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Feasible sets of small bicolorable STSs

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Abstract

A bicolorable STS(v) is a Steiner triple system whose vertices are colored in such way that every block receives precisely two colors. A k-bicoloring of a STS is a vertex coloring using each of k colors, and the feasible set Ω is a set of integers k for which k-bicolorings exist. In this paper, we study feasible sets of STS(v)s of all orders $v < 50$.

1 Introduction

A *Steiner triple system* STS(v) of *order* v is a pair $(X, \mathcal{B})$, where X is a finite set called vertices, $|X| = v$, and $\mathcal{B}$ is a family of subsets of X, called *blocks*, such that each block contains three vertices, and any two distinct vertices of X appear together in precisely one block. It is well-known that $v \equiv 1$ or $3 (\mathrm{mod}\ 6)$ for every STS(v).

A *coloring* of $(X, \mathcal{B})$ is a surjective mapping ϕ from X onto a finite set C whose elements are called *colors*. If $|C| = k$ we say that ϕ defines a *k-coloring*. For each $c \in C$, the set $\phi^{-1}(c) = \{x : \phi(x) = c\}$ is a *color class*. A coloring ϕ of $(X, \mathcal{B})$ is a *bicoloring* if $|\phi(B)| = 2$ for all $B \in \mathcal{B}$; i.e., B can not be a monochromatic or polychromatic block.

A STS(v) is *k-bicolorable* if it admits at least one k-bicoloring, and it is *unbicolorable* if no k-bicoloring exists for any $k \geq 1$. The first results on bicolorings of STSs and SQSs were obtained in [15] where it was also proved that there exist classes of unbicolorable STSs. More results can be found in [1, 6, 12, 13, 14, 17, 18].

Historically, the concept and terminology of bicoloring is originated from the coloring theory introduced by V. Voloshin in [19, 20, 21].

Let $S = \mathrm{STS}(v) = (X, \mathcal{B})$. An *independent set* of S is a subset of X which does not contain any block of $\mathcal{B}$. The cardinality of the largest independent set of S is denoted by $\beta(S)$. We refer to [4] the reader interested in more results of maximum size of an independent set in an $\mathrm{STS}(v)$. Since in any bicoloring of S no block is monochromatic, each color class is an independent set. It is well-known that there are no 2-bicolorable $\mathrm{STS}(v)$ for $v > 3$, since these systems do not have blocking sets [5].

Given a k-bicoloring C, if the cardinalities of color classes are $n_1, n_2, \ldots, n_k$, then for brevity we write $C = C(n_1, n_2, \ldots, n_k)$. In [11], we can find important necessary conditions for the existence of possible k-bicolorings for general STSs. One of them is given by the equation

$$v(v+2) = 3\sum_{j=1}^{k} n_j^2. \tag{1}$$

For a system $S = \mathrm{STS}(v)$, the set of integer numbers

$$\Omega(S) \;=\; \{k \mid \exists \text{ a } k\text{-bicoloring of } S\}$$

is called the *feasible set* of S. It contains all possible integers k for which k-bicolorings of S exist. We assume that the feasible set of an unbicolorable system S is the empty set. The maximum and minimum elements of $\Omega(S)$ are called the *upper* and *lower chromatic numbers* of S, denoted by $\overline{\chi}(S)$ and $\chi(S)$, respectively. In general, it is difficult to determine the lower and upper chromatic numbers of STSs; and, even if they are known, it is not always true that all the i with $\chi(S) \le i \le \overline{\chi}(S)$ are in $\Omega(S)$. For mixed hypergraph it was discovered in [10] that such particular *gaps* are possible. The problem to find a bicolorable STS with a gap in its feasible set still remains an open problem.

Given any integer v, such that $v \equiv 1$ or $3 \pmod 6$, the set

$$\Omega(v) \;=\; \{k \mid \exists \;\text{ a STS}(v) \text{ which is } k\text{-bicolorable }\}$$

contains all integers k for which there exists a $\mathrm{STS}(v)$ of a particular order v which has a k-bicoloring. We call it a *feasible set* for v. It can also be defined as $\Omega(v) = \bigcup_S \Omega(S)$, where the union is taken over all bicolorable $\mathrm{STS}(v)$s. The maximum and minimum elements of $\Omega(v)$ are, respectively, the upper and lower chromatic numbers for the order v, i.e. $\overline{\chi}(v)$ and $\chi(v)$.

Important results on bicolorings of $\mathrm{STS}(v)$s can be found in [16, 18] where the authors determine the best upper bound for the upper chromatic number and a lower bound for the color class cardinalities in an arbitrary k-bicoloring.

The following theorem summarizes some key results from [15] which are important in the proofs in subsequent sections.

