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Independent $[1, k]$-sets in graphs

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Abstract

A subset $S \subseteq V$ in a graph $G = (V, E)$ is a $[1, k]$-set for a positive integer k if for every vertex $v \in V \setminus S$, $1 \leq |N(v) \cap S| \leq k$, that is, every vertex $v \in V \setminus S$ is adjacent to at least one but not more than k vertices in S. We consider $[1, k]$-sets that are also independent, and note that not every graph has an independent $[1, k]$-set. For graphs having an independent $[1, k]$-set, we define the lower and upper $[1, k]$-independence numbers and determine upper bounds for these values. In addition, the trees having an independent $[1, k]$-set are characterized. Also, we show that the related decision problem is NP-complete.

1 Introduction

Let $G = (V, E)$ be a graph of *order* $n = |V|$ and *size* $m = |E|$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \mid uv \in E\}$ of vertices adjacent to v. Each vertex in $u \in N(v)$ is called a *neighbor* of v. The *degree* of a vertex v is $d(v) = |N(v)|$. The minimum and maximum degrees of a vertex in a graph G are denoted $\delta(G)$ and $\Delta(G)$, respectively. For a set S and a vertex v, we denote the number of neighbors of v in S as $d_S(v)$, that is, $d_S(v) = |N(v) \cap S|$. The *closed neighborhood* of a vertex $v \in V$ is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V$ of vertices is $N(S) = \bigcup_{v \in S} N(v)$, while the *closed neighborhood of a set* S is the set $N[S] = \bigcup_{v \in S} N[v]$.

A set S is *independent* if no two vertices in S are adjacent. The *vertex independence number* $\alpha(G)$ equals the maximum cardinality of an independent set in G. A set S is a *dominating set* of a graph G if $N[S] = V$, that is, for every $v \in V$, either $v \in S$ or $v \in N(u)$ for some vertex $u \in S$. A dominating set that is independent is an *independent dominating set*, and the minimum cardinality of an independent dominating set of G is the *independent domination number* of G, denoted $i(G)$. Since any maximal independent set is a dominating set, the independent domination number is also known as the lower independence number. For more on independent domination, we refer the reader to the excellent survey by Goddard and Henning [8]. For additional details on domination and terminology not defined here, the reader is referred to the book [11].

In [6], Chellali et al. define a subset $S \subseteq V$ in a graph $G = (V, E)$ to be a *$[j, k]$-set* if for every vertex $v \in V \setminus S$, $j \leq |N(v) \cap S| \leq k$, that is, every vertex in $V \setminus S$ is adjacent to at least j vertices, but not more than k vertices in S. For $j = 1$, a $[1, k]$-set S is a dominating set, since every vertex in $V \setminus S$ has at least one neighbor in S (is dominated by S), but every vertex in $V \setminus S$ has at most k neighbors in S. It was

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noted in [6] that $[j,k]$-sets are related to several other concepts in domination theory, including perfect domination, efficient domination, nearly perfect sets, 2-packings, and k-dependent sets. In [6], they determined bounds on the minimum cardinality of a $[1,2]$-set and investigated extremal graphs achieving these bounds. Using a result for $[1,3]$-sets, they also showed that grid graphs have equal domination and restrained domination numbers.

In this paper, we continue the study of $[j,k]$-sets and add the additional requirement that the set be independent. A $[j,k]$-set that is also independent is called an *independent* $[j,k]$*-set*. A dominating set S is an independent $[1,k]$-set of G if S is independent and $d_S(v) \leq k$ for all $v \in V \setminus S$. We note that an independent $[1,1]$-set S is an efficient dominating set, that is, for every vertex $v \in V$, $|N[v] \cap S| = 1$. Efficient dominating sets were introduced by Biggs in [5] in the context of error correcting codes, and later by Bange, Barkauskas and Slater in [1] in the context of graph theory. Efficient domination in graphs has received much interest, see [2, 3, 4, 9, 10, 12, 14], for example. The study of independent [1,1]-sets is closely related to the study of single error-correcting codes. Our generalization to independent $[1,k]$-sets is related as well. For example, an independent [1,2]-set would permit most words to be corrected, while those that cannot be corrected come from one of only two possible code words.

