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Algebraic invariants arising from the chromatic polynomials of theta graphs

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Abstract

This paper investigates some algebraic properties of the chromatic polynomials of theta graphs, i.e. graphs which have three internally disjoint paths sharing the same two distinct end vertices. We give a complete description of the Galois group, discriminant and ramification indices for the chromatic polynomials of theta graphs with three consecutive path lengths. We then do the same for theta graphs with three paths of the same length, by comparing them algebraically to the first family. This algebraic link extends naturally to generalised theta graphs with $k+1$ branches.

1 Introduction

The chromatic polynomial of an undirected graph $G = (V, E)$ interpolates the number of λ-colourings of its vertices V, under the usual restriction that adjacent vertices are assigned different colours. This polynomial has been extensively studied in graph theory and in statistical mechanics, where the chromatic polynomial is viewed as a specialisation of the Potts model partition function. An excellent overview of the relationship between the partition function and the chromatic polynomial is given in [1].

A *chromatic root* is a root of a chromatic polynomial. The study of chromatic roots is of particular interest to physicists, as the limit points of these roots give possible locations for physical phase transitions [25, 19, 26, 27]. This has motivated a large amount of research on identifying regions that are either root-free or root-dense for families of graphs [3, 2, 17, 9, 22, 21, 28, 19]. However, until recently there has been surprisingly little research on the algebraic nature of these roots; indeed as late as 2004, Sokal [27, Footnote 13] commented that the algebraic theory of chromatic roots remained "as yet rather undeveloped".

The factorisation of a polynomial into irreducibles is perhaps its most basic algebraic property. A graph G is said to have a *chromatic factorisation* if the chromatic polynomial of G can be expressed as the product of chromatic polynomials of lower degree (ignoring some linear factors). Any clique-separable graph, that is, a graph that can be obtained by identifying an r-clique in some graph H_1 with an r-clique in another graph H_2, has a chromatic factorisation. A graph G is *strongly non-clique-separable* if $P(G;\lambda)$ is not the chromatic polynomial of any clique-separable graph. In [15, 14] it was shown that there exist strongly non-clique-separable graphs exhibiting chromatic factorisations.

Another important invariant associated to a rational polynomial is its Galois group. Research into these objects was initiated by [8, 12, 13], and the Galois groups of all chromatic polynomials of degree ≤ 10 were calculated in [12, 13]. Although most small degree chromatic polynomials have irreducible factors with symmetric Galois group, there are also nice examples of infinite families of graphs with cyclic and dihedral Galois groups [13]. As a counterpoint, it was shown in [5] that the Galois group of the multivariate Tutte polynomial is always a direct product of symmetric groups, and it was conjectured that this result holds for the standard Tutte polynomial. Since the chromatic polynomial is a specialisation of the Tutte polynomial, and as most chromatic polynomials of degree ≤ 10 have Galois groups that are a direct product of symmetric groups, it is therefore worthwhile to discover families of graphs where this is not the case.

One family of graphs that have chromatic polynomials with non-symmetric Galois groups is the cycle graphs. It is easy to see the chromatic polynomial of the cycle of order $n+1$ has Galois group isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{\times}$, that is, the multiplicative group of units in the ring $\mathbb{Z}/n\mathbb{Z}$. The *theta graph* θ_{a_1,a_2,a_3} is the graph obtained from three disjoint paths $\{u_0, u_1, \ldots, u_{a_1}\}$, $\{v_0, v_1, \ldots, v_{a_2}\}$ and $\{w_0, w_1, \ldots, w_{a_3}\}$, by identifying the start vertices u_0, v_0 and w_0, and also identifying the end vertices u_{a_1}, v_{a_2} and w_{a_3} (see Figure 1 below).

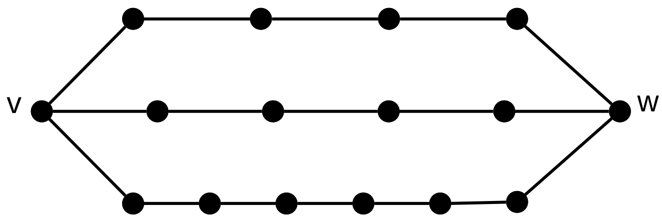

Figure 1: The theta graph $\theta_{5,5,7}$

Similarly, the *generalised theta graph* $\theta_{a_1,a_2,\ldots,a_t}$ is the graph obtained by identifying t internally disjoint paths of length $a_1, a_2, \ldots, a_t \geq 1$ respectively. Theta graphs can be regarded as being closely related to cycle graphs, since the graph θ_{a_1,a_2,a_3} is just the graph consisting of a cycle of length $a_1 + a_2$ together with an additional path of length a_3. Given that the Galois groups of chromatic polynomials of all cycle graphs are so easily determined, the next natural step is to look at the chromatic roots of theta graphs. Moreover generalised theta graphs are indispensable tools in proving quite general results on chromatic roots [23, 24, 7, 27, 4]. A notable example is Sokal's argument [27] that chromatic roots are dense in the complex plane. His proof shows that the family of generalised theta graphs with all paths of equal length have chromatic roots that are dense everywhere, with the possible exception of the disc $|\lambda - 1| < 1$.

In [13] the Galois groups of chromatic polynomials for theta graphs of order at most 19 with path lengths greater than 1 were calculated. It was found that with two exceptions (namely $\theta_{2,3,3}$ and $\theta_{2,3,5}$) all of these polynomials factorised into irreducibles that were chromatic factors of cycle graphs, and a single irreducible factor with symmetric Galois group. In this article we formulate a precise prediction (Conjecture 4.1) on the shape of the Galois group for all theta graphs, and demonstrate that this conjecture holds for two infinite families of theta graph.

1.1 Some algebraic questions

Recently, Cameron [8] made two conjectures on chromatic roots. The "$n\alpha$-conjecture" states that if α is a chromatic root, then so is $n\alpha$ for all $n \in \mathbb{N}$. The $n\alpha$-conjecture was shown to be true for *clique-theta graphs* [4]. A clique-theta graph is a generalised theta graph where the vertices are replaced by cliques and edges are replaced by all possible edges between adjacent cliques.

The "$\alpha + n$-conjecture" predicts that for every algebraic integer $\alpha \in \overline{\mathbb{Z}}$ there exists a constant $N(\alpha)$, such that $\alpha + n$ is a chromatic root for all integers $n \geq N(\alpha)$. If this conjecture is true, then every algebraic integer α should belong to the splitting field $\mathcal{K}_{P(G,\lambda)}$ of a chromatic polynomial $P(G, \lambda)$, for some finite graph G depending on α. Furthermore, if one recalls that

$$\mathrm{Gal}\left(\mathcal{K}_{P(G,\lambda)}/\mathbb{Q}\right) := \left\{\tau : \mathcal{K}_{P(G,\lambda)} \xrightarrow{\sim} \mathcal{K}_{P(G,\lambda)} \text{ such that } a^\tau = a \text{ for all } a \in \mathbb{Q}\right\}$$

one would expect the Galois group for (the normal closure of) each $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ to be realisable as the quotient of some $\mathrm{Gal}\left(\mathcal{K}_{P(G,\lambda)}/\mathbb{Q}\right)$, again for an appropriate choice of undirected graph G.