Theorem 1.1 *If S is a STS(v) with $v \leq 2^k - 1$, then $\bar{\chi}(S) \leq k$, and for any h-bicoloring $C = C(n_1, n_2, \ldots, n_h)$ of S the following inequalities hold: $n_1 \geq 2^0$, $n_2 \geq 2^1$, $n_3 \geq 2^2$, $\ldots$, $n_h \geq 2^{h-1}$. In particular, if $\bar{\chi} = k$, then:*

1. *$v = 2^k - 1$;*
2. *in any k-bicoloring of S the color classes have cardinalities*
$$2^0,\ 2^1,\ 2^2,\ \ldots,\ 2^{k-1}.$$
3. *S is obtained from the STS(3) by repeated application of a doubling plus one construction.* □

In the second section of this paper we find technical results useful for determining the feasible sets of STSs. The third section is devoted to the study of feasible sets for STS(v) with $v < 50$. In particular, for these systems, we determine all the possible feasible sets $\Omega(v)$ and $\Omega(S)$, and whether there are unbicolorable systems.

2 Technical results

We will use the well-known recursive construction called the *doubling construction* (other names: $v \to 2v + 1$ rule, doubling plus one construction, etc.) which starts with an STS(v) and ends up with an STS$(2v + 1)$.

To obtain such a construction, all that is needed, apart from the subsystem STS(v), is a 1-factorization of the complete graph K_{v+1}. Indeed, let $(X, \mathcal{F})$ be a 1-factorization of K_{v+1} where $\mathcal{F} = \{F_1, \ldots, F_v\}$, and $|X| = v + 1$ is even. If $(V, \mathcal{B})$, $V = \{a_1, \ldots, a_v\}$, is an STS(v), form the set of triples $\mathcal{C} = \{\{a_i, x, y\} : a_i \in V, \{x, y\} \in F_i\}$; then $(V \cup X, \mathcal{B} \cup \mathcal{C})$ is an STS$(2v + 1)$.

In [15], the authors show that the unique STS(v)s with $\bar{\chi} = k$ and $v \leq 2^k - 1$ are obtained from the STS(3) by repeated application of doubling constructions.

Let $S = (X, \mathcal{B}) =$ STS$(2v + 1)$ be obtained by a doubling construction from $S' = (X', \mathcal{B}') =$ STS(v), which is k-bicolorable with the bicoloring $\mathcal{C}' = \mathcal{C}'(n'_1, n'_2, \ldots, n'_k)$. We say that the system $S = STS(2v + 1)$ has a *k-extended bicoloring* of $\mathcal{C}'$ if there exists a k-bicoloring $\mathcal{C} = \mathcal{C}(n_1, n_2, \ldots, n_k)$ of S such that the subsystem S' is colored by $\mathcal{C}'$. This is equivalent to saying that in the k-bicoloring $\mathcal{C}$ of S the vertices of $X'' = X - X'$ are colored by the colors used in $\mathcal{C}'$.

The values $c_i = n_i - n'_i$, with $1 \leq i \leq k$, are the numbers of vertices in X'' which are colored with the color $i \in \mathcal{C}'$. Notice that $c_j = 0$ is possible for some $1 \leq j \leq k$.

It was proved in [2] that extended k-bicolorings do not exist if $v = 2^k - 1$ and $k < 10$. The case $k \geq 10$ was left as an open problem.

We will use notation and results on *extended bicoloring* found in [7]. For this reason, in the sequel we refer the reader to this cited paper.

In [11] it was proved that every bicoloring has one and only one color class with odd cardinality.

Proposition 2.1 *If $S = (X, \mathcal{B})$ is a bicolorable STS(v), then there exists no bicoloring with two color classes each of cardinality two.*

Proof. Let $\mathcal{C}$ be a bicoloring of an STS(v) with two color classes containing two vertices; i.e. $X_i = \{l_1, l_2\}$ and $X_j = \{m_1, m_2\}$. If we assume that block $\{l_1, l_2, m_1\} \in \mathcal{B}$, then the pair (l_1, m_2) cannot be contained in any block of $\mathcal{B}$. Similarly, if at least one of the two blocks $\{l_1, l_2, x\}$ and $\{m_1, m_2, y\}$ is in $\mathcal{B}$, then at least one of the pair (l_1, m_1) or (l_1, m_2) cannot be in a bicolored block, and the proposition is proved. □

The following proposition gives a necessary condition for the existence of an extended bicoloring.