We make some useful observations.

Observation 1 *Every independent* $[1,k]$*-set* S *of* G *is minimal since* $S \setminus S'$ *is not dominating for any non-empty subset* $S' \subseteq S$*, and maximal since* $S \cup U$ *is not independent for any non-empty subset* $U \in V \setminus S$*.*

Observation 2 *Every independent* $[1,k]$*-set is an independent* $[1,k']$*-set for each* $k' > k$*.*

A graph G may not have an independent $[1,k]$-set for some positive integer k. It is known, for example, that not every graph has an efficient dominating set. However, every graph G with $\Delta(G) = \Delta$ has an independent $[1,\Delta]$-set, since any independent dominating set is such a set. We define $\varphi(G)$ as the minimum positive integer k such that G admits an independent $[1,k]$-set. By Observation 2, G has an independent $[1,k]$-set for every $k \geq \varphi(G)$. For all $k \geq \varphi(G)$, we define the lower and upper $[1,k]$-independence numbers as follows. Let $i_{[1,k]}(G)$ equal the minimum cardinality of an independent $[1,k]$-set of G and $\alpha_{[1,k]}(G)$ the maximum cardinality of an independent $[1,k]$-set of G. The following observation is immediate.

Observation 3 *For any graph* G*,* $i(G) = i_{[1,\Delta]}(G) \leq i_{[1,\Delta-1]}(G) \leq ... \leq i_{[1,\varphi]}(G) \leq \alpha_{[1,\varphi]}(G) \leq ... \leq \alpha_{[1,\Delta-1]}(G) \leq \alpha_{[1,\Delta]}(G) = \alpha(G)$*.*

2 Independent $[1, k]$-sets

As previously mentioned, not all graphs have independent $[1, k]$-sets for a given positive integer k. For example, the complete bipartite graph $K_{3,3}$ does not have an independent $[1, 2]$-set. A *doublestar*, denoted $S_{r,s}$, is a tree that has exactly two nonleaf vertices, one of which is adjacent to $r \geq 1$ leaves and the other is adjacent to $s \geq 1$ leaves. The doublestar $S_{2,3}$ with order $n = 7$ is an example of a tree that has no independent $[1, 2]$-set. It can be seen that each of the two graphs $K_{3,3}$ and $S_{2,3}$ has an independent $[1, 3]$-set, but does not have an independent $[1, 2]$-set. This raises the following general problem:

Problem Characterize the graphs G with independent $[1, k]$-sets for a given positive integer k.

Another problem is to determine the smallest order of graphs that do not have an independent $[1, k]$-set. Our next result solves this problem and shows, in fact, that our example $K_{3,3}$ has the minimum order of a graph with no independent $[1, 2]$-set and the doublestar $S_{2,3}$ has the minimum for a tree with no independent $[1, 2]$-set.

Theorem 4 *(i) For every $k \geq 1$, every graph of order $n \leq 2k+1$ has an independent $[1, k]$-set, and there exists a connected graph of order $n = 2k+2$ with no independent $[1, k]$-set.*

(ii) For every $k \geq 1$, every tree of order $n = 2k + 2$ has an independent $[1, k]$-set, and there exists a tree of order $n = 2k + 3$ with no independent $[1, k]$-set.

Proof. Let $x \in V$ be a vertex of degree $\Delta(G) = \Delta$, let $A = V \setminus N[x]$, and let I be an independent dominating set of the subgraph $G[A]$ induced by A. The set $S = I \cup \{x\}$ is an independent dominating set of G of cardinality $|S| = |I| + 1 \leq |A| + 1 = n - \Delta$. If $n - \Delta \leq k$, then S is an independent $[1, k]$-set of G. Now assume that $n - \Delta > k$.