Question 1. *Is there an efficient method to determine the Galois group of the chromatic polynomial for G, bypassing the #P-hard problem of computing $P(G,\lambda)$?*

In other words, we are asking if the composition

$$\left\{\text{finite graphs}\right\} \overset{P(-,\lambda)}{\longrightarrow} \left\{\text{elements of } \mathbb{Z}[\lambda]\right\} \overset{\mathrm{Gal}(\mathcal{K}_{-}/\mathbb{Q})}{\longrightarrow} \left\{\text{finite groups}\right\}$$

is less algorithmically complex to represent than the functors $P(-,\lambda)$ and $\mathrm{Gal}(\mathcal{K}_{-}/\mathbb{Q})$.

Of course the Galois group is just one important invariant of a polynomial. Instead if one considers discriminants for these splitting field extensions, one is naturally led to study the ramification behaviour occurring inside $\mathcal{K}_{P(G,\lambda)}/\mathbb{Q}$.

Question 2. *What is the relationship between the path lengths in a graph G, and the prime numbers which ramify in the splitting field extension $\mathcal{K}_{P(G,\lambda)}$?*

We answer the second of these questions for two distinct infinite classes of theta graph which are chromatically inequivalent, but whose splitting fields are algebraically linked. It should be pointed out that just knowing the Galois group and ramification indices is not of itself enough to determine $\mathcal{K}_{P(G,\lambda)}$, up to isomorphism. (As a nice illustration, the two polynomials $\lambda^3-21\lambda+28$ and $\lambda^3-21\lambda-35$ share the same Galois group and both have discriminant 3969, yet they generate non-isomorphic cubic splitting fields.)

The relationship between polynomials that not only have the same Galois group and discriminant, but in fact have an identical splitting field structure (up to isomorphism) is very much stronger, which leads us to pose the following

Question 3. *Can we find two disjoint families of graphs such that each individual pair of graphs is splitting field, but not chromatically, equivalent?*

1.2 Galois groups of theta graphs

Henceforth we shall exclusively consider the generalised θ-graphs $\theta_{a_1,a_2,\ldots,a_t}$ with $a_j \geq 2$. The chromatic polynomial of a θ-graph is easy to work out via addition-identification. For instance, it is calculated in [7, Eqn(2.6a)] (and for the three path case in [18]) that $P\left(\theta_{a_1,a_2,\ldots,a_t},\lambda\right)$ equals

$$\frac{\lambda-1}{\lambda^{t-1}}\left(\prod_{j=1}^{t}\left((\lambda-1)^{a_j}-(-1)^{a_j}\right) \;+\; (\lambda-1)^{t-1}\prod_{j=1}^{t}\left((\lambda-1)^{a_j-1}+(-1)^{a_j}\right)\right) \quad (1)$$

which is a monic polynomial of degree $a_1+a_2+\cdots+a_t-t+2$, with integer coefficients.

Remark: In general, one can easily extract the cyclotomic factors from this polynomial. However determining the factorisation into irreducibles of the quotient polynomial is highly non-trivial; we have therefore restricted our study to three-branch θ-graphs whose path lengths a_1, a_2, a_3 are either equal, or else yield consecutive integers.

Even from these basic graphs many intriguing patterns emerge, intertwining the splitting fields of chromatic polynomials which should (at first glance) have no obvious connection. We will address Questions (2) and (3) for the families of graphs $G = \theta_{a,a+1,a+2}$ and $G = \theta_{a,a,a}$, before making a conjecture on the general case in the final section.

Theorem 1.1. *For all integers $a \geq 2$,*

$$\mathrm{Gal}\left(\mathcal{K}_{P(\theta_{a,a+1,a+2},\lambda)}/\mathbb{Q}\right) \;\cong\; \begin{cases} \left(\mathbb{Z}/(a^2+a)\mathbb{Z}\right)^{\times} \times \mathfrak{S}_{a+1} & \text{if } a \not\equiv 1 \pmod{3}, \\ \left(\mathbb{Z}/(a^2+a)\mathbb{Z}\right)^{\times} \times C_2 \times \mathfrak{S}_{a-1} & \text{if } a \equiv 1 \pmod{3}. \end{cases}$$

Prior to stating our second result, we remind the reader of some definitions from algebraic number theory. If L/K denotes a normal, finite extension of number fields and $\mathfrak{p}$ is a prime ideal in the ring of integers $\mathcal{O}_K$, there is a decomposition

$$\mathfrak{p}\cdot\mathcal{O}_L \;=\; \left(\prod_{i=1}^{g} \mathfrak{P}_i\right)^{\mathbf{e}_{\mathfrak{p}}(L/K)}$$

where $\mathbf{e}_{\mathfrak{p}}(L/K) > 0$ is called the *ramification index at* $\mathfrak{p}$, and the $\mathfrak{P}_i$'s are distinct prime ideals in $\mathcal{O}_L$ (if $K = \mathbb{Q}$ and $\mathfrak{p} = p\cdot\mathbb{Z}$, we shall just abbreviate this index by $\mathbf{e}_p(L)$).

Recall that $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ denotes Euler's totient function. For any given prime p, one normalises the p-adic valuation $\left\|-\right\|_p : \mathbb{Q} \longrightarrow p^{\mathbb{Z}} \cup \{0\}$ by the rule $\left\|p\right\|_p = 1/p$.

Theorem 1.2. (i) *If $a \geq 2$ satisfies $a \not\equiv 1 \pmod{3}$, the ramification indices are*

$$\mathbf{e}_p\left(\mathcal{K}_{P(\theta_{a,a+1,a+2},\lambda)}\right) \;=\; \begin{cases} \phi\left(\left\|a^2+a\right\|_p^{-1}\right) & \text{if } p \,\middle|\, (a^2+a) \\ 2 & \text{if } p \,\middle|\, \left((a+1)^{a+1}+(-a)^a\right) \\ 1 & \text{otherwise;} \end{cases}$$

(ii) *If $a \geq 2$ satisfies $a \equiv 1 \pmod{3}$, the corresponding ramification indices are*

$$\mathbf{e}_p\left(\mathcal{K}_{P(\theta_{a,a+1,a+2},\lambda)}\right) \;=\; \begin{cases} \phi\left(\left\|a^2+a\right\|_p^{-1}\right) & \text{if } p \,\middle|\, (a^2+a) \\ 2 & \text{if } p \,\middle|\, \left(\frac{(a+1)^{a+1}+(-a)^a}{3\times\left(a+\frac{(a-1)^2}{3}\right)^2}\right) \\ 2 & \text{if } p = 3 \\ 1 & \text{otherwise.} \end{cases}$$

Turning our attention to the case of equal path length, that is, for the θ-graphs $G = \theta_{a,a,a}$ we have the following analogue of the two theorems above.