Proposition 2.2 *Let $S = (X, \mathcal{B}) =$ STS$(2v+1)$ be a system obtained by a doubling construction from $S' = (X', \mathcal{B}') =$ STS(v) which is h-bicolorable with the bicoloring $\mathcal{C}' = \mathcal{C}'(n'_1, n'_2, \ldots, n'_h)$. If $\mathcal{C} = \mathcal{C}(n_1, n_2, \ldots, n_h)$ is an extended h-bicoloring of $\mathcal{C}'$ on S where $n_i = n'_i + c_i$, then the following two equalities hold:*

$$\begin{cases} \sum_{i=1}^{h} c_i^2 + 2\sum_{i=1}^{h} n'_i c_i = (v+1)^2 \\ \\ \sum_{i=1}^{h} c_i = v+1. \end{cases} \tag{2}$$

Proof. It is evident that $\sum_{i=1}^{h} c_i = v+1$, and $n_i = n'_i + c_i$ and for $1 \leq i \leq h$. The equation (1) for the systems S' and S can be written as it follows:

$$3\sum_{i=1}^{h} n_i'^2 \;=\; v(v+2);$$

$$3\sum_{i=1}^{h} (n'_i + c_i)^2 \;=\; (2v+1)(2v+3).$$

Subtracting the first equation from the second one, we obtain:

$$\sum_{i=1}^{h} c_i^2 + 2\sum_{i=1}^{h} n'_i c_i \;=\; (v+1)^2.$$

□

The following proposition permits one to characterize solutions of system (2) which do not give extended bicolorings.

Proposition 2.3 *Let $\mathcal{C}$ be an extended bicoloring of $\mathcal{C}' = \mathcal{C}'(n'_1, n'_2, \ldots, n'_h)$ for the system $S = (X, \mathcal{B}) =$ STS$(2v+1)$, obtained by doubling construction from $S' = (X', \mathcal{B}') =$ STS(v). If in a solution $(c_1, c_2, \ldots, c_h)$ with respect to $\mathcal{C}'$ there are two c_i and c_j, such that $c_i > 0$ and $c_j > 0$, $1 \le i, j \le h$ and $i \ne j$, then $c_i \le n'_i + n'_j$ and $c_j \le n'_i + n'_j$.*

Proof. Let us consider $x' \in X''$ colored with i. It is contained in c_j non monochromatic pairs colored with i and j, all contained in different factors. These factors correspond to vertices $x_l \in X'$ colored only with either i or j, and there are at most $n'_i + n'_j$ of them, so $c_j \le n'_i + n'_j$. Analogously, we obtain that $c_i \le n'_i + n'_j$. □

The proposition above is most useful when in $\mathcal{C}'$ there exists one $n'_i = 1$ and $c_i > 0$, because in this case $c_j \le n'_j + 1$ for every $c_j > 0$ with $1 \le j \le h$ and $i \ne j$. In the general case, if we want to obtain the best evaluation of a solution to system (2) with respect to the conditions of Proposition 2.3, we have to find in $\mathcal{C}'$ the value n'_i with $c_i > 0$, $n'_i \le n'_j$ for every $1 \le j \le h$, such that $c_j > 0$ and $i \ne j$. This particular choice of n'_i permits us to optimize the inequalities $c_j \le n'_j + n'_i$ with $i \ne j$.

Proposition 2.4 *Let $S = (X, \mathcal{B}) =$ STS$(2v+1)$, be a system obtained by a doubling construction from the system $S' = (X', \mathcal{B}') =$ STS(v), which admits the bicoloring $\mathcal{C}' = \mathcal{C}'(n'_1, n'_2, \ldots, n'_h)$. If $(c_1, c_2, \ldots, c_h)$ is a solution to system (2) with respect to $\mathcal{C}'$, with $c_l > 0$ for $1 \le l \le h$, and $c_i = (v+1)/2$, and with $c_j > 0$ such that $(\sum_k n'_k) \cdot \lfloor c_j/2 \rfloor < c_j(c_j - 1)/2$, with $k \ne i$ and j, then the solution $(c_1, c_2, \ldots, c_h)$ does not determine an extended h-bicoloring $\mathcal{C}'$.*

Proof. In X'', the solution $(c_1, c_2, \ldots, c_h)$ defines $c_j(c_j - 1)/2$ monochromatic pairs of color j which are not in the factors corresponding to the vertices $x_l \in X'$ colored with i and j. These pairs have to be in $\sum_k n'_k$ factors, with $k \ne i, j$ and corresponding to the vertices $x_l \in X'$ of color k. In each of these factors there are at most $\lfloor c_j/2 \rfloor$ monochromatic pairs of color j therefore $(\sum_k n'_k) \cdot \lfloor c_j/2 \rfloor \ge c_j(c_j - 1)/2$. □

The following theorem gives a sufficient condition for the existence of extended h-bicolorings of STS$(2v+1)$ when there is an h-bicoloring $\mathcal{C}'$ of system STS(v), and it allows us to find extended bicolorings without solving the system (2).

