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A characterisation of eccentric sequences of maximal outerplanar graphs

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Abstract

Let G be a connected graph. The eccentricity of a vertex v is defined as the distance in G between v and a vertex farthest from v. The nondecreasing sequence of the eccentricities of the vertices of G is the eccentric sequence of G. In this paper, we characterize eccentric sequences of maximal outerplanar graphs.

1 Overview

Let G be a connected graph. The eccentricity $\mathrm{ecc}_G(v)$ of a vertex v is defined as the distance in G between v and a vertex farthest from v. The nondecreasing sequence $s_1, s_2, \ldots, s_n$ of the eccentricities of the vertices of G is the eccentric sequence of G. Clearly, s_1 is the radius of G, and s_n is the diameter of G.

A sequence of integers is called eccentric if it is the eccentric sequence of some graph. The problem of characterising all eccentric sequences appears very difficult. Eccentric sequences were first considered by Lesniak [8]. She showed that in the eccentric sequence of every graph each entry, except possibly the smallest, appears at least twice, and she characterised sequences that are eccentric sequences of trees. Lesniak further showed that a nondecreasing sequence S of positive integers containing m distinct values is eccentric if and only if some subsequence of S with m distinct values is eccentric. This result suggests the following definition: An eccentric sequence is called minimal if it has no proper eccentric subsequence with the same number of distinct eccentricities. Minimal eccentric sequences with two values, i.e., minimal eccentric sequences of the form a^h, b^k (i.e., a repeated h times and b repeated k times) were considered by Hrnčiar and Monoszová [6]. They showed that there are exactly seven minimal eccentric sequences of the form $4^h, 5^k$, viz. $4^7, 5^2$; $4^6, 5^4$; $4^5, 5^6$; $4^4, 5^8$; $4^3, 5^9$; $4^2, 5^{12}$; and $4, 5^{14}$. They conjectured that in general there exist $2a-1$ minimal eccentric sequences of the form $a^h, (a+1)^k$. Buckley [1] reports that Nandakumar [9] determined all minimal eccentric sequences with least eccentricity 1 or 2. All 13 minimal eccentric sequences with least eccentricity 3 were determined by Haviar, Hrnčiar and Monoszová [5]. To date no characterisation of minimal eccentric sequences has been found.

A different approach to the problem of characterising eccentric sequences is to consider a restriction to suitable graph classes. Lesniak's characterisation of the eccentric sequences of trees [8] remains the only example for this approach. In this paper, we present a characterisation of the eccentric sequences of maximal outerplanar graphs.

The notation we use is as follows. G always denotes a maximal outerplanar graph (MOP) of order at least 3 with vertex set V and edge set E. The number of vertices of G is usually denoted by n. For a subset $A \subseteq V$ we define $N_G(A)$ to be the set of all vertices that are adjacent to some vertex in A, but are not themselves in A. If $A = \{v\}$, then we write $N_G(v)$ for $N_G(\{v\})$. The distance between two vertices u and v, i.e., the minimum number of edges on a (u, v)-path, is denoted by $d(u, v)$.

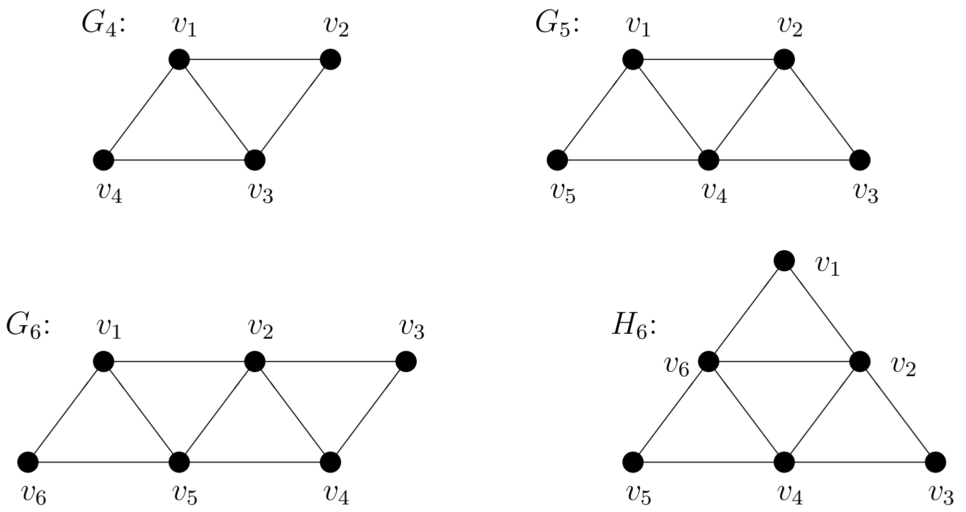

Figure 1: Four of the seven possible centers of a MOP.

The eccentricity of a vertex v, denoted by $\mathrm{ecc}(v)$, is the distance to a vertex farthest away from v, and a vertex at distance $\mathrm{ecc}(v)$ from v is called an eccentric vertex of v. For each $v \in V$ we choose an eccentric vertex of v and denote it by $\overline{v}$. The smallest and largest eccentricity of the vertices of G are the radius and the diameter of G, denoted by $\mathrm{rad}(G)$ and $\mathrm{diam}(G)$, respectively. The nondecreasing sequence of the eccentricities of the vertices of G is the eccentric sequence of G. For a nonnegative integer i we define e_i to be the number of vertices of eccentricity i in G. Clearly the eccentric sequence determines the e_i ($\mathrm{rad}(G) \leq i \leq \mathrm{diam}(G)$) and vice versa. So for a characterisation of eccentric sequences it suffices to characterise the sequence of the e_i.

A centre vertex of G is a vertex of minimum eccentricity. The centre of G, denoted by C, is the set of all centre vertices. The centre subgraph of G is the subgraph induced by C.

It is a well-known fact that every graph is the centre subgraph of some graph. However, Proskurowski [11] showed that the centre of a MOP is always isomorphic to one of seven graphs. We denote by K_n the complete graph on n vertices.

Theorem 1. [11] *Let G be a MOP. Then the centre of G is isomorphic to one of the following seven graphs: K_1, K_2, K_3, G_4, G_5, G_6, H_6, where G_4, G_5, G_6, and H_6 are shown in Figure 1.*

The present authors showed in [4] that in a 2-connected chordal graph every eccentricity strictly between the radius and the diameter occurs at least four times. Since every MOP is 2-connected and chordal, we have the following theorem which we will use extensively.