Theorem 1.3. (i) *If the integer $a \geq 2$, then* $\mathrm{Gal}\big(\mathcal{K}_{P(\theta_{a,a,a},\lambda)}/\mathbb{Q}\big) \cong \mathfrak{S}_{3(a-1)}$*;*

(ii) *If the integer $a \geq 2$, the ramification indices at each prime p are*

$$\mathbf{e}_p\big(\mathcal{K}_{P(\theta_{a,a,a},\lambda)}\big) \;=\; \begin{cases} 2 & \textit{if } p \,\Bigg|\, \left(\frac{(3a-1)^{3a-1}+(2-3a)^{3a-2}}{3\times\left(3a^2-3a+1\right)^2}\right) \\ 1 & \textit{if } p = 3 \\ 1 & \textit{otherwise.} \end{cases}$$

The proof of this result exploits an intriguing link between the chromatic polynomials for the $\theta_{a,a,a}$ and the $\theta_{3a-2,3a-1,3a}$ graphs, namely that there is a splitting field isomorphism

$$\mathcal{K}_{P(\theta_{3a-2,3a-1,3a},\lambda)} \;\cong\; \mathbb{Q}\Big(e^{2\pi i/(9a^2-9a+2)}, \sqrt{-3}\Big) \vee \mathcal{K}_{P(\theta_{a,a,a},\lambda)}\,.$$

In fact this phenomenon extends seamlessly to generalised theta graphs with $k+1$ paths, and we shall devote Section 3 to a detailed investigation. It seems highly unlikely that this relationship is the only such occurrence amongst theta-graphs, and one future direction is to look for common structural properties which might explain these dualities.

Based on further MAGMA [6] calculations, in Section 4 we shall make a general conjecture concerning the form of $\mathrm{Gal}\big(\mathcal{K}_{P(\theta_{a_1,a_2,a_3},\lambda)}/\mathbb{Q}\big)$ for any triple (a_1,a_2,a_3) of positive integers. This conjecture has been verified computationally by us at least for the 171,000 examples with path lengths $a_1,a_2,a_3 \leq 101$ and whose 'interesting factor' has degree < 80.

2 The argument for the polynomial $P\big(\theta_{a,a+1,a+2},\lambda\big)$

In this section we prove Theorems 1.1 and 1.2. We start with two preliminary results. Let G denote the graph $\theta_{a,a+1,a+2}$, and one assumes throughout that the integer $a \geq 2$. Applying the formulae in (1) for path lengths $a, a+1, a+2$, we find $P(G,\lambda)$ equals

$$\frac{\lambda-1}{\lambda^2}\left(\prod_{j=0}^{2}\Big((\lambda-1)^{a+j}-(-1)^{a+j}\Big) \;+\; (\lambda-1)^2\prod_{j=0}^{2}\Big((\lambda-1)^{a+j-1}+(-1)^{a+j}\Big)\right)$$
$$= \quad \lambda(\lambda-1)\times\left(\frac{(\lambda-1)^a+(-1)^{a+1}}{\lambda}\right)\times\left(\frac{(\lambda-1)^{a+1}+(-1)^a}{\lambda}\right)$$
$$\times\Big((\lambda-1)^{a+1}+(-1)^a(\lambda-1)+(-1)^{a+1}\Big).$$

The splitting field of $\frac{(\lambda-1)^a+(-1)^{a+1}}{\lambda}$ coincides with $\mathbb{Q}\big(e^{2\pi i/a}\big)$, whilst the splitting field of $\frac{(\lambda-1)^{a+1}+(-1)^a}{\lambda}$ is given by $\mathbb{Q}\big(e^{2\pi i/(a+1)}\big)$. Moreover, the final term satisfies

$$\Big((\lambda-1)^{a+1}+(-1)^a(\lambda-1)+(-1)^{a+1}\Big) \;=\; (-1)^{a+1}\times\big(X^{a+1}+X+1\big)\Big|_{X=1-\lambda}$$

so one immediately obtains

Lemma 2.1. *The total splitting field $\mathcal{K}_{P(G,\lambda)}$ must be the field compositum of $\mathbb{Q}\big(e^{2\pi i/(a^2+a)}\big)$ spliced with the splitting field $\mathcal{K}_{f_a(X)}$, where $f_a(X) := X^{a+1}+X+1$.*

The discriminant of $\mathbb{Q}\big(e^{2\pi i/(a^2+a)}\big)$ is well known (e.g. see [29, Prop. 2.7]), namely

$$\operatorname{disc}\Big(\mathbb{Q}\big(e^{2\pi i/(a^2+a)}\big)\Big) \;=\; \prod_{\text{primes } p}\frac{\big(a^2+a\big)^{\phi(a^2+a)}}{p^{\phi(a^2+a)/(p-1)}}$$

while the polynomial discriminant for $f_a(X)$ is computed via the formula

$$\operatorname{disc}\big(f_a(X)\big) \;=\; (-1)^{\frac{a^2+a}{2}}\times\big((a+1)^{a+1}+(-a)^a\big).$$

In particular, any prime dividing $\operatorname{disc}\big(\mathbb{Q}(e^{2\pi i/(a^2+a)})\big)$ must then divide a or $a+1$, and thus cannot divide $\operatorname{disc}\big(f_a(X)\big)$. Consequently these discriminants are coprime, in which case no primes can ramify in $\mathbb{Q}\big(e^{2\pi i/(a^2+a)}\big)$ and $\mathcal{K}_{f_a(X)}$ simultaneously. A famous result of Minkowski implies $\mathbb{Q}\big(e^{2\pi i/(a^2+a)}\big)\cap\mathcal{K}_{f_a(X)}=\mathbb{Q}$, as there exist no everywhere unramified extensions of the rationals.

Corollary 2.2. $\operatorname{Gal}\big(\mathcal{K}_{P(G,\lambda)}/\mathbb{Q}\big) \;\cong\; \operatorname{Gal}\big(\mathbb{Q}(e^{2\pi i/(a^2+a)})/\mathbb{Q}\big)\times\operatorname{Gal}\big(\mathcal{K}_{f_a(X)}/\mathbb{Q}\big)$.