Theorem 2.1 *Let $S' = (X', \mathcal{B}') =$ STS(v), be a system h-bicolorable with $\mathcal{C}' = \mathcal{C}'(n'_1, n'_2, \ldots, n'_h)$. If there exist p integers n'_{k_i}, with $1 \le i \le p$ and $p < h$, where $n'_{k_1} + n'_{k_2} = (v+1)/2^{p-1}$ is an even integer, and $n'_{k_i} = (v+1)/2^{p-i+1}$, for $3 \le i \le p$, are all even, then $S = (X, \mathcal{B}) =$ STS$(2v+1)$, obtained by doubling construction from S', has an extended h-bicoloring of $\mathcal{C}'$.*

Proof. Set $c_{k_1} = (v+1)/2^{p-1}$, $c_{k_2} = (v+1)/2^{p-1}$, $c_{k_i} = (v+1)/2^{p-i+1}$ for $3 \le i \le p$, and $c_j = 0$ for all $j \ne k_i$ with $1 \le i \le p$. It is necessary that these assignments give correct numbers of monochromatic and non monochromatic pairs in a factorization

$\mathcal{F}$ of X'' which defines a bicoloring of STS$(2v+1)$, i.e., it is necessary to check whether $(c_1, c_2, \ldots, c_h)$ is a solution with respect to $\mathcal{C}'$ of the system

$$\begin{cases} \sum_{i=1}^{p} c_{k_i}^2 + 2\sum_{i=1}^{p} n'_{k_i} c_{k_i} &= (v+1)^2 \\ \sum_{i=1}^{p} c_{k_i} &= (v+1). \end{cases}$$

Replacing the values of c_{k_i} and n'_{k_i}, for $1 \leq i \leq p$, the first equality becomes

$$\sum_{i=1}^{2p-2} \frac{(v+1)^2}{2^i} + \frac{(v+1)^2}{2^{2p-2}} = (v+1)^2,$$

and it is trivially true.

Setting $s = \sum_{i=2}^{p}(v+1)/2^{p-i+1} + (v+1)/2^{p-1}$, it is simple to verify that $s = v+1$, so the second equality is also true.

In this second part of the proof we construct a factorization $\mathcal{F}$ specifying how we distribute both the vertices in X'' and the colors k_i, with $1 \leq i \leq p$, in such a way that the rules of an extended bicoloring of $\mathcal{C}'$ are respected.

Let us color exactly $(v+1)/2$ vertices by color k_p and the other $(v+1)/2$ vertices by colors k_i for $1 \leq i \leq p-1$. Easily, it is possible to build $(v+1)/2$ factors with non monochromatic pairs using for each one of them colors k_p and k_i, with $1 \leq i \leq p-1$. These factors are connected with the $(v+1)/2$ vertices $x_l \in X'$ colored with k_p. The set of $(v+1)/2$ vertices in X'' colored with k_p defines a factorization $\mathcal{F}^{(1)}$ of $(v-1)/2$ factors all containing $(v+1)/2^2$ monochromatic pairs of color k_p. The factors in $\mathcal{F}^{(1)}$ are placed on the bottom of the $(v-1)/2$ factors in $\mathcal{F}$ corresponding to the vertices $x_l \in X'$ colored with all the colors distinct from k_p. Now, let us consider $(v+1)/2^2$ vertices of X'' colored with the color k_{p-1}; all the non monochromatic pairs that use the colors k_{p-1} and k_i, with $1 \leq i \leq p-2$, define $(v+1)/2^2$ factors all containing $(v+1)/2^2$ pairs. These last factors cover completely $(v+1)/2^2$ factors of $\mathcal{F}$ which contain $(v+1)/2^2$ monochromatic pairs of color k_p of $\mathcal{F}^{(1)}$, and they are connected with the vertices $x_l \in X'$ colored with k_{p-1}. The $(v+1)/2^2$ vertices of X'' colored with k_{p-1} define a factorization $\mathcal{F}^{(2)}$ of $(v+1)/2^2-1$ factors. The pairs of these factors are added to the other incomplete factors of $\mathcal{F}$ which contain $(v+1)/2^2$ monochromatic pairs colored with k_p and contained in the factors of $\mathcal{F}^{(1)}$. Therefore in these last factors there are $(v+1)/2^2$ pairs of color k_p and $(v+1)/2^3$ pairs of colork_{p-1}. We repeat this procedure until the $(v+1)/2^{p-2}$ vertices in X'' colored with k_3 define $(v+1)/2^{p-2}$ factors of non monochromatic pair colored with the colors k_3 and k_i, with $i = 1$ or 2. They completely cover $(v+1)/2^{p-2}$ factors of $\mathcal{F}$ corresponding to the vertices $x_l \in X'$ colored with k_3. Also the vertices of X'' colored with k_3 define a factorization $\mathcal{F}^{(p-2)}$ of $(v+1)/2^{p-2}-1$ factors of $(v+1)/2^{p-1}$ monochromatic pairs colored with k_3. The factors in $\mathcal{F}^{(p-2)}$ are posed on the remaining $(v+1)/2^{p-2}-1$ factors of $\mathcal{F}$ which are not complete and containing monochromatic pairs of colors k_i for $3 \leq i \leq p$. Finally the $(v+1)/2^{p-1}$ vertices of color k_1 and the $(v+1)/2^{p-1}$ vertices of colors k_2 define $(v+1)/2^{p-1}$ factors containing non monochromatic pairs