Theorem 2. ([4]) *Let G be a MOP with radius r, diameter d. Then $e_i \geq 4$ for $i \in \{r+1, r+2, \ldots, d-1\}$.*

If G is a MOP of order at most 5, then G is one of the graphs K_1, K_2, K_3, G_4, and G_5, and so a sequence of at most five integers is the eccentric sequence of a MOP if and only if it is the eccentric sequence of one of these five graphs. We may thus restrict our attention to MOPs of order at least 6. Our main result is the following theorem:

Theorem 3. *Let $S : s_1, s_2, \ldots, s_n$ be a nondecreasing sequence of positive integers, where $n \geq 6$, and, for $i \geq 1$, let e_i be the number of entries of S with value exactly i. Let $r = s_1$ and $d = s_n$. Then S is the eccentric sequence of some MOP if and only if $e_d \geq 2$, $e_i \geq 4$ for all i with $r < i < d$, and one of the following holds:*

(1) $n \geq 6r - 6$, $r \geq 2$, $d = 2r - 2$, $e_r = 6$, $e_i \geq 6$ for all i with $r < i < d$, and $e_d \geq 3$,

(2) $r \geq 3$, $d = 2r - 1$, and $e_r = 6$,

(3) $r \geq 2$, $d = 2r - 1$, and $e_r = 5$,

(4) $r \geq 2$, $d = 2r - 1$, and $e_r = 4$,

(5) $n \geq 4r - 1$, $d = 2r - 1$, and $e_r = 3$,

(6) $r \geq 2$, $d = 2r$, and $e_r = 2$,

(7) $n \geq 4r + 1$, $d = 2r$, and $e_r = 1$.

The rest of the paper is the proof of the seven cases in Theorem 3: In Section 2, we prove the necessity — that the eccentric sequence of a MOP of order at least 6 always satisfies the conditions of Theorem 3 — while in Section 3, we prove the sufficiency by showing how to construct MOPs satisfying these conditions.

2 Necessary Conditions

We denote the set of all centre vertices of a given MOP by C, and the individual vertices of C are $v_1, v_2, \ldots, v_{|C|}$, where the vertices are labeled as shown in Figure 1. Subscripts will always be taken modulo $|C|$.

Now let G be a MOP with $|C| \geq 3$. Consider a component H of $G - C$. Since G is outerplanar, there are adjacent vertices v_i, v_j of C such that $N_G(V(H)) \subset \{v_i, v_j\}$. Moreover, v_iv_j is on the boundary of the unbounded face of $G[C]$ and thus of the form v_iv_{i+1}, and there is no further component H' of $G - C$ with $N_G(V(H')) = \{v_i, v_{i+1}\}$. We denote the set of vertices $V(H) \cup \{v_i, v_{i+1}\}$ by U_i. Hence each vertex of $G - C$ is in exactly one U_i. We will frequently use the fact that a path in G joining a vertex in U_i and a vertex in U_j always contains a vertex in $\{v_i, v_{i+1}\}$ and a vertex in $\{v_j, v_{j+1}\}$.

In Lemmas 1 to 7 we prove the necessity of each of the seven conditions stated in Theorem 3.

Lemma 1. *Let G be a MOP of diameter d and radius r whose centre is isomorphic to H_6. Then $d = 2r - 2$ and* *(i)* $e_r = 6$,

(ii) $e_i \geq 6$ *for* $r < i < d$,

(iii) $e_d \geq 3$,

(iv) $n \geq 6r - 6$.

Proof. Claim 1: $d(v, C) \leq r - 2$ for all vertices v.

Let $v \in V$ be an arbitrary vertex. Without loss of generality we assume that v is in U_1. Then every (v, v_5)-path passes through $\{v_1, v_2\}$, hence

$$r \geq d(v, v_5) \geq d(v, \{v_1, v_2\}) + d(\{v_1, v_2\}, v_5) = d(v, C) + 2,$$

which yields Claim 1.

Claim 2: Let $i \in \{1, 3, 5\}$. Then $\overline{v_{i+3}} \in U_{i-1} \cup U_i$ and $d(\overline{v_{i+3}}, v_i) = r - 2$, and $d(\overline{v_{i+3}}, v_j) = r - 1$ for $j \in \{i - 1, i + 1\}$.

We show the statement for $i = 1$, the other statements are proved analogously. Consider vertex v_4. Since the vertices v_2, v_3, v_4, v_5, v_6 are within distance 1 of v_4, it follows from Claim 1 and the triangle inequality that every vertex in $H_2 \cup H_3 \cup H_4 \cup H_5$ is within distance $r - 1$ of v_4; hence, $\overline{v_4}$ is in $H_6 \cup H_1$.

We may assume that $\overline{v_4} \in U_1$; the case $\overline{v_4} \in U_6$ is analogous. Then every $(\overline{v_4}, v_4)$-path goes through $\{v_1, v_2\}$. Since v_2 is adjacent to v_4, it follows that $d(\overline{v_4}, v_2) \geq r - 1$. In conjunction with Claim 1 this implies that $d(v_1, \overline{v_4}) = r - 2$ and $d(v_2, \overline{v_4}) = r - 1$. Clearly we have also $d(v_6, \overline{v_4}) = r - 1$.

Claim 3: $d = 2r - 2$ and $e_d \geq 3$.

By Claim 1, every vertex is within distance $r - 2$ of its nearest vertex in C, and any two vertices in C are at most distance 2 apart; hence, $d \leq 2(r - 2) + 2 = 2r - 2$. It follows from Claim 2 that $d(\overline{v_4}, \overline{v_6}) = d(\overline{v_6}, \overline{v_2}) = d(\overline{v_2}, \overline{v_4}) = 2r - 2$. Hence $d = 2r - 2$ and $e_d \geq 3$.
We now prove that for any j with $r < j < d$ we have $e_j \geq 6$. Since the eccentricities of adjacent vertices differ by at most 1, the vertices of eccentricity j that belong to $U_{i-1} \cup U_i$ separate C and $\overline{v_{i+3}}$ for $i \in \{1, 3, 5\}$. Since G is 2-connected, $U_{i-1} \cup U_i$ contains at least two vertices of eccentricity j in G. Hence G contains at least six vertices of eccentricity j.

Claim 4: $n \geq 6r - 6$.