It therefore remains to compute the right-most Galois group, and thereby complete the proof of Theorem 1.1. The remainder of this section is devoted to the proof of

Proposition 2.3.

$$\operatorname{Gal}\big(\mathcal{K}_{f_a(X)}/\mathbb{Q}\big) \;\cong\; \begin{cases}\mathfrak{S}_{a+1} & \text{if } a\not\equiv 1 \ (\bmod 3)\\ C_2\times\mathfrak{S}_{a-1} & \text{if } a\equiv 1 \ (\bmod 3).\end{cases}$$

The problem naturally splits into two complementary situations.

Case (I) – *The path length $a\not\equiv 1\ (\bmod 3)$.*

Exploiting an old result of Selmer [20, Thm 1], the trinomial $f_a(X)$ is irreducible over $\mathbb{Q}$ whenever $a+1\not\equiv 2\ (\bmod 3)$. Further, Osada [16, Thm 1] then establishes that the Galois group of $\mathcal{K}_{f_a(X)}$ over $\mathbb{Q}$ is the full symmetric group i.e. $\mathfrak{S}_{a+1}$, thus Proposition 2.3 must hold true here.

The only primes that will ramify in $\mathcal{K}_{f_a(X)}/\mathbb{Q}$ are those that divide the discriminant $(-1)^{\frac{a^2+a}{2}} \times \big((a+1)^{a+1} + (-a)^a\big)$. However the extension $\mathcal{K}_{f_a(X)}/\mathbb{Q}\big(\sqrt{\mathrm{disc}(f_a(X))}\big)$ is unramified at all finite places by [16, Cor. 1], which implies that the ramification index for each such prime is precisely 2.

The primes p ramifying in $\mathbb{Q}(e^{2\pi i/(a^2+a)})$ have ramification index equal to the degree of the totally ramified extension $\left[\mathbb{Q}\left(e^{2\pi i/p^{N_p^{(a)}}}\right) : \mathbb{Q}\right]$ with $N_p^{(a)} = \left\|a^2 + a\right\|_p^{-1}$; the latter coincides with the quantity in Theorem 1.2(i), so we are done with the first case.

Case (II) – *The path length $a \equiv 1 \pmod 3$.*

This situation is trickier. The polynomial $f_a(X)$ is no longer irreducible in $\mathbb{Q}[X]$, rather it splits into $f_a(X) = (X^2+X+1)\times g_a(X)$; again using [20, Thm 1] the second factor is irreducible. We therefore need to extend techniques of Llorente, Nart and Vila [11], which themselves only deal with irreducible trinomials, in order to work out the Galois group for this irreducible component.

Lemma 2.4. **(i)** $\mathrm{disc}\left(X^2 + X + 1\right) = -3$*;*

(ii) $\mathrm{disc}\left(g_a(X)\right) = (-1)^{\frac{a^2+a+2}{2}} \times \frac{(a+1)^{a+1}+(-a)^a}{3\times\left(a+\frac{(a-1)^2}{3}\right)^2}$ *;*

(iii) *The prime* 3 *does not divide* $\mathrm{disc}\left(g_a(X)\right)$*.*

Proof. Part (i) is elementary. To show part (ii), let us write $\alpha_1, \ldots, \alpha_{a-1}$ for the roots of $g_a(X)$; then

$$\begin{aligned}\mathrm{disc}\left(f_a(X)\right) &= (\zeta-\overline{\zeta})^2 \times \prod_{1\le j<k\le a-1}(\alpha_k-\alpha_j)^2 \times \prod_{1\le j\le a-1}\Big((\zeta-\alpha_j)^2 \times (\overline{\zeta}-\alpha_j)^2\Big)\\ &= -3 \times \mathrm{disc}\left(g_a(X)\right) \times \Big(g_a(\zeta)\times g_a(\overline{\zeta})\Big)^2 \qquad \text{where } \zeta = e^{2\pi i/3}.\end{aligned}$$

Now expanding $g_a(X)$ as a geometric series,

$$\begin{aligned}g_a(X) &= 1 + \frac{X^{a+1}-X^2}{X^2+X+1} = 1 + X^2(X-1)\times\frac{X^{a-1}-1}{X^3-1}\\ &= 1 + (X^3-X^2)\times\sum_{m=0}^{(a-4)/3} X^{3m}\end{aligned}$$

thence $g_a(\zeta) = 1 + (\zeta^3-\zeta^2)\times\frac{a-1}{3} = z + 1 - z\overline{\zeta}$ where the natural number $z = \frac{a-1}{3}$. By an identical argument, $g_a(\overline{\zeta}) = z + 1 - z\zeta$.

Taking the product of these two quantities, one calculates that

$$\begin{aligned}g_a(\zeta)\times g_a(\overline{\zeta}) &= (z+1)^2 + z^2\zeta\overline{\zeta} - z(z+1)(\zeta+\overline{\zeta})\\ &= z^2+2z+1+z^2-(z^2+z)\times(-1) = a + \frac{(a-1)^2}{3}\end{aligned}$$

in which case

$$\operatorname{disc}\big(g_a(X)\big) \;=\; \frac{\operatorname{disc}\big(f_a(X)\big)}{-3\big(g_a(\zeta)\times g_a(\overline{\zeta})\big)^2} \;=\; \frac{(-1)^{\frac{a^2+a}{2}}\times\big((a+1)^{a+1}+(-a)^a\big)}{-3\times\big(a+\frac{(a-1)^2}{3}\big)^2}\,.$$

Lastly to prove (iii), it is enough to show that $9\nmid\big((a+1)^{a+1}+(-a)^a\big)$ if $a\equiv 1\ (\bmod 3)$. First observe that a and $a+1$ are both coprime to 3, and by Fermat's little theorem $x^6\equiv 1\ (\bmod 9)$ for all $x\in\big(\mathbb{Z}/9\mathbb{Z}\big)^\times$. Therefore it suffices to check that

$$(x+1)^{x+1}+(-x)^x \;\equiv\; 3 \text{ or } 6\ \ (\bmod 9) \quad \text{at every such } x\le \operatorname{lcm}(6,9)=18.$$

This means we need to verify it for $x\in\{1,4,7,10,13,16\}$, which is easily done. □

Returning to the proof of Proposition 2.3 when $a\equiv 1\ (\bmod 3)$, we see immediately by Lemma 2.4(i),(iii) that X^2+X+1 and $g_a(X)$ have relatively prime discriminants; applying Minkowski's result once more, we deduce that $\mathbb{Q}\big(e^{2\pi i/3}\big)$ and $\mathcal{K}_{g_a(X)}$ are linearly disjoint over $\mathbb{Q}$. As a direct consequence,

$$\operatorname{Gal}\big(\mathcal{K}_{f_a(X)}/\mathbb{Q}\big) \;\cong\; \operatorname{Gal}\big(\mathbb{Q}(e^{2\pi i/3})/\mathbb{Q}\big)\times\operatorname{Gal}\big(\mathcal{K}_{g_a(X)}/\mathbb{Q}\big) \;\cong\; C_2\times\operatorname{Gal}\big(\mathcal{K}_{g_a(X)}/\mathbb{Q}\big).$$

Therefore to prove Proposition 2.3 in Case (II), it remains to establish the following

Lemma 2.5. $\operatorname{Gal}\big(\mathcal{K}_{g_a(X)}/\mathbb{Q}\big)\cong\ \mathfrak{S}_{a-1}$.