(i) If $n \leq 2k + 1$, then $k + 1 \leq n - \Delta \leq 2k - \Delta + 1$, and thus, $k \geq \Delta$. Hence, G has an independent $[1, k]$-set.

Let $G = K_{k+1,k+1}$ of order $n = 2k + 2$. The only independent dominating sets of G are the two partite sets A and B. Since each vertex of G has $k + 1$ neighbors in either A or B, A and B are not independent $[1, k]$-sets and G has no independent $[1, k]$-set.

(ii) Now let G be a tree T of order $n = 2k + 2$. Then, as in (i), $k + 1 \leq n - \Delta = 2k + 2 - \Delta$. Thus, $k \geq \Delta - 1$. If $k \geq \Delta$, then every maximal independent set in T is an independent $[1, k]$-set. Assume $k = \Delta - 1$, and thus, $n = 2\Delta$, and $|A| = k$. If A is not independent, then $|I| < |A|$, $|S| \leq |A|$, and S is an independent $[1, k]$-set. If A is independent, then each vertex of A is adjacent to exactly one vertex of $N(x)$. Since $|A| = k < \Delta = |N(x)|$, the set $B = N(x) \setminus N(A)$ is independent and satisfies $0 < |B| \leq \Delta - 1 = k$. Therefore, $A \cup B$ is an independent $[1, k]$-set of T.

Consider the doublestar $S_{k,k+1}$, where a and b are the non-leaf vertices. Let A be the set of leaves adjacent to a and B be the set of leaves adjacent to B. The independent dominating sets of $S_{k,k+1}$ are $B \cup \{a\}$, $A \cup \{b\}$, and $A \cup B$. None of these is a $[1, k]$-set. Hence, $S_{k,k+1}$ has order $2k + 3$ and no independent $[1, k]$-set. □

Corollary 5 *The smallest orders of a connected graph and of a tree having no independent* $[1, k]$*-sets are, respectively,* $2k + 2$ *and* $2k + 3$.

The next result shows that there is no forbidden subgraph characterization of graphs with independent $[1, k]$-sets.

Proposition 6 *Every graph* G *is an induced subgraph of some graph having an independent* $[1, 1]$*-set, and thus an independent* $[1, k]$*-set for* $k \geq 1$.

Proof. Let G be an arbitrary graph, and let G' be the graph obtained from G by adding a new vertex, say y, and joining y to every vertex of G. Clearly, G is an induced subgraph and $\{y\}$ is an independent $[1, 1]$-set of G'. By Observation 2, $\{y\}$ is an independent $[1, k]$-set of G' for all $k \geq 1$. □

The *corona* of graphs G and H, denoted $G \circ H$, is the graph formed from one copy of G and $|V(G)|$ copies of H, where the i^{th} vertex in $V(G)$ is adjacent to every vertex in the i^{th} copy of H. Next we characterize the coronas $G \circ H$ having an independent $[1, k]$-set.

Proposition 7 *Let* k *be a positive integer and* $G \circ H$ *be the corona of graphs* G *and* H*. Then* $G \circ H$ *has an independent* $[1, k]$*-set if and only if each component of* G *is an isolated vertex or* $i(H) \leq k$.

Proof. Assume that the corona of graphs G and H has an independent $[1, k]$-set D. If $G = \overline{K}_n$, then $V(G)$ is an independent $[1, 1]$-set of $G \circ H$. Thus, suppose that some component of G, say G', is non-trivial. It follows that there exists at least one vertex of G' that does not belong to D. Let x be any vertex of G' such that $x \notin D$. Let H_x the copy of H for which every vertex is adjacent to x and $D_x = D \cap V(H_x)$. Since D is an independent $[1, k]$-set and $x \notin D$, every vertex of $V(H_x)$ is either in D_x or has at most k neighbors in D_x. Hence, D_x is a maximal independent set of H_x. Moreover, since $x \notin D$, $|D_x| \leq k$, for otherwise, x would have more than k neighbors in D_x, which contradicts the fact that D is an independent $[1, k]$-set of $G \circ H$. Consequently, $i(H) \leq k$.