and colored with k_1 and k_2. These factors completely cover all the $(v+1)/2^{p-1}$ factors of $\mathcal{F}$ corresponding to the vertices $x_l \in X'$ colored with k_1 and k_2. The vertices of X'' colored with k_1 and k_2 define respectively two factorizations $\mathcal{F}^{(p-1)}$ and $\mathcal{F}^{(p)}$ of $(v+1)/2^{p-1} - 1$ factors all containing $(v+1)/2^p$ monochromatic pairs of colors k_1 and k_2. These factors cover completely all the remaining $(v+1)/2^{p-1} - 1$ factors of $\mathcal{F}$ corresponding to the vertices $x_l \in X'$ colored with $j \neq k_i$ with $1 \leq i \leq p$.
We obtain a factorization $\mathcal{F}$ of $\sum_{i=1}^{p-1}(v+1)/2^i + (v+1)/2^{p-1} - 1 = v$ factors which gives a correct bicoloring for STS$(2v+1)$ obtained by a doubling construction, and the theorem follows. □

In the previous theorem, the factorization $\mathcal{F}$ is possible to obtain since all the quantities $(v+1)/2^i$ with $1 \leq i \leq p-1$ are even. Therefore, it is always possible to construct the factorizations $\mathcal{F}^{(i)}$, for $1 \leq i \leq p-1$. Notice that the solution with respect to $\mathcal{C}'$ has at least one $c_l = 0$ with $1 \leq l \leq h$, since in $\mathcal{F}$ there are factors with only monochromatic pairs.

It is necessary to emphasize that the theorem above permits us to find a considerable number of STSs with extended bicolorings. For example, for $50 \leq v \leq 200$, from tables on pages 10, 14 and 15 of [3], we immediately see that it is applicable to 48 bicolorings related to STSs of orders $v = 55, 77, 79, 87, 103, 111, 127, 135, 151, 159, 175$, and 199. For all these systems, we obtain feasible sets with $\chi \neq \overline{\chi}$.

3 Feasible sets for STS(v) with $v < 50$

In this section we determine all the possible feasible sets $\Omega(v)$ and $\Omega(S)$, and if there are unbicolorable systems for $v < 50$ as shown in the Table at the end of this section. We refer to [3] the reader interested in all the possible bicolorings for $v < 100$.

The following theorem summaries all the results when $v = 2^k - 1$ and $k = 3$, 4 and 5.

Theorem 3.1 *If S is a STS(v) with $v = 7$, 15 and 31, then $\Omega(7) = \{3\}$, $\Omega(15) = \{4\}$ and $\Omega(31) = \{3, 5\}$. There is no $S =$ STS(31) with the feasible set $\Omega(S) = \{3, 5\}$. There are unbicolorable STS(15)s and STS(31)s.*

Proof. The unique STS(7) can be colored only with a 3-bicoloring, so $\Omega(7) = \{3\}$. In [11], it was proved that the unique bicolorable BSTS(15)s are obtained by a sequence of doubling constructions and they are only 4-bicolorable, so $\Omega(15) = \{4\}$. All the other STS(15)s are unbicolorable.

In [3], it is shown that all 3-bicolorable STS(31)s have type $\mathcal{C}(\forall, \exists, \infty\triangle)$, and that there are no solutions to equation (1) for 4-bicoloring. The 3-bicolorable STS(31)s cannot be 5-bicolorable, since, by Theorem 1.1, any 5-bicoloring has to be of type $\mathcal{C}(1, 2, 4, 8, 16)$; i.e., these systems are obtained by a sequence of doubling constructions, so they are only 5-bicolorable.

