lemmas 3.1 and Lemma 2.1 there is a 4-placement of T, a contradiction. Thus $N(u)$ contains an end vertex v_4. Again, $\Delta(T - v_4) \geq 5$ otherwise removing any two additional end vertices produces a contradiction. But then T must contain either T_6 or T_{11} as a shrub, both contradictions.

Therefore no such T exists and the Lemma is true. □

4 Tri-path trees

If T is a tree with exactly three distinct nodes then Lemma 2.5 cannot be applied. Fortunately, trees with three distinct nodes have a common structure, that is they each have a shrub consisting of three paths meeting at a single vertex. Define $Q(n_1, n_2, n_3)$ as the tree of order $n = n_1 + n_2 + n_3 + 1$ consisting of a single vertex a that begins three disjoint (except for a) paths of length n_1, n_2, and n_3, respectively, (see Figure 5). This section will show that each of these *tri-path trees* has a 4-placement such that each of the end points is 4-placed. It will be assumed that $1 \leq n_1 \leq n_2 \leq n_3$.

Lemma 4.1. *Let T be the tree $Q(n_1, n_2, n_3)$ with order n. If $n \geq 10$ and $n_1 \leq n - 9$, then there is 4-placement of T such that each end point of T is 4-placed.*

Proof: Let z_1, z_2, and z_3 be the end vertices of the n_1, n_2, and n_3 length paths in T, respectively. Let G be the graph of order n obtained from T by adding the edge z_2z_3. Finally, let $H \cong K_n$ and let $V(H) = \{v_1, v_2, ..., v_n\}$.

Here, a 4-placement of G is constructed by a method similar to one used in Lemma 2.3. First, suppose that $n - 1 = 2t$ for some positive integer t. For each $i = 1, 2, 3, 4$, define the path $P^i = v_iv_{i+1}v_{i-1} \cdots v_{i-t+1}v_{i+t}$, where the subscripts of the v_j's are taken modulo $n-1$ in $\{1, 2, ..., n-1\}$. Again for $i = 1, 2, 3, 4$, let $b_i = v_i$, $c_i = v_{i+t}$, and a_i be such that the distance between a_i and b_i along path P^i is n_1. It is straightforward to see that the elements of $\{a_i, c_i : i = 1, 2, 3, 4\}$ are distinct since $n_1 \leq n - 9$. For $i = 1, 2, 3, 4$, let $E^i = E(P^i) \cup \{a_iv_n, c_iv_n\}$. Since the set of a_i's and c_i's are distinct, then $E^i \cap E^j = \emptyset$ when $i \neq j$ and the subgraph induced by each E^i is isomorphic to G (see Figure 4).

For $i = 1, 2, 3, 4$, let γ_i be an embedding of G into H such that $\gamma_i(E(G)) = E^i$ and let $\Gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. Note that it can be assumed that all vertices of G are 4-placed by Γ except a single vertex x that is placed on v_n. Moreover, it may be assumed that $x \notin \{z_1, z_2, z_3\}$. Clearly, Γ is also a 4-placement of T with each end vertex 4-placed.

A similar argument can be used if $n = 2t$ for some positive integer t. □

Lemma 4.2. *Let T be the tree $Q(n_1, n_2, n_3)$ with order n. If $n \geq 8$ then there is a 4-placement of T such that each end vertex of T is 4-placed.*

Proof: By Lemma 3.1, it may be assumed that $n \geq 9$. There are exactly nine tri-path trees with $n > 8$ that do not satisfy the conditions for Lemma 4.1: $Q(1, 1, 6)$, $Q(1, 2, 5)$, $Q(1, 3, 4)$, $Q(2, 2, 4)$, and $Q(2, 3, 3)$ for $n = 9$; $Q(2, 2, 5)$, $Q(2, 3, 4)$, and $Q(3, 3, 3)$ for $n = 10$; and $Q(3, 3, 4)$ for $n = 11$.