Proof. Recalling that $g_a(X)=\frac{X^{a+1}+X+1}{X^2+X+1}$, high school calculus shows

$$g_a'(X) \;=\; \frac{\mathrm{d}\left(\frac{X^{a+1}+X+1}{X^2+X+1}\right)}{\mathrm{d}X} \;=\; \frac{(a+1)X^a+1-(2X+1)\times g_a(X)}{X^2+X+1}\,.$$

Suppose we fix some prime p dividing $\operatorname{disc}\big(g_a(X)\big)$, so that p may ramify inside $\mathcal{K}_{g_a(X)}$; in particular, the prime $p\,\big|(a+1)^{a+1}+(-a)^a$ hence p can divide neither of a nor $a+1$. Let us further assume there exists $\beta\in\overline{\mathbb{F}}_p$ satisfying

- $g(\beta)=0$;
- $g_a'(\beta)=0$.

Our strategy is to show that β is exactly a double root of $g_a(X)$ modulo p, and at most one such β can occur.

Plugging β into our above expression for the derivative,

$$g_a'(\beta) \;\equiv\; \frac{(a+1)\beta^a+1-(2\beta+1)\times g(\beta)}{\beta^2+\beta+1} \;\equiv\; 0 \quad (\bmod p)$$

which implies $(a+1)\beta^a + 1 \equiv 0 \pmod{p}$, so that $\beta^a \equiv -(a+1)^{-1} \pmod{p}$. Furthermore as $g_a(\beta) \equiv 0 \pmod{p}$ one finds $\beta^{a+1} + \beta + 1 \equiv 0 \pmod{p}$, thus

$$\beta\big(\beta^a + 1\big) \equiv \beta\big(-(a+1)^{-1} + 1\big) \equiv -1, \quad \text{i.e. } \beta\big(-1 + (a+1)\big) \equiv -(a+1) \pmod{p}.$$

It follows that $\beta \equiv -\left(\frac{a+1}{a}\right) \in \mathbb{F}_p$ can be the only double root for $g_a(X)$ modulo p. As a corollary, $g_a(X) \equiv (X-\beta)^2 \times h_a(X) \pmod{p}$ where $h_a(X) \in \mathbb{F}_p[X]$ has no multiple roots, except possibly for $X = \beta$.

Remark: In fact, one can quickly see why $X = \beta$ simply cannot be a root of $h_a(X)$. If it were a root then $(X-\beta)^2$ would have to divide $g'_a(X)$, in which case

$$(X-\beta)^2 \quad \text{would divide} \quad (a+1)X^a + 1 - (2X+1) \times g_a(X).$$

Now $(X-\beta)^2$ divides $g_a(X)$, implying $(X-\beta)^2$ divides $j_a(X) := (a+1)X^a + 1$; but the only root of $j'_a(X) = a(a+1)X^{a-1}$ is $X = 0$, and $0 \not\equiv -\left(\frac{a+1}{a}\right) \pmod{p}$.

In summary, we have so far established that either $g_a(X)$ has no multiple roots mod p, or at worst $g_a(X) \equiv (X-\beta)^2 \times h_a(X) \pmod{p}$ where $h_a(X)$ has no multiple roots and $h_a(\beta) \not\equiv 0$, for every prime $p \mid \mathrm{disc}\big(g_a(X)\big)$. Consequently the inertia subgroup at any place $\mathfrak{p}$ of $\mathcal{K}_{g_a(X)}$ above p is either trivial, or else is a group generated by a transposition.

If we write $\mathcal{I}_{\mathfrak{p}}$ for each inertia group at $\mathfrak{p}$, then the set $\big\{\mathcal{I}_{\mathfrak{p}}\big\}_{\mathfrak{p} \mid \mathrm{disc}(g_a)}$ generates a subgroup $\mathcal{J}$ of $\mathcal{G} = \mathrm{Gal}\big(\mathcal{K}_{g_a(X)}/\mathbb{Q}\big)$. The fixed field of $\mathcal{J}$ is an unramified extension of $H^0\big(\mathcal{G}, \mathcal{K}_{g_a(X)}\big) = \mathbb{Q}$, which must equal the ground field $\mathbb{Q}$ (via Minkowski again). It follows that $\mathcal{J} = \mathcal{G}$, hence this whole group is generated by transpositions. However the irreducibility of $g_a(X)$ implies $\mathcal{G}$ operates transitively on the roots $\alpha_1, \ldots \alpha_{a-1}$; indeed there is only one outcome, namely one deduces $\mathcal{G} \cong \mathfrak{S}_{a-1}$ as claimed. $\square$

The same arguments as in Case (I) then allow us to compute the ramification indices $\mathbf{e}_p$ listed in Theorem 1.2(ii). Note that $\mathbf{e}_3\big(\mathcal{K}_{f_a(X)}\big) = 2$ because $\mathbb{Q}\big(e^{2\pi i/3}\big)$ is always a subfield of the splitting field of $f_a(X) = (X^2 + X + 1) \times g_a(X)$, while the prime 3 is unramified in $\mathcal{K}_{g_a(X)}$ courtesy of Lemma 2.4(iii).

The proof of Theorems 1.1 and 1.2 is complete.

3 The analysis for θ-graphs with equal path length

We now give the proof of Theorem 1.3, which exploits an algebraic connection between the chromatic polynomials of the $\theta_{a,a,a}$ and $\theta_{3a-2,3a-1,3a}$ graphs (the number theory of the latter object is covered by the previous section). Since this connection generalises to theta-graphs with $k+1$-branches, we shall therefore work in this more general setting.