According to [4], there are STS(31)s with $\beta = 12$. All these systems are unbicolorable because 5 and 3-bicolorings have a color class of cardinalities 16 and 14, respectively. □

Theorem 3.2 *If S is a STS(v), then:*

1. *for $v = 9$, 13 and 21, the feasible set for v is $\Omega(v) = \{3\}$, and there are unbicolorable systems for $v = 21$ only;*
2. *for $v = 19$, there are three possible feasible sets: $\Omega(S) = \{3\}$, $\Omega(S) = \{4\}$ and $\Omega(S) = \{3, 4\}$. There exist unbicolorable systems.*

Proof. There is a unique STS(9) up to isomorphism. The only solutions of the equation (1) are $(1, 4, 4)$ and $(2, 2, 5)$. The solution $(2, 2, 5)$ is not a bicoloring by proposition 2.1, while $(1, 4, 4)$ is a 3-bicoloring, so $\Omega(9) = \{3\}$.

For $v = 13$, we have two distinct STS(v)s up to isomorphism, and $(2, 5, 6)$ is the unique solution of (1). It is easy to prove that the two systems are 3-bicolorable.

The cases $v = 19, 21$ are discussed in [11]. As it is shown in [4], there are STS(19)s and STS(21)s with $\beta \leq 8$. □

From now on, all the table references refer to tables in [8].

Theorem 3.3 *For order $v = 25$, the feasible set $\Omega(25) = \{3, 4\}$, and there are unbicolorable systems.*

Proof. By Theorem 1.1 for an $S =$ STS(25), we have $\overline{\chi} \leq 4$. The only solutions to (1) are $(2, 4, 6, 13)$, $(1, 4, 8, 12)$ and $(5, 10, 10)$. The solution $(2, 4, 6, 13)$ cannot be a bicoloring because an independent set of order 13 does not exist in a STS(25). The system S_1 in Table 1 in [8] is bicolorable with bicoloring $\mathcal{C}'(1, 4, 8, 12)$. The system S_2 in Table 2 of [8] is bicolorable with bicoloring $\mathcal{C}''(5, 10, 10)$.

There are unbicolorable STS(25)s, since in [4] it was proved that some of them have $\beta < 10$. □

The system S_1 in Table 1 is only 4-bicolorable; in fact numerical analysis shows that this system does not contain two disjoint independent sets of order 10, so $\Omega(S_1) = \{4\}$. The system S_2 in Table 2 is only 3-bicolorable because it does not have independent sets of order 12, and therefore $\Omega(S_2) = \{3\}$. To find out if there exists a system S with $\Omega(S) = \{3, 4\}$ is still an open problem.

Theorem 3.4 *For order $v = 27$, the feasible set $\Omega(27) = \{3, 4\}$. In particular, there are $S =$ STS(27)s with $\Omega(S) = \{3\}$, $\Omega(S) = \{4\}$ and $\Omega(S) = \{3, 4\}$. There are unbicolorable systems.*

Proof. There are no 5-bicolorable STS(27)s since by Theorem 1.1, for a 5-bicoloring, $\sum_{i=1}^{5} n_i \geq \sum_{i=1}^{5} 2^{i-1} > 27$. The only solutions to (1) are $(6, 9, 12)$, $(1, 4, 10, 12)$, and $(2, 5, 6, 14)$. The systems S_3 and S_4 in [8] are, respectively, bicolorable with $\mathcal{C}_1(6, 9, 12)$ and $\mathcal{C}_2(1, 4, 10, 12)$ as it is shown in Tables 3 and 4. The system S_3, by numerical analysis, does not have two disjoint independent sets of order 12 and 10, and it does contain a subsystem STS(13), so it is only 3-bicolorable. The system S_4 has a unique independent set of order 12 which defines for it color class X_4. All independent sets of cardinality 9 disjoint from X_4 are contained in color classes X_3, and since there is a unique vertex in color class X_1, this system is not 3-bicolorable.

The two distinct non isomorphic STS(13)s are only bicolorable with 3-bicoloring $\mathcal{C}(2, 5, 6)$; Table 5 shows an extended bicoloring $C_3(6, 9, 12)$ of the system S_5 obtained by doubling construction from an STS(13). This system is also 4-bicolorable with $\mathcal{C}_4(2, 5, 6, 14)$. Consequently, $\Omega(S_5) = \{3, 4\}$.

In [4], it is shown that there are STS(27)s with $\beta < 12$; all such systems are unbicolorable. □

Theorem 3.5 *For order $v = 33$, we have the feasible set i $\Omega(33) = \{4\}$. There are unbicolorable systems.*

Proof. The solutions of (1) are: $(1, 8, 8, 16)$, $(2, 4, 13, 14,)$, $(2, 5, 10, 16)$ and $(8, 4, 4, 17)$. The first three solutions are bicolorings for STS(33)s [3], while $(8, 4, 4, 17)$ does not define a bicoloring because there are no STS(33)s with $\beta > 16$.