Suppose without loss of generality that $\overline{v_4} \in U_1$. Then by Claim 2 and the fact that G is 2-connected, there are internally disjoint $\overline{v_4} - v_1$ and $\overline{v_4} - v_2$ paths in U_1 having lengths at least $r - 2$ and $r - 1$, respectively. Thus $U_1 - \{v_1, v_2\}$ contains at least $1 + 2r - 5$ vertices. Applying the same argument to $\overline{v_2}$ and $\overline{v_6}$, we get $n \geq 6r - 6$. □

In what follows we make use of the following result, due to Chang:

Theorem 4. *[3] If G is a chordal graph in which the diameter is twice the radius, then the center of G is a clique.*

Lemma 2. *Let G be a MOP of diameter d and radius r whose centre is isomorphic to G_6. Then $r \geq 3$ and $d = 2r - 1$. Moreover*
(i) $e_r = 6$,
(ii) $e_i \geq 4$ *for* $r < i < d$,
(iii) $e_d \geq 2$.

Proof. In view of Theorems 2 and 4, it suffices to show that $r \geq 3$ and that $d \geq 2r-1$. The former follows from $d(v_3, v_6) = 3$.

Claim 1: (i) $d(v, \{v_5, v_1\}) \leq r - 2$ for all $v \in U_5 \cup U_6$,
(ii) $d(v, \{v_2, v_4\}) \leq r - 2$ for all $v \in U_2 \cup U_3$,
(iii) $\max(d(v, v_1), d(v, v_2)) \leq r - 1$ for all $v \in U_1$,
(iv) $\max(d(v, v_4), d(v, v_5)) \leq r - 1$ for all $v \in U_4$.

To prove (i) consider vertex v_3. Since every path from a vertex $v \in U_5 \cup U_6$ to v_3 goes through $\{v_1, v_5\}$, and since $d(v_3, v_1) = d(v_3, v_5) = 2$, we have $d(v, \{v_5, v_1\}) = d(v, v_3) - 2 \leq r - 2$, and (i) follows. Part (ii) is proved similarly by considering vertex v_6.
To prove (iii) let v be an arbitrary vertex in U_1. Since $d(v_4, v) \leq r$, and every (v, v_4)-path goes through $\{v_1, v_2\}$, we have $d(v_2, v) \leq r - 1$. Similarly, every shortest (v, v_6)-path goes through $\{v_1, v_2\}$, so $d(v_1, v) \leq r - 1$, and (iii) follows. Similarly we prove (iv).

Claim 2: $d \geq 2r - 1$.

By Claim 1 every vertex in $U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6$ is within distance $r - 1$ of v_5, so $\overline{v_5} \in U_1$. Clearly, $d(\overline{v_5}, v_1) \geq r - 1$ and $d(\overline{v_5}, v_2) \geq r - 1$. By Claim 1 we get $d(\overline{v_5}, v_1) = d(\overline{v_5}, v_2) = r - 1$.
Similarly, $\overline{v_2} \in U_4$ and $d(\overline{v_2}, v_4) = d(\overline{v_2}, v_5) = r - 1$. From these statements we get that

$$\begin{aligned} d(\overline{v_5}, \overline{v_2}) &\geq d(\overline{v_5}, \{v_1, v_2\}) + d(\{v_1, v_2\}, \{v_4, v_5\}) + d(\{v_4, v_5\}, \overline{v_2}) \\ &= (r - 1) + 1 + (r - 1) = 2r - 1, \end{aligned}$$

as desired. □

Lemma 3. *Let G be a MOP of diameter d and radius r whose centre is isomorphic to G_5. Then $d = 2r - 1$. Moreover,*
(i) $e_r = 5$,
(ii) $e_i \geq 4$ *for* $r < i < d$,
(iii) $e_d \geq 2$.

Proof. In view of Theorems 2 and 4 it suffices to show that $d \geq 2r - 1$.

Claim 1: (i) $d(v, C) \leq r - 2$ for all $v \in U_5 \cup U_2$,
(ii) $d(v, v_4) \leq r - 1$ for all $v \in U_3 \cup U_4$,
(iii) $\max(d(v, v_1), d(v, v_2)) \leq r - 1$ for all $v \in U_1$.

Parts (i) and (ii) are proved as above by using the fact that for $v \in U_5$ ($v \in U_2$, $v \in U_3$, $v \in U_4$) the distance between v and v_3 (v_5, v_1, v_2) is not more than r. Part

(iii) follows from the fact that $d(v, v_3) \leq r$ and $d(v, v_5) \leq r$ for all $v \in U_1$.

Claim 2: (i) $\overline{v_4} \in U_1$ and $d(\overline{v_4}, v_1) = d(\overline{v_4}, v_2) = r - 1$.
(ii) We can choose $\overline{v_2}$ such that $\overline{v_2} \in U_3 \cup U_4$.
(iii) $d(\overline{v_2}, v_3) = d(\overline{v_2}, v_4) = r-1$ if $\overline{v_2} \in U_3$, and $d(\overline{v_2}, v_4) = d(\overline{v_2}, v_5) = r-1$ if $\overline{v_2} \in U_4$.

Consider vertex v_4. It follows from Claim 1 that every vertex in $U_5 \cup U_6 \cup U_2 \cup U_3$ is within distance $r - 1$ of v_4, so $\overline{v_4} \in U_1$. Then $d(\overline{v_4}, v_1) \geq r - 1$ and $d(\overline{v_4}, v_2) \geq r - 1$, and in conjunction with Claim 1 we get $d(\overline{v_4}, v_1) = d(\overline{v_4}, v_2) = r - 1$, and part (i) follows.
To prove (ii) consider a common neighbour u of v_1 and v_2 in U_1. Then $\mathrm{ecc}(u) = r+1$ since u is not a centre vertex. It follows from Claim 1 that $\overline{u} \notin U_1 \cup U_2 \cup U_5$, so $\overline{u} \in U_3 \cup U_4$. Since $d(v_2, \overline{u}) \geq r$, $\overline{u}$ is an eccentric vertex of v_2.
Part (iii) is proved as above.

Claim 3: $d \geq 2r - 1$.

We may assume that $\overline{v_2} \in U_4$. By Claim 2 we have

$$d(\overline{v_2}, \overline{v_4}) \geq d(\overline{v_2}, \{v_4, v_5\}) + 1 + d(\{v_1, v_2\}, \overline{v_4}) \geq 2r - 1,$$

as desired. □

Lemma 4. *Let G be a MOP of diameter d and radius r whose centre is isomorphic to G_4. Then $d = 2r - 1$. Moreover,*
(i) $e_r = 4$,
(ii) $e_i \geq 4$ *for* $r < i < d$,
(iii) $e_d \geq 2$.

Proof. In view of Theorems 2 and 4, it suffices to show that $d \geq 2r - 1$.