Expanding on our previous notation, we now write $\theta^{(k+1)}_{\underline{a},\,\|}$ for the theta-graph containing $k+1$ paths all of equal length $a \geq 2$. Using the formula in (1), its chromatic polynomial factorises into

$$P\big(\theta^{(k+1)}_{\underline{a},\,\|}, \lambda\big) \;=\; (-1)^{k+1} \times \lambda(\lambda-1) \times \mathcal{H}_{\underline{a},\|}(1-\lambda)$$

where the last factor

$$\mathcal{H}_{\underline{a},\|}(X) \;=\; \frac{(X^a-1)^{k+1} - X^k(X^{a-1}-1)^{k+1}}{(X-1)^{k+1}}\,.$$

The second family we consider are theta-graphs obtained by identifying the endpoints for consecutive paths of length $(k+1)a-k$, $(k+1)a-k+1, \ldots, (k+1)a$. If we denote this new graph with the label $\theta^{(k+1)}_{\underline{a},\,\uparrow}$ then using Equation (1) again, one calculates

$$P\big(\theta^{(k+1)}_{\underline{a},\,\uparrow}, \lambda\big) \;=\; (-1)^{(k+1)a}(\lambda-1) \times \frac{\prod_{j=0}^{k} P(\mathcal{C}_{(k+1)a-j}, \lambda)}{\lambda^k(\lambda-1)^k} \times \mathcal{H}_{\underline{a},\uparrow}(1-\lambda)$$

where the quadrinomial term

$$\mathcal{H}_{\underline{a},\uparrow}(X) = X^{(k+1)a} - X^{(k+1)a-1} + X^k - 1$$

and each $\mathcal{C}_r$ denotes a cycle graph of length r.

Lemma 3.1. *The interesting factor $\mathcal{G}_{\underline{a},\uparrow}$ dividing $\mathcal{H}_{\underline{a},\uparrow}$ is given by the quotient*

$$\mathcal{G}_{\underline{a},\uparrow}(X) \;:=\; \begin{cases} \frac{(X-1)\,\mathcal{H}_{\underline{a},\uparrow}(X)}{(X^{k+1}-1)(X^d+1)} & \textit{if } k \textit{ is odd and } d = \gcd(k-1, 2a-1) > 1 \\ \frac{\mathcal{H}_{\underline{a},\uparrow}(X)}{X^{k+1}-1} & \textit{otherwise,} \end{cases}$$

and is an irreducible polynomial over $\mathbb{Q}$.

Proof. The roots of $\mathcal{H}_{\underline{a},\uparrow}(X) = X^{(k+1)a} - X^{(k+1)a-1} + X^k - 1$ are inverses of the roots of $\mathcal{P}(X) = X^{(k+1)a} - X^{(k+1)a-k} + X - 1$. From [10, Theorem 1] if $\mathcal{P}(X)$ does not vanish at roots of unity then $\mathcal{P}(X)$ is irreducible; otherwise $\mathcal{P}(X)$ can be factored into two rational factors, one of which is irreducible, and the other factor vanishes only at roots of unity.

Applying [10, Theorem 2] the only possible zeros of $\mathcal{H}_{\underline{a},\uparrow}(X)$ that are also roots of unity must be simple, and are found amongst the zeros of:

- $X^{k+1} = \pm 1$
- $X = \pm 1$
- $X^d - 1$ where $d = \gcd\big((k+1)a-k, (k+1)a-1\big) = \gcd(k-1, 2a-1)$.

(a) All roots of $X^{k+1}-1$ are roots of $\mathcal{H}_{\underline{a},\uparrow}(X)$. If ω is a root of $X^{k+1}-1$, then ω is a root of the factor $X^{(k+1)a}-X^{(k+1)a-1}+X^k-1$ because

$$\omega^{(k+1)a}-\omega^{(k+1)a-1}+\omega^k-1 \;=\; 1^a-1^a\omega^{-1}+\omega^{-1}-1 \;=\; 0.$$

(b) No roots of $X^{k+1}+1$ are roots of $\mathcal{H}_{\underline{a},\uparrow}(X)$. If ω is a root of $X^{k+1}+1$ then

$$\begin{aligned}\mathcal{H}_{\underline{a},\uparrow}(\omega) \;&=\; \omega^{(k+1)a}-\omega^{(k+1)a-1}+\omega^k-1 \;=(-1)^a-(-1)^a\omega^{-1}-\omega^{-1}-1\\ &= \begin{cases}-2\omega^{-1} & \text{if } a \text{ is even}\\ -2 & \text{if } a \text{ is odd.}\end{cases}\end{aligned}$$

(c) The roots ± 1. It is easy to see that $X=1$ is a root of $\mathcal{H}_{\underline{a},\uparrow}(X)$ for all k, and that $X=-1$ is a root of $\mathcal{H}_{\underline{a},\uparrow}(X)$ if and only if k is odd. These are also roots of $X^{k+1}-1$, and as the roots of unity at which $\mathcal{H}_{\underline{a},\uparrow}(X)$ vanishes are simple roots by [10, Theorem 2], it suffices to consider this factor only.

(d) For d as above, the roots of X^d-1 (excluding 1) are not roots of $\mathcal{H}_{\underline{a},\uparrow}(X)$. If $\omega\neq 1$ is a root of X^d-1, then

$$\begin{aligned}\mathcal{H}_{\underline{a},\uparrow}(\omega) \;&=\; \omega^{(k+1)a}-\omega^{(k+1)a-1}+\omega^k-1\\ &=\; \omega^{(k+1)a-1}\omega-\omega^{(k+1)a-1}+\omega^{k-1}\omega-1\\ &=\; \omega-1+\omega-1 \;=\; 2(\omega-1) \;\neq\; 0.\end{aligned}$$

(e) Non-integer roots of X^d+1 will be roots of $\mathcal{H}_{\underline{a},\uparrow}(X)$ if and only if k is odd. We first compute the sign of $(-1)^d$. If $d=\gcd(k-1,2a-1)$ then $2a\equiv 1 \pmod d$; in particular, the divisor d must be odd and $(-1)^d=-1$. If ω is a root of X^d+1 then

$$\begin{aligned}\mathcal{H}_{\underline{a},\uparrow}(\omega) \;&=\; \omega^{(k+1)a}-\omega^{(k+1)a-1}+\omega^k-1\\ &=\; \omega^{(k+1)a-1}(\omega-1)+\omega^{k-1}\omega-1\\ &=\; (-1)^{\frac{(k+1)a-1}{d}}(\omega-1)+(-1)^{\frac{k-1}{d}}\omega-1.\end{aligned}$$

Using a parity argument together with the fact d is odd, one deduces that

$$\mathcal{H}_{\underline{a},\uparrow}(\omega) \;=\; \begin{cases}-\omega+1+\omega-1= \;\;0 & \text{if } k \text{ is odd}\\ \;\;\omega-1-\omega-1=-2 & \text{if } k \text{ is even and } a \text{ is odd}\\ -\omega+1-\omega-1=-2\omega & \text{if } k \text{ is even and } a \text{ is even,}\end{cases}$$

hence X^d+1 divides $\mathcal{H}_{\underline{a},\uparrow}(X)$ if and only if k is odd.

In summary the only factor of $\mathcal{H}_{\underline{a},\uparrow}(X)$ arising from roots of unity is $X^{k+1}-1$, except when k is odd and $d>1$ in which case X^d+1 also divides $\mathcal{H}_{\underline{a},\uparrow}(X)$. $\square$

Proposition 3.2. **(i)** *We have an inclusion of splitting fields* $\mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)} \subset \mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}$*;*

(ii) *The group* $\mathrm{Gal}\big(\mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}/\mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)}\big)$ *is always solvable;*

(iii) *If* $\mathrm{Gal}\big(\mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}/\mathbb{Q}\big) \cong \mathfrak{S}_n$ *with* $n = \deg(\mathcal{G}_{\underline{a},\uparrow}(X)) > 4$*, then* $\mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)} = \mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}$.