As shown in [4], there are STS(33) with $\beta < 14$; all are unbicolorable. □

Theorem 3.6 *For order $v = 37$, the feasible set is $\Omega(37) = \{3, 4\}$. There are unbicolorable systems.*

Proof. The only solutions of the equation (1) are $(9, 12, 16)$ and $(2, 5, 14, 16)$. The system S_6 in Table 6 of [8] is 3-bicolorable with the bicoloring $\mathcal{C}(9, 12, 16)$, while the system S_7 in Table 7 is 4-bicolorable with the bicoloring $\mathcal{C}(2, 5, 14, 16)$.

In [4] unbicolorable systems were described with $\beta < 15$. □

Numerical analysis shows that the system S_6 is only 3-bicolorable because it does not contain two disjoint independent sets of cardinalities 16 and 14. The system S_7 has a unique independent set of order 16 corresponding to the color class X_4. All independent sets of order 12 are contained in X_3, therefore, because $X_1 \cup X_2$ is not an independent set, S_7 cannot be 3-bicolorable.

The existence of a system S with $\Omega(S) = \{3, 4\}$ remains an open problem.

Theorem 3.7 *For order $v = 39$, the feasible set is $\Omega(39) = \{3, 4, 5\}$. In particular, there are $S =$ STS(39)s with $\Omega(S) = \{3\}$, $\Omega(S) = \{4\}$, $\Omega(S) = \{5\}$, $\Omega(S) = \{3, 4\}$, and $\Omega(S) = \{3, 4, 5\}$. There are unbicolorable systems.*

Proof. In [11], it is shown that there are STS(19)s which are either only 3-bicolorable by $\mathcal{C}_1(4,6,9)$ or only 4-bicolorable by $\mathcal{C}_2(1,2,8,8)$; there are also STS(19)s admitting both the bicoloring $\mathcal{C}_1(4,6,9)$ and $\mathcal{C}_2(1,2,8,8)$. By Theorem 2.1, there are STS(39)s with the extended bicolorings of $\mathcal{C}_1$ and $\mathcal{C}_2$. Starting with these three kinds of STS(19)s by doubling construction we obtain STS(19)s of the respective kind as claimed.

In [8], the systems in Tables 8 and 10 are only 3 and 5-colorable, respectively. In fact, by numerical analysis, it is possible to verify that they have no independent sets of cardinality 18 (all the possible 4-bicolorings have a coloring class of cardinality 18). Then, the system S_8 does not have two disjoint independent sets of cardinality 16 (the unique possible 5-bicoloring has two bicoloring classes of cardinality 16), and the system S_{10} does not admit two distinct disjoint independent sets of cardinalities $9,14,16$ and $10,12,17$, i.e., it is not 3-bicolorable. The STS(39) of Table 9 is only 4-colorable, since, by numerical analysis, it has neither two disjoint independent sets of cardinality 16 nor two disjoint independent sets of cardinalities $9,14,16$ and $10,12,17$.

It follows from [4] that there are unbicolorable systems with $\beta < 14$. □

Theorem 3.8 *For order* $v = 43$*, the feasible set is* $\Omega(43) = \{3,4\}$*. There are* STS(43)*s with* $\Omega(S) = \{3\}$*,* $\Omega(S) = \{4\}$*, and* $\Omega(S) = \{3,4\}$*. There are unbicolorable systems.*

Proof. Since $\sum_{i=1}^{6} n_i \geq \sum_{i=1}^{6} 2^{i-1} > 43$, by Theorem 1.1, for every STS(43) we obtain $\overline{\chi} \leq 5$. However, further analysis shows that there are no 5-bicolorable STS(43). Indeed, the only solutions of the equation (1) are $(10,16,17)$, $(1,8,16,18)$, $(2,10,10,21)$, $(4,4,17,18)$, $(1,10,12,20)$, $(5,6,10,22)$, and $(4,8,9,22)$. In [3, 11] it was found that an STS(21) can only be 3-bicolorable with the bicolorings $\mathcal{C}_1(5,6,10)$ or $\mathcal{C}_2(4,8,9)$. So the last two solutions are obtained by doubling construction while all the other solutions are bicolorings found in [3].

In Table 11 of [8], we can see that C_1 of STS(21) is extendable to a 3-bicoloring $C = C(10,16,17)$ of a $S =$ STS(43), for which we have $\chi = 3$ and $\bar{\chi} = 4$; i.e. $\Omega(S) = \{3,4\}$.

In the Appendix [8], by numerical analysis we can see that the system S_{11} in Table 11 has independent set of order $\beta < 18$, hence this system is only 3-bicolorable.