In the 4-placement of $T_{17} \cong Q(2, 2, 3)$ given in Lemma 3.1 the edges $b_{17}c_{17}$, $a_{17}e_{17}$, and $a_{17}g_{17}$ are 4-placed (see Figure 3). Using this and Lemma 2.2 there are 4-placements of $Q(2, 3, 3)$, $Q(2, 2, 4)$, $Q(3, 3, 3)$, $Q(2, 3, 4)$, and $Q(3, 3, 4)$ with each end vertex 4-placed. An embedding of each remaining tree is shown in Figure 5 and these embeddings can be used to generate a dispersed 4-placements by rotating each embedding clockwise by one, two, and three vertices. □

5 Proof of Theorem 1

The necessity of Theorem 1.1 is shown by Lemma 2.6. Assume to contradict the theorem is not true and let T be a counterexample of minimum order n. By Lemmas 3.1 and 3.3 it may be assumed that $n \geq 12$. Clearly, T has more than one distinct node and by Lemmas 2.1 and 3.1 T contains no shrub in $\mathbb{T} \cup \{F_1, F_2, F_4, F_5\}$.

Case 1: T has exactly 2 distinct nodes u_1 and u_2. Let U be the shrub of T obtained by removing all end vertices. Clearly, $U \cong P_s$ for some $s \geq 2$ and by Lemmas 2.1 and 2.3 $s \leq 5$. Note $s \neq 2$ since $\Delta(T) \leq n - 4$ and T_6 is not a shrub of T. Similarly $s \neq 4$ since T_{21} is not a shrub of T and $T \ncong S_n^4$. Suppose that $s = 5$. Then T_{22} is a shrub of T and $\{u_1, u_2\} = \{a_{22}, g_{22}\}$ and there is 4-placement of

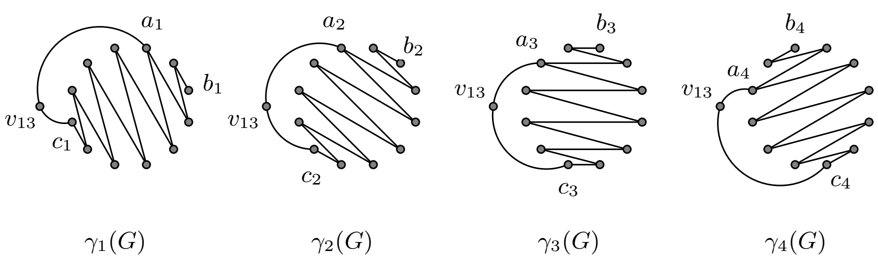

Figure 4: The 4-placement of G in Lemma 4.1 with $n = 13$ and $n_1 = 3$

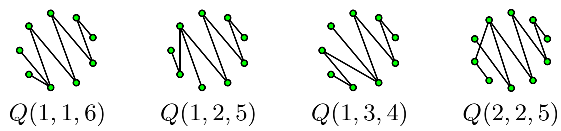

Figure 5: Embeddings that produce dispersed 4-packings by rotation.

T using Lemmas 3.1 and 2.1. Now suppose that $s = 3$. Then F_3 is a shrub of T since $\Delta(T) \leq n-4$ and $T \not\cong Y_n$. Similarly, a 4-placement of T can be obtained from Lemmas 3.1 and 2.1.

Case 2: T has exactly 3 distinct nodes u_1, u_2, and u_3. Let U be the shrub of T obtained by removing all end-vertices of T and let $s = |V(U)|$. If $s \geq 8$, then by Lemmas 4.2 and 2.1 there is a 4-placement of T, so $s \leq 7$. Since T_{14}, T_{17}, and T_{20} are not shrubs of T, then $U \cong P_s$. Furthermore, since T_{23} is not a shrub of T then $s \leq 5$. Assume without loss of generality that u_2 is not an end vertex of U. Suppose first $s = 5$. Then T_{19} is a shrub of T since T_{20} is not. However, by Lemmas 3.1 and 2.1 there is a 4-placement of T, a contradiction. Similarly, if $s = 4$ then either T_{10}, T_{16}, or T_{18} is a shrub of T, all contradictions. Finally, suppose $s = 3$. Since T_7 is not a shrub of T and $\Delta(T) \leq n-4$, then $d_T(u_2) = 3$. Moreover, since $T \not\cong T_{13}$, without loss of generality $d_T(u_1) \geq 4$. But then T_8 is a shrub of T, a contradiction.