Conversely, if each component of G is trivial, then clearly, $V(G)$ dominates H and is an independent $[1, k]$-set of $G \circ H$. Now if $i(H) \leq k$, then by considering any maximal independent set of size at most k from each copy of H, we form an independent $[1, k]$-set for of $G \circ H$. □

Graphs that are $K_{1,k+1}$-free provide another example of graphs with independent $[1,k]$-sets.

Theorem 8 *If G is a $K_{1,k+1}$-free graph, then every independent dominating set is an independent $[1,k]$-set.*

Proof. Let S be any independent dominating set of G. Since G is $K_{1,k+1}$-free, every vertex in $V \setminus S$ has at most k neighbors in S. Hence, S is an independent $[1,k]$-set. □

Corollary 9 *If G is a $K_{1,k+1}$-free graph, then $i_{[1,k]}(G) = i(G)$ and $\alpha_{[1,k]}(G) = \alpha(G)$.*

Even for $K_{1,4}$-free graphs G, we note that the difference between $i_{[1,k]}(G)$ and $\alpha_{[1,k]}(G)$ can be arbitrarily large. To see this consider the tree T_{2t} formed by a path P_{2t} on $2t$ vertices, where for each vertex v of P_{2t}, two new vertices v' and v'' are added with edges vv' and $v'v''$. It is easy to see that for any positive integer $k \geq 2$, $i_{[1,k]}(G) = i_{[1,2]}(G) = 2t$ and $\alpha_{[1,k]}(G) = \alpha_{[1,2]}(G) = 3t$.

3 Trees with Independent $[1,k]$-sets

Bange et al. [1] constructively characterize the trees having an independent $[1,1]$-set (i.e., an efficient dominating set) and give a linear time algorithm to find such a set. In [10], Grinstead and Slater present a recurrence template that gives linear time algorithms to determine several domination related parameters, including efficient dominating sets, for (generalized) series-parallel graphs.

In this section, we give a constructive characterization of the trees having an independent $[1,k]$-set, for $k \geq 2$. Since any non-trivial tree T is a bipartite graph, it has a unique bipartition (X,Y,E). For $k \geq 2$, we say that T is a *pk-tree* if every vertex in one of the partite sets has degree at most k. We call such a partite set, a *pk*-set. If T is a tree with unique bipartition (X,Y,E) and X is a *pk*-set of T, then Y is an independent $[1,k]$-set of T. First we characterize the *pk*-trees.

A *star* is either the trivial graph or the complete bipartite graph $K_{1,t}$ for $t \geq 1$. For the purposes of our discussion, we abuse notation slightly to say that every star has exactly one vertex as its center. In other words, although the star $K_{1,1}$ is self-centered, we designate exactly one of the two central vertices as its center. We define the family of trees $\mathcal{T}_k$ to include all trees T that can be constructed from a forest F of $r \geq 1$ stars as follows: Let S be the set of centers of the stars of F (note that each star contributes exactly one vertex to S, and if a star is the trivial graph, then this vertex is in S). Add a set S' of $q \leq r-1$ new vertices and $r+q-1$ new edges (each between a vertex in S and a vertex in S'), such that T is a tree and every new vertex is adjacent to at least two, but no more than k, vertices in S. We note that T is a tree and the sum of the degrees of the new vertices satisfies the condition $\sum_{i=2}^{k} ia_i = r + q - 1$, where a_i is the number of new vertices of degree i.

Lemma 10 *For any integer $k \geq 2$, a non-trivial tree T is a pk-tree if and only if $T \in \mathcal{T}_k$.*

Proof. Let $T \in \mathcal{T}_k$ and F be the underlying forest of T in the construction of T, where S is the center of the stars of F. Then each of S and $V(T) \setminus S$ is an independent set, and so $\{S, V(T) \setminus S\}$ is the unique bipartition of T. Moreover, by the construction, every vertex in $V(T) \setminus S$ has at most k neighbors in S, and so S is an independent $[1, k]$-set. Thus, T is a *pk*-tree.