Claim 1: (i) $d(v, v_1) \leq r - 1$ for all $v \in U_4 \cup U_1$,
(ii) $d(v, v_3) \leq r - 1$ for all $v \in U_2 \cup U_3$,

Parts (i) and (ii) are proved as above by using the fact that for $v \in U_4$ ($v \in U_1$, $v \in U_2$, $v \in U_3$) the distance between v and v_2 (v_4, v_4, v_2) is not more than r. This also implies the following.

Claim 2: (i) $\overline{v_1} \in U_2 \cup U_3$ and $d(\overline{v_1}, v_3) = r - 1$.
(ii) If $\overline{v_1} \in U_2$ ($\overline{v_1} \in U_3$) then $d(\overline{v_1}, v_2) \geq r - 1$ ($d(\overline{v_1}, v_4) \geq r - 1$).

Claim 3: $d \geq 2r - 1$.

We may assume that $\overline{v_1} \in U_2$. Then U_2 contains a common neighbour w of v_2 and v_3. Since $\mathrm{ecc}(w) = r+1$, it follows from Claim 1 that every vertex in U_2 has distance at most r from w; so $\overline{w} \notin U_2$. Hence every $(\overline{w}, \overline{v_1})$-path passes through $\{v_2, v_3\}$. Now $d(\overline{w}, \{v_2, v_3\}) \geq \mathrm{ecc}(w) - 1 = r$, and in conjunction with Claim 1 we get

$$d(\overline{w}, \overline{v_1}) \geq d(\overline{w}, \{v_2, v_3\}) + d(\{v_2, v_3\}, \overline{v_1}) \geq r + (r - 1) = 2r - 1,$$

as desired. □

Lemma 5. *Let G be a MOP of order n, diameter d and radius r whose centre is isomorphic to K_3. Then $d = 2r - 1$. Moreover*
- *(i)* $e_r = 3$,
- *(ii)* $e_i \geq 4$ *for* $r < i < d$,
- *(iii)* $n \geq 4r - 1$.

Proof. In view of Theorem 2 it suffices to show that $n \geq 4r - 1$ and $d = 2r - 1$. If $r = 1$, then since no outerplanar graph contains K_4 as a subgraph, we must have $G = K_3$. If $r \geq 2$, then $C = K_3$ is a proper subgraph of G and we may then assume that $|U_1| > 2$. We claim that $|U_2| > 2$ or $|U_3| > 2$; if this is not the case, then since v_1 and v_2 have a neighbour $x_1 \in U_1$, we have $e(x_1) \leq e(v_3)$, a contradiction. We hence assume that $|U_2| > 2$. Since for every vertex $y \in U_1$, we have $d(x_1, y) \leq d(v_3, y) \leq r$, we must (without loss of generality) have $\overline{x_1} \in U_2$. Furthermore, every $x_1 - \overline{x_1}$ path contains v_1 or v_2; since $d(v_1, \overline{x_1}), d(v_2, \overline{x_1}) \leq r$, this implies that $d(x_1, \overline{x_1}) = r + 1$. In fact, since every $v_1 - \overline{x_1}$ path contains v_2 or v_3 and v_2 is adjacent to v_3, we must have $d(v_2, \overline{x_1}) = r$. Since $d(v_1, \overline{x_1}) \leq r$, this then implies that $d(v_3, \overline{x_1}) = r - 1$. Since G is 2-connected, there are internally disjoint $\overline{x_1} - v_2$ and $\overline{x_1} - v_3$ paths, which implies that $U_2 - \{v_2, v_3\}$ contains at least $1 + r - 2 + r - 1 = 2r - 2$ vertices. A similar argument proves that $U_1 - \{v_1, v_2\}$ contains at least $2r - 2$ vertices. Hence G has order at least $3 + 2(2r - 2) = 4r - 1$.

Finally, every vertex of G is distance at most $r - 1$ from C, so $d \leq 2r - 1$. Let x_2 be a common neighbour of v_2 and v_3 in U_2. Then $\overline{x_2} \in U_1$ or $\overline{x_2} \in U_3$. If $\overline{x_2} \in U_1$, then a similar argument to the preceding shows that $d(v_2, \overline{x_2}) = r$ and $d(v_1, \overline{x_2}) = r - 1$, which implies that $d(x_2, \overline{x_2}) = 2r - 1$. The case when $\overline{x_2} \in U_3$ is handled similarly. □

Lemma 6. *Let G be a MOP of diameter d and radius r whose centre is isomorphic to K_2. Then $d = 2r$. Moreover,*
- *(i)* $e_r = 2$,
- *(ii)* $e_i \geq 4$ *for* $r < i < d$,
- *(iii)* $e_d \geq 2$.

Proof. In view of Theorem 2 it suffices to show that $d = 2r$.

Claim 1: $G - C$ has exactly two components.

It suffices to show that edge $v_1 v_2$ is not on the boundary of the unbounded face. Suppose it is. Then v_1 and v_2 have a common neighbour v. Now for every vertex w of G there exist paths of length at most r from w to v_1 and to v_2. Using these paths it is easy to construct a $w - v$ path of length at most r for every $w \in V$, which contradicts the fact that v is not a centre vertex of G.

Claim 2: $d = 2r$.

Clearly, $d \leq 2r$. The graph $G - C$ has two components H and H' with vertex sets U and U', respectively. Then H contains a common neighbour v of v_1 and v_2. Clearly, $\mathrm{ecc}(v) = d(v, \overline{v}) = r + 1$. We show that

$$\overline{v} \in U' \quad \text{and} \quad d(\overline{v}, C) = r.$$

Suppose to the contrary that $\overline{v} \notin U'$. Then $d(v_1, \overline{v}) = d(v_2, \overline{v}) = r$, and so there exist a $(v_1, \overline{v})$-path P_1 and a $(v_2, \overline{v})$-path P_2, both of length r. If v is on P_1 or P_2 then $d(v, \overline{v}) \le r$, a contradiction; hence v is on neither path. But then the union of P_1 and P_2, together with the edge v_1v_2 contains a cycle through v_1 and v_2 so that v is on the inside of this cycle. This contradicts the outerplanarity of G, and so $\overline{v} \in U'$. Now let u be a common neighbour of v_1 and v_2 in H'. With a similar argument we show that $\overline{u} \in U$ and $d(\overline{u}, C) = r$. Since $\overline{u}$ and $\overline{v}$ are not in the same component of $G - C$, every $(\overline{u}, \overline{v})$-path goes through C. Hence

$$d(\overline{u}, \overline{v}) \ge d(\overline{u}, C) + d(C, \overline{v}) = 2r,$$

and so $d = 2r$ as desired. □

Lemma 7. *Let G be a MOP of diameter d and radius r whose centre is isomorphic to K_1. Then $d = 2r$. Moreover,*
- *(i)* $e_r = 1$,
- *(ii)* $e_i \ge 4$ *for* $r < i < d$,
- *(iii)* $e_d \ge 2$,
- *(iv)* $n \ge 4r + 1$.