Case 3: T has 4 distinct nodes u_1, u_2, u_3, and u_4. For $i = 1, 2, 3, 4$, let v_i be an end vertex adjacent to u_i, $V = \{v_1, v_2, v_3, v_4\}$, and let $U = T - V$. Suppose first that $\Delta(U) > (n-4) - 4$, then U is one of five trees: S_{n-4}, S^2_{n-4}, S^{2+}_{n-4}, $S^{2,2}_{n-4}$, or S^3_{n-4}. However this isn't possible since then at least one of T_6, T_7, T_8, T_{10}, F_1, or F_4 is a shrub of T, a contradiction. Thus $\Delta(U) \leq (n-4) - 4$. Therefore $U \in W$ since otherwise U has a 4-placement and by Lemma 2.5 so does T.

Case 3a: Suppose to contradict that $U = T_9$. Since neither F_1 nor F_2 are shrubs of T, then $a_9, e_9 \notin N(V)$. But then $N(V) \cap \{f_9, g_9, h_9\} \neq \emptyset$ and T_{17} is a shrub of T, a contradiction.

Case 3b: Suppose to contradict that $U = T_{13}$. If $d_{13} \notin N(V)$ then T_{18} is a shrub of T. If $d_{13} \in N(V)$ then T_{14} is a shrub of T, both contradictions.

Case 3c: Suppose $U = S^4_{n-4}$. Label the P_5 path in U as $y_1y_2y_3y_4y_5$ with $d_U(y_1) = n-9$ and let R_1 be the set of remaining (end) vertices and $r_1 = |N(V) \cap R_1|$. Suppose first $y_5 \notin N(V)$. Note that $r_1 \neq 0$ since T_{21} is not a shrub of T. Similarly $r_1 \notin \{1, 2, 3\}$ since T_{10} is not a shrub of T. Thus $r_1 = 4$. Let $U' = T - \{y_5, v_2, v_3, v_4\}$. Thus U' is a shrub of T not in W and so it has a 4-placement. But then T has a 4-placement by Lemma 2.5, a contradiction. Thus $y_5 \in N(V)$ and it may be assumed $v_1y_5 \in E(T)$. Again $r_1 \neq 0$ since otherwise $N(V) \cap \{y_2, y_4\} \neq \emptyset$ and T_{20} is a shrub of T. Similarly $r_1 \notin \{1, 2\}$ since T_{20} is not a shrub of T. Thus $r_1 = 3$ and F_5 is a shrub of T, another contradiction.

Case 3d: Suppose to contradict that $U = Y_{n-4}$. Label the shrub isomorphic to P_3 in Y_{n-4} as $x_1x_2x_3$ where $\mathrm{d}_U(x_1) = n-9$. Let R_2 (R_3) be the set of end vertices adjacent to x_1 (x_3) and let $r_2 = |N(V) \cap R_2|$ ($r_3 = |N(V) \cap R_3|$). Suppose to contradict $r_3 = 2$. If $r_2 > 0$ then T_{18} is a shrub of T and if $r_2 = 0$ then T_{17} is a shrub of T, both contradictions. Thus $r_3 < 2$. Note $r_2 \neq 0$ since then $x_2 \in N(V)$ and T_8 is a shrub of T. Similarly $r_2 \notin \{1, 2, 3\}$ since T_{10} is not a shrub of T. But then $r_2 = 4$ and F_1 is a shrub of T, a contradiction.

This completes the proof. □

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