Assume that T is a *pk*-tree, and let $\{X, Y\}$ be the unique bipartition of T. Then, without loss of generality, X is a *pk*-set, that is, every vertex in X has at most k neighbors in Y. Since T is a non-trivial tree, every vertex in X has at least one neighbor in Y, implying that Y is an independent $[1, k]$-set. If T is a star, then $T \in \mathcal{T}_k$. Hence, we may assume that T is not a star. Since T is connected, every vertex in Y is adjacent to at least one vertex of degree at least 2. Consider the forest F formed from T by removing the vertices of X of degree at least 2. Since the remaining vertices in X now have exactly one neighbor in Y, it follows that F is a collection of stars with centers in Y. Since T is a tree, there are q ($q \leq |Y| - 1$) vertices of degree at least 2 and at most k in X, and there are $q + |Y| - 1$ edges between these q vertices and the vertices of Y. Hence, $T \in \mathcal{T}_k$. □

We next define a family $\mathcal{F}_k$ of trees T such that T is the trivial graph or T can be constructed as follows. Begin with a forest of non-trivial *pk*-trees T_i, for $1 \leq i \leq t$, where the unique bipartition of T_i is $\{X_i, Y_i\}$ and X_i is a *pk*-set of T_i. Add $t - 1$ edges where each edge joins vertices in two different sets X_i and X_j such that T is connected.

Theorem 11 *For any integer $k \geq 2$, a tree T has an independent $[1, k]$-set if and only if $T \in \mathcal{F}_k$.*

Proof. Assume that $T \in \mathcal{F}_k$. If T is a *pk*-tree, then Lemma 10 implies that T has an independent $[1, k]$-set. Suppose that T is not a *pk*-tree. Then T is constructed from a forest of *pk*-trees T_i, for $1 \leq i \leq t$, where the unique bipartition of T_i is $\{X_i, Y_i\}$ and X_i is a *pk*-set of T_i, by adding $t - 1$ edges where each edge joins vertices in two different sets X_i and X_j such that T is connected. Let $X = \bigcup_i^t X_i$ and $Y = \bigcup_i^t Y_i$. Then Y is an independent set, and for every vertex $x \in X$, $1 \leq d_Y(x) \leq k$. Thus, Y is an independent $[1, k]$-set of T, as desired.

Assume that T has an independent $[1, k]$-set, say S. If T is trivial, then $T \in \mathcal{F}_k$. Hence, assume that $V(T) \setminus S \neq \emptyset$. If $V(T) \setminus S$ is independent, then T is a non-trivial

pk-tree, and so $T \in \mathcal{F}_k$. Thus, we may assume that $V(T) \setminus S$ contains at least one edge, and let E' be the set of edges in the subgraph induced by $V(T) \setminus S$. Then removing E' from T produces a forest with $|E'|+1$ non-trivial components. Consider a component, say T_i, of F. Let $X_i = V(T_i) \cap (V(T) \setminus S)$ and $Y_i = V(T_i) \cap S$. Now Y_i is an independent $[1,k]$-set of T_i. Moreover, X_i is independent, implying that $\{X_i, Y_i\}$ is the unique bipartition of T_i. Hence, T_i is a non-trivial pk-tree. Since T_i is an arbitrary component of F and all the edges removed from T to form F were between vertices in $V(T) \setminus S$, by construction, $T \in \mathcal{F}_k$. □

4 Bounds

In this section, we determine upper bounds on the $[1,k]$-independence numbers for graphs having independent $[1,k]$-sets.

We define families $\mathcal{L}'$ and $\mathcal{L}$ as follows: The graphs $G' \in \mathcal{L}'$ are bipartite with partite sets S and $V \setminus S$, where $|S| = kq/\delta$ and $|V \setminus S| = q$, q is any positive integer such that δ divides kq. Moreover, $d_{G'}(v) = \delta$ for all $v \in S$ and $d_{G'}(v) = k$ for all $v \in V \setminus S$.