Deferring its proof momentarily, let us first explain why Theorem 1.3 is a consequence. If $k = 2$ the polynomial $\mathcal{G}_{\underline{a},\uparrow}(X) = g_{3a-2}(X)$ in the notation of §2, and so by Lemma 2.5 one knows that $\mathrm{Gal}\left(\mathcal{K}_{g_{3a-2}(X)}/\mathbb{Q}\right) \cong \mathfrak{S}$ with $n = 3a - 3$.

Providing $a > 2$ we can apply the third part of the above result, and thereby deduce

$$\mathcal{K}_{g_{3a-2}(X)} \;=\; \mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)} \stackrel{\text{by } 3.2\text{(iii)}}{=} \mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)} \;=\; \mathcal{K}_{P\left(\theta^{(3)}_{\underline{a},\|},\,\lambda\right)} \;=\; \mathcal{K}_{P(\theta_{a,a,a},\lambda)}$$

in which case Theorem 1.3(i) follows immediately. Likewise the ramification indices for $\mathcal{K}_{P(\theta_{a,a,a},\lambda)}$ are identical to those of $\mathcal{K}_{g_{3a-2}(X)}$, and the latter indices were determined at the end of the previous section; hence 1.3(ii) follows too.

(Alternatively if $a = k = 2$, Theorem 1.3 can be verified computationally using MAGMA [6].)

Proof of Proposition 3.2. Recall that the polynomial $\mathcal{G}_{\underline{a},\uparrow}(X)$ is irreducible in $\mathbb{Q}[X]$, thus we can write $\mathcal{R} = \{\alpha_1, \ldots, \alpha_n\}$ to denote its set of distinct roots. If $\alpha \in \mathcal{R}$ then

$$(\alpha - 1)^{k+1} \times \mathcal{H}_{\underline{a},\|}\left(\alpha^{k+1}\right) \;=\; \left(\alpha^{(k+1)a} - 1\right)^{k+1} - \left(\alpha^{(k+1)a-1} - \alpha^{k}\right)^{k+1}.$$

However $\mathcal{H}_{\underline{a},\uparrow}(\alpha) = 0$ whence $\alpha^{(k+1)a-1} - \alpha^k = \alpha^{(k+1)a} - 1$, and so $(\alpha - 1)^{k+1} \times \mathcal{H}_{\underline{a},\|}\left(\alpha^{k+1}\right)$ must be zero. Since $\alpha \neq 1$ by the irreducibility of $\mathcal{G}_{\underline{a},\uparrow}$, it follows α^{k+1} is a root of $\mathcal{H}_{\underline{a},\|}(X)$. Conversely, by reversing the argument above, one sees that all the roots of $\mathcal{H}_{\underline{a},\|}(X)$ arise as $(k+1)$-st powers of elements in $\mathcal{R}$.

As a corollary, $\mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)} = \mathbb{Q}(\alpha_1, \ldots, \alpha_n) \supset \mathbb{Q}(\alpha_1^{k+1}, \ldots, \alpha_n^{k+1}) = \mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)}$ which proves statement (i). Furthermore, by basic Kummer theory $\mathrm{Gal}\big(\mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}/\mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)}\big)$ will be a subquotient of the semi-direct product

$$\big(\mathbb{Z}/(k+1)\mathbb{Z}\big)^{\times} \ltimes \big(C_{k+1} \times \cdots \times C_{k+1}\big) \qquad \text{with } n\text{-copies of } C_{k+1},$$

as the larger splitting field is obtained by adjoining $(k+1)$-st roots of elements in the ground field. Since the latter group is solvable, its subquotient $\mathrm{Gal}\big(\mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}/\mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)}\big)$ must also be solvable and part (ii) is established.

Finally to prove statement (iii), we will assume that $\mathrm{Gal}\big(\mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}/\mathbb{Q}\big) \cong \mathfrak{S}_n$ with $n > 4$. Since n is at least 5, the only normal subgroups of $\mathfrak{S}_n$ are the alternating group $\mathfrak{A}_n$, the trivial group $\{\mathrm{id}\}$, and $\mathfrak{S}_n$ of course. By the fundamental theorem of Galois theory,

$$\mathrm{Gal}\big(\mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)}/\mathbb{Q}\big) \;\cong\; \frac{\mathrm{Gal}\big(\mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}/\mathbb{Q}\big)}{H} \qquad \text{where } H = \mathrm{Gal}\big(\mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}/\mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)}\big).$$

If $H \cong \mathfrak{S}_n$ then $\mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)} = \mathbb{Q}$, which is impossible since $\mathcal{H}_{\underline{a},\|}(X)$ has no rational roots.

If $H \cong \mathfrak{A}_n$ then $\mathrm{Gal}\big(\mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)}/\mathbb{Q}\big) \cong C_2$ which is cyclic and therefore solvable; however $\mathrm{Gal}\big(\mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}/\mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)}\big)$ is also solvable by (ii), in which case $\mathrm{Gal}\big(\mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}/\mathbb{Q}\big)$ must be solvable as "G/H and H both solvable $\iff$ G is solvable". This yields yet another contradiction, as it is well known that $\mathfrak{S}_n$ is a solvable group if and only if $n = 1, 2, 3, 4$.

By a process of elimination, the only remaining possibility is that $H = \{\mathrm{id}\}$ in which case $\mathrm{Gal}\big(\mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)}/\mathbb{Q}\big) \cong \mathrm{Gal}\big(\mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}/\mathbb{Q}\big)$; in addition $\mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)} \subset \mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}$ by part (i), and one concludes that $\mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)}$ and $\mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}$ are identical splitting fields. □

Remark: Each graph $\theta_{3a-2,3a-1,3a}$ falls under Case (II) of Proposition 2.3, which involves the additional factor $X^2 + X + 1$ corresponding to the interesting factor in the chromatic polynomial of the cycle C_4. As $\mathcal{K}_{\mathcal{H}_{\underline{a},\|}(X)}$ and $\mathcal{K}_{\mathcal{G}_{\underline{a},\uparrow}(X)}$ are identical splitting fields, it is clear that the graph $\theta_{a,a,a} \cup C_{3a-1} \cup C_{3a} \cup C_4$ (i.e. the graph with components $\theta_{a,a,a}$ and the three cycles of order $3a-1$, $3a$ and 4) shares the same splitting field as the graph $\theta_{3a-2,3a-1,3a}$. Thus we have found two disjoint families of graphs such that each individual pair of graphs is splitting field equivalent, thereby yielding a positive an answer to Question 3. However we note that Question 1 remains unresolved in general, although our results at least give us a way to obtain the splitting field of one family of graphs from the other family.