Proof. In view of Theorem 2 it suffices to show that $d = 2r$ and that $n \ge 4r + 1$.

Let $w_1, w_2, \ldots, w_k$ be the neighbours of v_1 in clockwise order, and such that v_1w_1 and v_1w_k are the edges incident with v_1 that lie on the unique hamiltonian cycle of G. Since G is maximal outerplanar, the edges $w_1w_2, w_2w_3, \ldots, w_{k-1}w_k$ are present in G. Similar to the notation used above, we denote by U_i the set of vertices in the component of $G - \{v_1, w_1, w_2, \ldots, w_k\}$ adjacent to $\{w_i, w_{i+1}\}$, together with $\{w_i, w_{i+1}\}$.

Claim 1: $d(w, \{w_i, w_{i+1}\}) \le r - 1$ for all $w \in U_i$, $1 \le i \le k$.

The proof is as above.

Claim 2: $d = 2r$.

Since $d \le 2r$, it suffices to show that there exists a pair of vertices at distance at least $2r$. First assume that there exists a set U_i containing a vertex u with $\max\{d(u, w_i), d(u, w_{i+1})\} = r$. Without loss of generality let $d(u, w_i) = r$. Then $d(u, w_{i+1}) = r - 1$ by Claim 1. We claim that $d(u, \overline{w_{i+1}}) = 2r$. A shortest $(u, \overline{w_{i+1}})$-path contains w_i or w_{i+1}. If it contains w_{i+1}, then

$$d(u, \overline{w_{i+1}}) = d(u, w_{i+1}) + d(w_{i+1}, \overline{w_{i+1}}) = (r - 1) + (r + 1) = 2r,$$

and if it contains w_i then $d(u, \overline{w_{i+1}}) = d(u, w_i) + d(w_i, \overline{w_{i+1}}) \ge r + r = 2r$, so in both cases we conclude that $d(u, \overline{w_{i+1}}) \ge 2r$, as desired.
Hence we may assume that, for all $i \in \{1, 2, \ldots, k\}$ and for all $u \in U_i$, we have $d(u, w_i) \le r - 1$ and $d(u, w_{i+1}) \le r - 1$. Fix an eccentric vertex $\overline{v_1}$ of the centre vertex v_1. Let $\overline{v_1} \in U_j$, say. Then $d(\overline{v_1}, w_j) = d(\overline{v_1}, w_{j+1}) = r - 1$. If now there is a vertex w which is a common eccentric vertex of w_j and of w_{j+1}, then $w \notin U_j$ and so, by $\mathrm{ecc}(w_j) = \mathrm{ecc}(w_{j+1}) = r + 1$, we have $d(\overline{v_1}, w) \ge d(\overline{v_1}, \{w_j, w_{j+1}\}) +$

$d(\{w_j, w_{j+1}\}, w) = (r-1)+(r+1) = 2r$, as desired. Hence we may assume that there is no common eccentric vertex of w_j and w_{j+1}. Then $d(w_j, \overline{w_{j+1}}) = d(w_{j+1}, \overline{w_j}) = r$, and so w_j is on some shortest $(w_{j+1}, \overline{w_{j+1}})$-path P_{j+1}, and w_{j+1} is on some shortest $(w_j, \overline{w_j})$-path P_j. Clearly, P_{j+1} and P_j do not pass through v_1. We conclude that $\overline{w_j} \in U_{j+1} \cup U_{j+2}$, and similarly that $\overline{w_{j+1}} \in U_{j-1} \cup U_{j-2}$. Now $\overline{w_{j+1}} \notin U_{j-1}$ since otherwise, if $\overline{w_{j+1}} \in U_{j-1}$, we would obtain the contradiction $d(\overline{w_{j+1}}, w_{j+1}) = d(\overline{w_{j+1}}, w_j) + d(w_j, w_{j+1}) \leq (r-1) + 1 = r$. Therefore we have $\overline{w_{j+1}} \in U_{j-2}$, and similarly $\overline{w_j} \in U_{j+2}$. This implies that there is a shortest $(\overline{w_{j+1}} - \overline{w_j})$-path that contains v_1. Since $d(v_1, \overline{w_{j+1}}) = d(v_1, \overline{w_j}) = r$, we conclude that $d(\overline{w_{j+1}}, \overline{w_j}) = r + r = 2r$, as desired.

Claim 3: $n \geq 4r + 1$.

Let x and y be vertices of G with $d(x, y) = d = 2r$. Since G is 2-connected, there are internally disjoint $x - y$ paths P_1 and P_2, each of length at least $2r$, which implies that $n \geq 4r$. Suppose that $n = 4r$. Then $P_1 \cup P_2$ is the unique hamilton cycle of G, and both P_1 and P_2 are $x - y$ geodesics of length $2r$. Suppose without loss of generality that $v_1 \in V(P_1)$; then $d(v_1, x) = d(v_1, y) = r$. If $\deg v_1 = 2$, then since G is a MOP, the two neighbours of v_1 on P_1 are adjacent, contradicting the fact that P_1 is a geodesic. For the same reason, the vertex v_1 cannot be adjacent to three vertices of P_1. It follows that v_1 is adjacent to a vertex z of $P_2 - \{x, y\}$. The vertex z is distance at most r away from every vertex of $P_1 - \{x, y\}$. Since $\mathrm{ecc}(z) > \mathrm{ecc}(v_1)$, it follows that $\overline{z} \in V(P_2)$, which implies, without loss of generality, that $\overline{z} = x$. Let z' be the neighbour of z on P_2 that lies between z and x. If z' is adjacent to v_1, then we obtain $\mathrm{ecc}(z') = r$, a contradiction. On the other hand if z' is not adjacent to v_1, then since G is a MOP, the vertex z must be adjacent to the neighbour of v_1 on P_1 that lies between v_1 and x. But then $d(z, x) \leq r$, a contradiction. It follows that $n \geq 4r + 1$. □

3 Constructions

We now prove the sufficiency of Theorem 3 by showing, for every sequence S satisfying one of the conditions (1)–(7) of Theorem 3, how to construct a MOP with eccentric sequence S.