In the system S_{12}, admitting the bicoloring $\mathcal{C}(1,8,16,18)$, all the independent sets of cardinality 17 are inside the color class X_4 and X_3 is the unique independent set of order 16 disjoint from the independent sets of cardinalities 17. Since $X_1 \cup X_2$ is not an independent set, S_{12} is not 3-bicolorable.

In [4], the authors found unbicolorable systems with $\beta < 17$. □

It was found in [3] that an STS(45) can be bicolorable only with the 4-bicolorings $C(2,8,14,21)$ and $C(4,6,13,22)$. In [4], we can find STSs with $\beta < 21$. Consequently, we can formulate the following result.

Theorem 3.9 *For order $v = 45$, the feasible set is $\Omega(45) = \{4\}$. There are unbicolorable systems.* □

Theorem 3.10 *For order $v = 49$, the feasible set is $\Omega(49) = \{3, 4, 5\}$. There are unbicolorable systems.*

Proof. The unique solutions of the equation (1) are $(12, 17, 20)$, $(14, 14, 21)$, $(2, 8, 18, 21)$, $(5, 6, 14, 24)$ $(1, 4, 4, 20, 20)$. In [3] it was found that all these solutions are 3 and 4-bicolorings. □

In Appendix [8], the system S_{13} in Table 13 is an STS(49) colored by a 3-bicoloring $(14, 14, 21)$. Numerical analysis shows that it has only one independent set of cardinality 21 and none of cardinality 24. It does not have two disjoint independent sets of cardinalities 21 and 18. All independent sets of cardinality 20 are inside the color class of order 21. This means that the feasible set is given by $\Omega(S_{13}) = \{3\}$. The system S_{14} is an STS(49) colored with a 4-bicoloring $\mathcal{C}(5, 6, 14, 24)$. This system has all independent sets of cardinality 20 contained in the color class X_4, and no independent set of cardinality 17 with at least four vertices of the color class X_4. Moreover, it does not have three disjoint independent sets of cardinalities 20, 14 and 14. This implies that there is no way to color S_{14} with 3 and 5-bicolorings, i.e., $\Omega(S_{11}) = \{4\}$. The system S_{15} is an STS(49) colored with a 5-bicoloring $\mathcal{C}(1, 4, 4, 20, 20)$. This system contains a subsystem STS(9) and $\beta = 20$. The only two independent sets of cardinality 20 correspond to the coloring classes X_3 and X_4. All the independent sets of cardinality 17 are contained inside X_3 and X_4. This means that $\Omega(S_{15}) = \{5\}$.

It is still an open problem to determine whether there exists a system of order 49 with distinct upper and lower chromatic numbers.

The following table shows all the feasible sets $\Omega(v)$ and $\Omega(S)$.

BSTS(v)	$\Omega(v)$	$\Omega(S)$	Open problem $\Omega(S)$	Unbicolorable S
BSTS(7)	{3}	{3}	–	no
BSTS(9)	{3}	{3}	–	no
BSTS(13)	{3}	{3}	–	no
BSTS(15)	{4}	{4}	–	yes
BSTS(19)	{3, 4}	{3} {4} {3, 4}	–	yes
BSTS(21)	{3}	{3}		yes
BSTS(25)	{3, 4}	{3} {4}	{3, 4}	yes
BSTS(27)	{3, 4}	{3} {4} {3, 4}	–	yes
BSTS(31)	{3, 5}	{3} {5}	–	yes
BSTS(33)	{4}	{4}	–	yes
BSTS(37)	{3, 4}	{3} {4}	{3, 4}	yes
BSTS(39)	{3, 4, 5}	{3} {4} {5} {3, 4, 5}	{3, 5}	yes
BSTS(43)	{3, 4}	{3} {4} {3, 4}	–	yes
BSTS(45)	{4}	{4}	–	yes
BSTS(49)	{3, 4, 5}	{3} {4} {5}	{3, 4} {3, 5} {4, 5} {3, 4, 5}	yes

Table

4 Conclusion

In this article, we have studied feasible sets of all STS(v) for $v \leq 50$. It was shown that such methods as doubling construction, extension of bicolorings and numerical analysis with scientific computations are successful in finding new information about feasible sets. For all admissible values of $v \leq 50$, the feasible sets are determined. The following partial problems remain open:

1. Determine whether there exist systems STS(25) and STS(37) with the feasible set $\Omega(S) = \{3, 4\}$.
2. Determine whether there exists STS(49) with distinct upper and lower chromatic numbers.
3. Determine whether extended k-bicolorings exist for $v = 2^k - 1$ and $k \geq 10$.
4. Determine whether there exist systems S with a gap in their $\Omega(S)$.

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