A graph G belongs to $\mathcal{L}$ if it is obtained from a graph G' of $\mathcal{L}'$ by possibly adding edges between vertices of $V \setminus S$ in such a way that $d_G(v) \geq \delta$ for all $v \in V \setminus S$. A graph $G \in \mathcal{L}$ has minimum degree $\delta(G) = \delta$, and S is an independent $[1,k]$-set of G with cardinality $\frac{k|V\setminus S|}{\delta} = \frac{k(n-|S|)}{\delta}$. Hence, $\alpha_{[1,k]}(G) \geq |S| = \frac{kn}{\delta+k}$.

Theorem 12 *Let G be a graph of order n with no isolated vertices. If $k \geq \varphi(G)$, then $i_{[1,k]}(G) \leq \alpha_{[1,k]}(G) \leq \frac{kn}{\delta+k}$. Equality, $\alpha_{[1,k]}(G) = \frac{kn}{\delta+k}$, is achieved if and only if $G \in \mathcal{L}$.*

Proof. If $k \geq \varphi(G)$, then G has an independent $[1,k]$-set, say S. Let t denote the number of edges joining the vertices of S to the vertices of $V \setminus S$. Since there are no isolated vertices in G, every vertex of S has at least $\delta(G) \geq 1$ neighbors in $V \setminus S$, and so $t \geq \delta(G)\,|S|$. Moreover, since every vertex of $V \setminus S$ has at most k neighbors in S, we have $t \leq k\,|V \setminus S|$. Therefore, $\delta(G)\,|S| \leq t \leq k\,|V \setminus S|$, and so, $|S| \leq \frac{k|V|}{\delta+k}$. Hence, every independent $[1,k]$-set of G has cardinality at most $\frac{k|V|}{\delta+k}$, and the bound follows.

If $\alpha_{[1,k]}(G) = \frac{kn}{\delta+k}$, then $d_G(v) = \delta$ for all $v \in S$ and $d_G(v) = k$ for all $v \in V \setminus S$. Hence, $G \in \mathcal{L}$. Conversely, if $G \in \mathcal{L}$, then $\frac{kn}{\delta+k} \leq \alpha_{[1,k]}(G) \leq \frac{kn}{\delta+k}$. □

Figure 1 is an example of a graph of $\mathcal{L}$, where $V \setminus S$ is a clique. In this case, $S = \{x_i \,|\, 1 \leq i \leq 6\}$ is the unique independent $[1,k]$-set of G and $i_{[1,k]}(G) = \alpha_{[1,k]}(G) = \frac{kn}{\delta+k} = 6$. In the special case of $\delta(G) = 1$, more can be said about the graphs achieving the equality in the upper bound of Theorem 12.

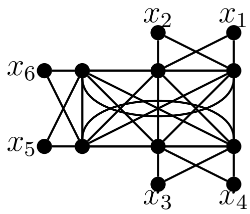

Figure 1: A graph in $\mathcal{L}$.

Theorem 13 *Let G be a graph of order n with minimum degree $\delta(G) = 1$. If G has an independent $[1, k]$-set, then the following hold:*

(a) $\alpha_{[1,k]}(G) = \frac{kn}{k+1}$ if and only if G is the corona $H \circ \overline{K}_k$, where H is any graph.

(b) If $\alpha_{[1,k]}(G) = \frac{kn}{k+1}$ and no component of G is a star $K_{1,k}$, then $i_{[1,k]}(G) = \alpha_{[1,k]}(G)$.

Proof. Part (a) follows from Theorem 12 and the definition of the family $\mathcal{L}$.

(b) If an independent $[1, k]$-set X of G is different from the set S of leaves, let x be a vertex of H in X and $y \in N_H(x)$. Then $y \notin X$ and the k leaves of G attached at y are in X. The vertex y of $V \setminus X$ has at least $k + 1$ neighbors in X, a contradiction. Hence, S is the unique independent $[1, k]$-set of G. □

Note that for the star $K_{1,k}$, $\alpha_{[1,k]}(K_{1,k}) = k$ and $i_{[1,k]}(K_{1,k}) = 1$.