3.1 Cases 2, 3, 4, and 6

We make use of the following gadget in our construction. Let $m \geq 1$ and let $a_0 = 2, a_1, \ldots, a_m$ be a sequence of integers with $a_s \geq 2$ for all $s \in \{1, \ldots, m-1\}$ and $a_m \geq 1$. We define a graph $P(2, a_1, \ldots, a_m)$ of order $\sum_{s=0}^{m} a_s$ as follows.
The vertex set is $\{\, x_{s,t} : 0 \leq s \leq m \text{ and } 1 \leq t \leq a_s \,\}$. The edge set consists of

(i) all edges of the form $x_{s,t}x_{s,t+1}$, where $0 \le s \le m$ and $1 \le t \le a_s - 1$;
(ii) all edges of the form $x_{s,2}x_{s+1,t}$, where $0 \le s \le m-2$ and $1 \le t \le a_{s+1}$;
(iii) all edges of the form $x_{m-1,1}x_{m,t}$, where $1 \le t \le a_m$;
(iv) all edges of the form $x_{s,1}x_{s+1,1}$, where $0 \le s \le m-2$; and
(v) the edge $x_{m-1,2}x_{m,1}$.

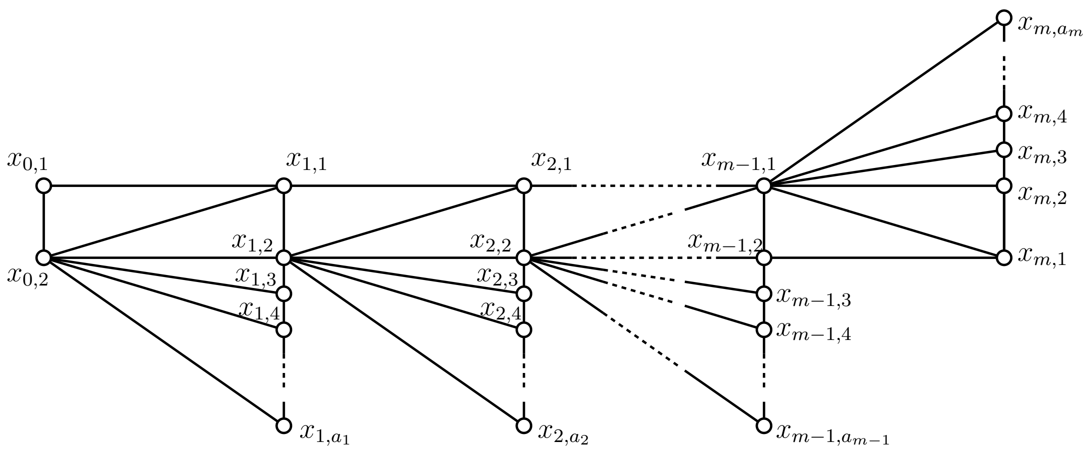

Figure 2: The construction of $P(2, a_1, \ldots, a_m)$

Lemma 8. *Let* $W = P(2, a_1, \ldots, a_m)$. *Then*
(a) W *is a MOP;*
(b) if $m \ge 2$, *then* $d(x_{0,2}, x_{s,t}) = s$ *for all* $1 \le s \le m$ *and* $1 \le t \le a_s$*; and*
(c) if $m \ge 2$, *then* $d(x_{0,1}, x_{m,t}) = m$ *for all* $1 \le t \le a_m$.

Proof. (a) We can think of building W as follows. Start with two adjacent vertices, and repeatedly add a path L, joining everything on L to some vertex y and joining one end of L to one neighbor of y. This graph is a MOP at each stage.

(b) Easy by induction on s.

(c) Easily checked. □

We now present the constructions.

Case 2: $r \ge 3$, $d = 2r-1$, $e_r = 6$, $e_i \ge 4$ for all i with $r < i < d$, and $e_d \ge 2$.

Let $a_s = 2$ for $1 \le s \le r-2$ and let $a_{r-1} = 1$. For $1 \le s \le r-1$, let $b_s = e_{r+s} - a_s$. Let $W_A = P(2, a_1, \ldots, a_{r-1})$ and $W_B = P(2, b_1, \ldots, b_{r-1})$. We denote the vertices of W_A by $x^A_{s,t}$ and the vertices of W_B by $x^B_{s,t}$. Let $G(S)$ be the graph constructed from G_6 (which, as before, has vertex set $\{v_1, v_2, \ldots, v_6\}$), W_A, and W_B, by identifying $v_1 = x^A_{0,2}$, $v_2 = x^A_{0,1}$, $v_4 = x^B_{0,2}$, and $v_5 = x^B_{0,1}$. The construction is illustrated in Figure 3.

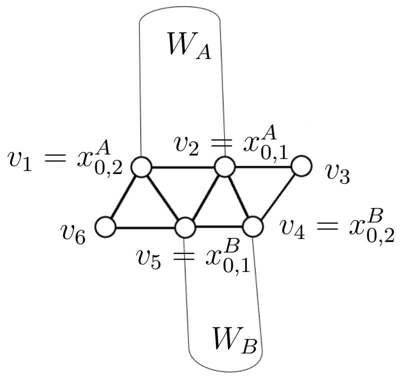

Figure 3: The construction for Case 2.

Clearly, $G(S)$ is a MOP. Since $r \geq 3$, by Lemma 8, both v_1 and v_2 are within distance $r-1$ of every vertex of W_A, and both v_4 and v_5 are within distance $r-1$ of every vertex of W_B. Hence, every vertex of G_6 has eccentricity at most r. Furthermore, $d(v_1, x^B_{r-1,1}) = d(v_2, x^B_{r-1,1}) = d(v_3, x^B_{r-1,1}) = r$, and $d(v_4, x^A_{r-1,1}) = d(v_5, x^A_{r-1,1}) = d(v_6, x^A_{r-1,1}) = r$; so every vertex of G_6 has eccentricity r. Consider now a vertex $x^A_{s,t}$; then $x^A_{s,t}$ is distance s from v_2, which implies that $e(x^A_{s,t}) \leq r+s$. Furthermore $d(x^A_{s,t}, x^B_{r-1,1}) = r+s$, so $e(x^A_{s,t}) = r+s$. Similarly, for every vertex $x^B_{s,t}$ we have $e(x^B_{s,t}) = r+s$. It follows that $G(S)$ has eccentricity sequence S, as required.

Case 3: $r \geq 2$, $d = 2r-1$, $e_r = 5$, $e_i \geq 4$ for all i with $r < i < d$, and $e_d \geq 2$.

If $r \geq 3$, then simply delete v_3 from the construction for Case 2. If $r = 2$, then one can similarly replicate the construction from Case 2 and again delete v_3. This is equivalent to taking $G_6 - \{v_3\}$, adding one vertex $x^A_{1,1}$ adjacent to v_1 and v_2, and adding a path L^B of length $e_d - 1$ with v_5 adjacent to every vertex on L^B and with v_4 adjacent to one end of L^B. It is easily checked that $G(S)$ has the desired properties.