4 The general case

In this conclusion, we present some speculations concerning the structure of the Galois group of a general theta graph. Fix integers $a_1, a_2, a_3 \geq 2$, and consider the finite set

$$\Sigma_{a_1,a_2,a_3} := \Big\{ d \in \mathbb{N} \text{ such that } d \,\Big|\, \gcd(a_i, a_j - 1) \text{ for some } i \neq j \Big\}.$$

Using the formulae in (1), it is not too difficult to check that $\prod_{d\in\Sigma_{a_1,a_2,a_3}} \Phi_d(1-\lambda)$ always divides into $P(\theta_{a_1,a_2,a_3}, \lambda)$, where $\Phi_d(X)$ denotes the d-th cyclotomic polynomial.

Conjecture 4.1. *Assume the theta graph θ_{a_1,a_2,a_3} is neither $\theta_{2,3,3}$ nor $\theta_{2,3,5}$. Then*

(i) *In the polynomial factorisation*

$$P(\theta_{a_1,a_2,a_3}, \lambda) \;=\; (-1)^{a_1+a_2+a_3-1}(\lambda - 1) \times \prod_{d\in\Sigma_{a_1,a_2,a_3}} \Phi_d(1-\lambda) \times \mathcal{F}_{a_1,a_2,a_3}(\lambda) \quad \textit{say,}$$

the right-most term $\mathcal{F}_{a_1,a_2,a_3}(X)$ is a monic irreducible inside $\mathbb{Z}[X]$;

(ii) *The interesting factor $\mathcal{F}_{a_1,a_2,a_3}(X)$ has symmetric Galois group, and its splitting field is disjoint from the splitting field of $\prod_{d\in\Sigma_{a_1,a_2,a_3}}\Phi_d(X)$;*

(iii) *The Galois group of the general theta graph θ_{a_1,a_2,a_3} is a direct product*

$$\mathrm{Gal}\big(\mathcal{K}_{P(\theta_{a_1,a_2,a_3},\lambda)}/\mathbb{Q}\big) \;\cong\; \Big(\mathbb{Z}/N_{a_1,a_2,a_3}\mathbb{Z}\Big)^{\times} \times \mathfrak{S}_{a_1+a_2+a_3-2-\sum_{d\in\Sigma}\phi(d)}$$

where the integer $N_{a_1,a_2,a_3} = \mathrm{lcm}_{d\in\Sigma_{a_1,a_2,a_3}}\{d\}$.

For example, if $a_1 = a_2 = a_3 = a$ then the set $\Sigma_{a,a,a} = \{1\}$ and we recover the isomorphism $G \cong \mathfrak{S}_{3a-3}$ in Theorem 1.3(i). Likewise choosing $(a_1, a_2, a_3) = (a, a+1, a+2)$ instead, Theorem 1.1 is then an immediate consequence of 4.1(iii) above.

The reasoning behind this conjecture is as follows. Firstly, one predicts after factorising out the maximal cyclotomic piece $\prod_{d\in\Sigma_{a_1,a_2,a_3}}\Phi_d(1-\lambda)$ from $P(\theta_{a_1,a_2,a_3},\lambda)$, the remaining polynomial $\mathcal{F}_{a_1,a_2,a_3}(\lambda)$ forms part of an algebraically uniform family of polynomials, and the family is generically irreducible with symmetric Galois group. Moreover the discriminant of $\mathcal{F}_{a_1,a_2,a_3}$ should be coprime to $\phi(N_{a_1,a_2,a_3})$; the latter would imply that the field $\mathcal{K}_{\mathcal{F}_{a_1,a_2,a_3}}$ intersects trivially with the splitting field of the cyclotomic terms.

Certainly the first two assertions in the above conjecture imply the third, in which case one would expect a direct product decomposition between the cyclotomic Galois group and that of the interesting factor. One major obstruction to proving this conjecture is that we don't yet have a general method which can establish irreducibility over $\mathbb{Q}$ for polynomials of the form $\mathcal{F}_{a_1,a_2,a_3}(\lambda)$ at every triple (a_1, a_2, a_3). Another impediment is the lack an explicit formula expressing $\mathrm{disc}(\mathcal{F}_{a_1,a_2,a_3})$ as a function in a_1, a_2 and a_3.

Remarks: (a) We exclude the exceptional graphs $\theta_{2,3,3}$ and $\theta_{2,3,5}$ as the values of (a_1, a_2, a_3) are so small that the generic behaviour of $\mathcal{F}_{a_1,a_2,a_3}$ is not preserved at these specialisations. The chromatic polynomial of $\theta_{2,3,3}$ is equal to $\lambda(\lambda-1)(\lambda-2)(\lambda^4 - 5\lambda^3 + 11\lambda^2 - 13\lambda + 7)$, and the Galois group of the quartic factor is isomorphic to the dihedral group of order 8. Secondly $P(\theta_{2,3,5}, \lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^6 - 7\lambda^5 + 22\lambda^4 - 40\lambda^3 + 45\lambda^2 - 31\lambda + 11)$ where the sextic has Galois group isomorphic to $\mathfrak{S}_2 \wr \mathfrak{S}_3$ which is of order 48.

(b) There are 171,700 non-isomorphic theta graphs with path lengths in the range $[2, 101]$; MAGMA [6] confirmed that $P(\theta_{a_1,a_2,a_3}; \lambda)$ is always divisible by $(\lambda-1) \times \prod_{d\in\Sigma_{a_1,a_2,a_3}}\Phi_d(1-\lambda)$, and the quotient polynomial $\mathcal{F}_{a_1,a_2,a_3}(\lambda)$ is irreducible for all such $a_1, a_2, a_3 \in [2, 101]$.

(c) We also computed numerically the Galois group of the interesting factor $\mathcal{F}_{a_1,a_2,a_3}(\lambda)$ where $a_1, a_2, a_3 \in [2, 101]$, under the restriction that its degree was < 80, and found in all cases (except $\theta_{2,3,3}$ and $\theta_{2,3,5}$) that it was the full symmetric group as Conjecture 4.1 predicts. If the degree of $\mathcal{F}_{a_1,a_2,a_3}$ is more than about 80, then the MAGMA [6] algorithm takes too long a time to run.

In general, it is not possible to peel cycle graphs off a given graph by simple applications of the addition/contraction relation, as we did for theta graphs. In fact the data for graphs of small order n that have no separating clique [13] supports a conjecture that the chromatic polynomial of such graphs has a single non-linear irreducible factor which almost always has the Galois group $\mathfrak{S}_{n-\kappa}$, where κ is the chromatic number of the graph.

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