Theorem 14 *Let G be a graph of order n and size m. If G has an independent $[1, k]$-set, then $i_{[1,k]}(G) \leq \alpha_{[1,k]}(G) \leq \frac{2n+2k-1-\sqrt{8m+(2k-1)^2}}{2}$.*

Proof. Let S be a $\alpha_{[1,k]}(G)$-set and t the number of edges joining the vertices of S to the vertices of $V \setminus S$. Clearly, $t \leq k(n - |S|)$. If $\overline{m}$ denotes the number of edges of the complement graph of G, then $\overline{m} \geq \frac{|S|(|S|-1)}{2} + (n - |S|)\,|S| - t \geq \frac{|S|(|S|-1)}{2} + (n - |S|)\,|S| - k(n - |S|)$. Using the fact that $\overline{m} + m = \frac{n(n-1)}{2}$, we obtain

$$\frac{n(n-1)}{2} - m \geq -\frac{|S|^2}{2} + \left(n + \frac{2k-1}{2}\right)|S| - kn$$

which can be written as $|S|^2 - (2n + 2k - 1)|S| + n^2 + (2k - 1)n - 2m \geq 0$.

The equation $|S|^2 - (2n + 2k - 1)|S| + n^2 + (2k - 1)n - 2m = 0$ has two positive roots and n is between them. Hence

$$\begin{aligned} |S| &\leq \frac{1}{2}(2n + 2k - 1 - \sqrt{(2n + 2k - 1)^2 - 4(n^2 + (2k - 1)n - 2m)}) \\ &\leq \frac{1}{2}(2n + 2k - 1 - \sqrt{8m + (2k - 1)^2}). \end{aligned}$$

Equality is attained if and only if G is obtained from a split graph with S independent, $V \setminus S$ complete, and $d_S(v) = k$ for all $v \in V \setminus S$. □

We note that the bound in Theorem 14 is better than the bound of Theorem 12 for all graphs G with large size and small minimum degree.

5 Complexity

As previously noted, not all graphs have independent $[1, k]$-sets. This leads to the following decision problem:

> INDEPENDENT $[1, k]$-SET
> INSTANCE: Graph $G = (V, E)$, positive integers k and $t \leq |V|$.
> QUESTION: Does G have an independent $[1, k]$-set of cardinality at most t?

We note that for $k = 1$, independent $[1, k]$-sets coincide with efficient dominating sets for which the decision problem is NP-complete even for planar graphs with maximum degree three (see [7]). We also note that for $k \geq \Delta$, we have $i_{[1,k]}(G) = i(G)$, and it is well-known that determining the number $i(G)$ for an arbitrary graph is NP-complete.

We will show that INDEPENDENT $[1, k]$-SET is NP-complete for $k \geq 2$ by giving a transformation from the known NP-complete problem (see [13]), NOT-ALL-EQUAL p-SAT (NEpSAT).

> NEpSAT
> INSTANCE: A set $U = \{u_1, u_2, \ldots, u_n\}$ of variables and a set $C = \{C_1, C_2, \ldots, C_m\}$ of p-element subsets, $p \geq 3$, called *clauses*, where each clause C_i contains p distinct occurrences of either a variable u_i or its complement $\bar{u}_i$.
> QUESTION: Does C have a satisfying truth assignment, such that at least one variable, but not all p, in each clause is assigned the value True?

Theorem 15 *INDEPENDENT* $[1, k]$*-SET is NP-complete for each* $k \geq 1$.

Proof. Clearly, INDEPENDENT $[1, k]$-SET is in the class $\mathcal{P}$, since it is easy to verify a 'yes' instance of INDEPENDENT $[1, k]$-SET in polynomial time.