Case 4: $r \geq 2$, $d = 2r-1$, $e_r = 4$, $e_i \geq 4$ for all i with $r < i < d$, and $e_d \geq 2$.

Delete v_6 from the corresponding construction in Case 3.

Case 6: $r \geq 2$, $d = 2r$, $e_r = 2$, $e_i \geq 4$ for $r < i < d$, and $e_d \geq 2$.

Let $a_s = 2$ for $1 \leq s \leq r-1$ and let $a_r = 1$. For $1 \leq s \leq r$, let $b_s = e_{r+s} - a_s$. Let $W_A = P(2, a_1, \ldots, a_r)$ and $W_B = P(2, b_1, \ldots, b_r)$. As before, we denote the vertices of W_A by $x^A_{s,t}$ and the vertices of W_B by $x^B_{s,t}$. Let $G(S)$ be the graph constructed from W_A and W_B by identifying $x^A_{0,1} = x^B_{0,2}$ and $x^A_{0,2} = x^B_{0,1}$. The construction is illustrated in Figure 4.

3.2 Cases 1, 5 and 7

We make use of the following gadget in our construction. The fundamental difference between this gadget and P above is that in P we have $d(x_{0,t}, x_{m,t'}) = m$ for all

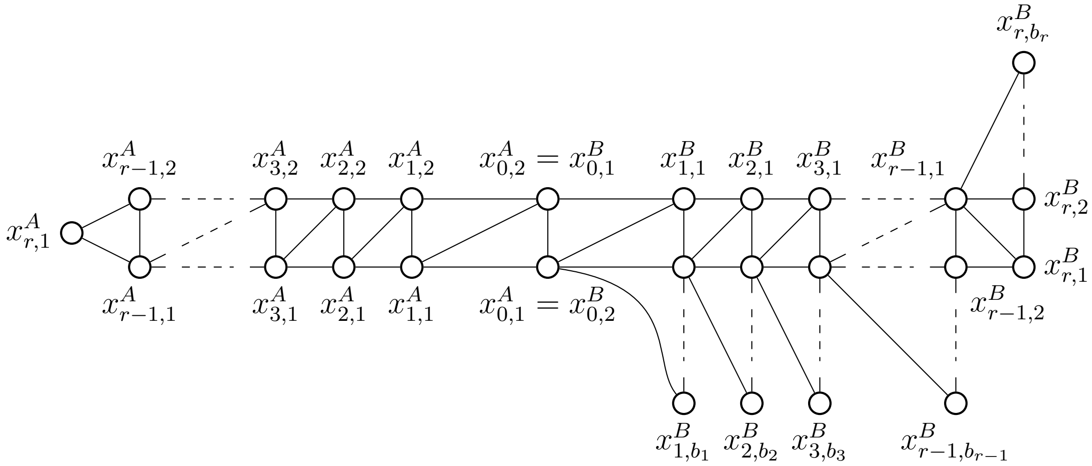

Figure 4: The construction for Case 6.

choices of t and t' (provided $m > 1$), whereas we now construct a graph such that $d(x_{0,t}, x_{m,t'}) > m$ for some choice of t and t'.

Let $m \geq 1$, and let $a_0 = 2, a_1, \ldots, a_m$ be a sequence of integers with $a_s \geq 2$ for all $s \in \{1, \ldots, m-1\}$ and $a_m \geq 1$. We define a graph $Q(2, a_1, \ldots, a_m)$ of order $\sum_{s=0}^{m} a_s$ as follows. The vertex set is $\{\, x_{s,t} : 0 \leq s \leq m \text{ and } 1 \leq t \leq a_s \,\}$. The edge set consists of
(i) all edges of the form $x_{s,t}x_{s,t+1}$, where $0 \leq s \leq m$ and $1 \leq t \leq a_s - 1$;
(ii) all edges of the form $x_{s,2}x_{s+1,t}$, where $0 \leq s \leq m-1$ and $1 \leq t \leq a_{s+1}$; and
(iii) for each s with $0 \leq s \leq m-1$, the edge $x_{s,1}x_{s+1,1}$ if $a_s = 2$, and the edge $x_{s,3}$ to $x_{s+1,a_{s+1}}$ otherwise.

To illustrate the construction, the graph $Q(2, 3, 2, 4, 2)$ is shown in Figure 5.

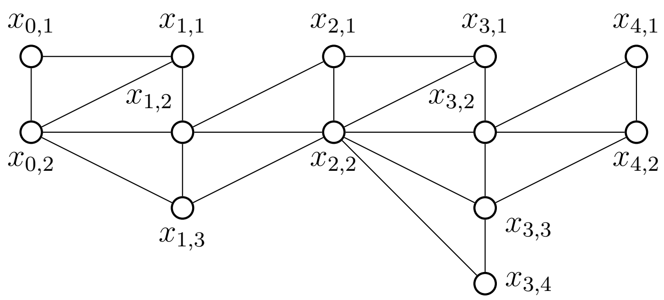

Figure 5: The graph $Q(2, 3, 2, 4, 2)$.

Lemma 9. *Let $W = Q(2, a_1, \ldots, a_m)$. Then*
(a) W is a MOP;
(b) $d(x_{0,2}, x_{s,t}) = s$ for all $1 \leq s \leq m$ and $1 \leq t \leq a_s$; and
(c) if $\sum_{s=1}^{m} a_s \geq 2m$, then $d(x_{0,1}, x_{m,a_m}) = m + 1$.

Proof. The proofs of (a) and (b) are as in the proof of Lemma 8.

To prove (c), notice that since $x_{0,1}$ and $x_{0,2}$ are adjacent, it follows from part (b) that $d(x_{0,1}, x_{m,a_m}) \leq m + 1$. The condition $\sum_{s=1}^{m} a_s \geq 2m$ means that $a_s > 2$ for some $1 \leq s < m$, or that $a_m > 1$. For each s, vertex $x_{s,1}$ has at most one neighbor of the form $x_{s+1,t}$, namely $t = 1$. Thus, if $a_m > 1$ then we get the claimed distance. Further, if any $a_s > 2$ for $s < m$, then $x_{s,1}$ is not adjacent to $x_{s+1,1}$. □

We now prove the remaining cases for Theorem 3.

Case 1: $n \geq 6r - 6$, $r \geq 2$, $d = 2r - 2$, $e_r = 6$, $e_i \geq 6$ for all i with $r < i < d$, and $e_d \geq 3$.

The graph is H_6 if $r = 2$. So assume $r \geq 3$. Then, one can construct three vectors $A = (2, a_1, \ldots, a_{r-2})$, $B = (2, b_1, \ldots, b_{r-2})$, and $C = (2, c_1, \ldots, c_{r-2})$ satisfying the following three conditions: (i) for all $s \in \{1, \ldots, r-2\}$ we have $a_s + b_s + c_s = e_{r+s}$; (ii) for all $s \in \{1, \ldots, r-3\}$ we have $a_s, b_s, c_s \geq 2$, and $a_{r-2}, b_{r-2}, c_{r-2} \geq 1$; and (iii) each of $\sum_{s=1}^{r-2} a_s$, $\sum_{s=1}^{r-2} b_s$ and $\sum_{s=1}^{r-2} c_s$ is at least $2r - 4$.

Let $W_A = Q(A)$, $W_B = Q(B)$, and $W_C = Q(C)$. We shall denote the vertices of W_i by $x^i_{s,t}$ as before. Let $G(S)$ be the graph constructed from H_6 (which, as before, has vertex set $\{v_1, v_2, \ldots, v_6\}$), W_A, W_B, and W_C, by identifying $v_1 = x^A_{0,2}$, $v_2 = x^A_{0,1}$, $v_3 = x^B_{0,2}$, $v_4 = x^B_{0,1}$, $v_5 = x^C_{0,2}$, $v_6 = x^C_{0,1}$ The construction is illustrated in Figure 6.

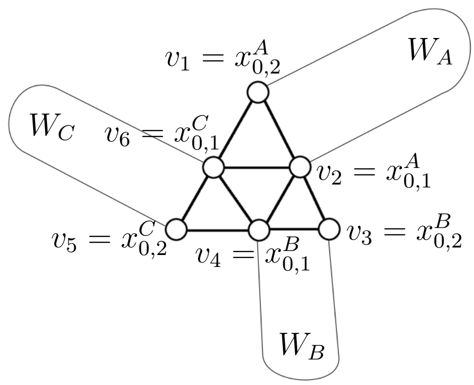

Figure 6: The construction for Case 1.

Clearly, $G(S)$ is a MOP. Notice that v_1, v_3, and v_5 are within distance $r - 2$ of every vertex in W_A, W_B, and W_C, respectively; so every vertex in H_6 has eccentricity at most r. Furthermore, $d(v_1, x^B_{r-2,1}) = d(v_3, x^C_{r-2,1}) = d(v_5, x^A_{r-2,1}) = r$. By Lemma 9, $d(v_2, x^C_{r-2,c_{r-2}}) = d(v_6, x^B_{r-2,b_{r-2}}) = d(v_4, x^A_{r-2,a_{r-2}}) = r$; hence $e(v_i) = r$ for all $i \in \{1, 2, \ldots, 6\}$. Consider now a vertex $x^i_{s,t}$ in W_i with $s \geq 1$. Then $x^i_{s,t}$ is within distance $s + 2$ of every vertex in H_6, which implies that $e(x^i_{s,t}) \leq r + s$. Furthermore $d(x^A_{s,t}, x^C_{r-2,c_{r-2}}) = d(x^B_{s,t}, x^A_{r-2,a_{r-2}}) = d(x^C_{s,t}, x^B_{r-2,b_{r-2}}) = r + s$. Consequently, $e(x^i_{s,t}) = r + s$. It follows that $G(S)$ has eccentricity sequence S, as required.

Case 5: $n \geq 4r - 1$, $d = 2r - 1$, $e_r = 3$, $e_i \geq 4$ for all i with $r < i < d$, and $e_d \geq 2$.

If $r = 1$, then the graph is K_3; so assume $r \geq 2$. Then, one can construct two vectors $A = (2, a_1, \ldots, a_{r-1})$ and $B = (2, b_1, \ldots, b_{r-1})$, satisfying the following three conditions: (i) for all $s \in \{1, \ldots, r-1\}$ we have $a_s + b_s = e_{r+s}$; (ii) for all $s \in \{1, \ldots, r-2\}$ we have $a_s, b_s \geq 2$, and, $a_{r-1}, b_{r-1} \geq 1$; and (iii) both $\sum_{s=1}^{r-1} a_s$ and $\sum_{s=1}^{r-1} b_s$ are at least $2r-2$.

Let $W_A = Q(A)$ and $W_B = Q(B)$. Let $G(S)$ be the graph constructed from K_3 with vertex set $\{v_1, v_2, v_3\}$, W_A and W_B, by identifying $v_1 = x^A_{0,2}$, $v_2 = x^B_{0,2}$, and $v_3 = x^A_{0,1} = x^B_{0,1}$.

Lemma 9 shows that there is some vertex in both W_A and W_B at distance r from v_3. On the other hand, by Lemma 9, v_1 can reach all vertices of W_A in $r-1$ steps, and similarly v_2 in W_B. It follows that v_1, v_2, and v_3 all have eccentricity r. From this it follows that all $x_{s,t}$ in both W_A and W_B have eccentricity $r+s$.

Case 7: $n \geq 4r+1$, $d = 2r$, $e_r = 1$, $e_i \geq 4$ for all i with $r < i < d$, and $e_d \geq 2$.

Then, one can construct two vectors $A = (2, a_1, \ldots, a_r)$, $B = (2, b_1, \ldots, b_r)$, satisfying the following three conditions: (i) for all $s \in \{1, \ldots, r\}$ we have $a_s + b_s = e_{r+s}$; (ii) for all $s \in \{1, \ldots, r-1\}$ we have $a_s, b_s \geq 2$, and, $a_r, b_r \geq 1$; and (iii) both $\sum_{s=1}^{r} a_s$ and $\sum_{s=1}^{r} b_s$ are at least $2r$.

Let $W_A = Q(A)$ and $W_B = Q(B)$. Let $G(S)$ be the graph constructed from W_A and W_B, by identifying $x^A_{0,2} = x^B_{0,2}$ and adding one edge joining $x^A_{0,1}$ to $x^B_{0,1}$. The justification is similar to that of Case 5.

4 Open questions

As has been noted previously, the general problem of characterizing the eccentric sequences of graphs appears to be difficult. Proskurowski's characterization of the centers of MOPs [11] plays a central role in our characterization of eccentric sequences of MOPs. Proskurowski has also characterized the centers of 2-trees [12], a class of graphs which properly contains MOPs. It would be interesting to see whether a characterization of eccentric sequences of 2-trees can be obtained via Proskurowksi's characterization of the centers of 2-trees.

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