Given an instance C of NEpSAT, where $p = k + 1 \geq 3$, we construct an instance $G(C)$ of INDEPENDENT $[1, k]$-SET as follows. For each variable u_i, construct a triangle with vertices labelled $u_i, \bar{u}_i, v_i$. For each clause C_j create a single vertex C_j, and add the p edges u_iC_j for $u_i \in C_j$.

We must show that C has a NEpSAT assignment if and only if the graph $G(C)$ has an independent $[1, k]$-set of cardinality $t \leq n$.

If C has a NEpSAT assignment, then construct a set S of vertices in $G(C)$ as follows: if a variable u_i is assigned the value True, then place $u_i \in S$. It is easy to see that the set S has cardinality n, and is a dominating set, since each triangle contains at least one vertex in S, every triangle vertex is adjacent to a vertex in S, and since every clause contains a vertex that is assigned the value True, each clause vertex is also adjacent to a vertex in S. It remains to show that the set S is an independent $[1, k]$-set. But each triangle vertex is adjacent to exactly one vertex in S, and each clause vertex C_j is adjacent to at most $p - 1 = k$ vertices in S, since the vertices in S are determined by a NEpSAT assignment, which means that no clause vertex is adjacent to p vertices in S.

Conversely, we must show that if $G(C)$ has an independent $[1, k]$-set of cardinality $t \leq n$, then C has a NEpSAT assignment. Notice first that if S is an independent $[1, k]$-set, then each triangle must contain exactly one vertex in S; it cannot contain two vertices in S, else S is not an independent set, and it must contain at least one vertex in S, since the vertices v_i must be either in S or adjacent to a vertex in S. But if $t \leq n$, then it must be the case that, in fact, $t = n$. It only remains to construct a NEpSAT assignment from S. For each vertex $u_i \in S$, assign the value True to variable u_i, otherwise assign the value False to u_i. Since every vertex C_j is adjacent to at least one vertex in S, this produces a truth assignment for C. We must show that this truth assignment is a NEpSAT assignment. This follows from the fact that S is an independent $[1, k]$-set, which means that no vertex, and in particular, no clause vertex is adjacent to $k + 1 = p$ vertices in S. Therefore, at least one variable in each clause corresponds to a vertex not in S. $\square$

6 Open problems

Problem 1 *Characterize the graphs G having independent $[1, 2]$-sets.*

Problem 2 *Determine necessary conditions for a graph to have an independent $[1, k]$-set.*

Problem 3 *Determine lower bounds on $i_{[1,k]}(G)$ and $\alpha_{[1,k]}(G)$ for graphs G.*

Problem 4 *Characterize the trees T with $i_{[1,2]}(T) = \alpha_{[1,2]}(T)$.*

Problem 5 *Which grids $P_m \square P_n$ have independent $[1, 2]$-sets?*

Problem 6 *Which Cartesian products* $P_m \square C_n$ *and* $C_m \square C_n$ *have independent* $[1,2]$*-sets?*

Problem 7 *If a graph* G *has an independent* $[1,k]$*-set, does the Cartesian product* $G \square H$ *have an independent* $[1,k]$*-set?*

Problem 8 *If a tree has an independent* $[1,2]$*-set, can* $i_{[1,2]}(T)$ *and* $\alpha_{[1,2]}(T)$ *be computed in polynomial time?*

Problem 9 *Suppose a graph* G *does not have an independent* $[1,k]$*-set. Define* $\partial\alpha_{[1,k]}(G)$ *to equal the maximum cardinality of a set* S *such that* S *is an independent* $[1,k]$*-set in the subgraph* $G[N[S]]$ *induced by* $N[S]$*. This is a measure of how close the graph* G *comes to having an independent* $[1,k]$*-set, and thus it is an approximation to* $\alpha_{[1,k]}(G)$*. One could also define* $\partial i_{[1,k]}(G)$ *to equal the minimum cardinality of a maximal set* S *which is an independent* $[1,k]$*-set in* $G[N[S]]$*.*

Problem 10 *From a coding theoretic perspective, it would be worthwhile to find an independent* $[1,2]$*-set having a minimum number of vertices dominated twice. What can you say about such sets?*

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