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Constructions for Hadamard matrices using Clifford algebras, and their relation to amicability / anti-amicability graphs

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In honour of Kathy Horadam

Abstract

It is known that the Williamson construction for Hadamard matrices can be generalized to constructions using sums of tensor products. This paper describes a specific construction using real monomial representations of Clifford algebras, and its connection with graphs of amicability and anti-amicability. It is proven that this construction works for all such representations where the order of the matrices is a power of 2. Some related results are given for small dimensions.

1 Introduction

Williamson's construction for Hadamard matrices [50] uses the real monomial representation for the unit quaternions. This construction has been generalized in a number of directions, including constructions based on a Cayley table for the octonions [49, 32]. Another direction of generalization is that of Goethals and Seidel [23]. One type of generalization of Williamson's construction seems to have been overlooked: to generalize from the Quaternions to real monomial representations of Clifford algebras. This is remarkable because all of the ingredients for this generalization have been in place for a long time. Clifford algebras have been used in the study of orthogonal designs since at least 1974 [51] [22, Sections 3.10, 5.3]. Gastineau-Hills, in his Ph.D. thesis of 1980, used the concept of *anti-amicability*, along with Kronecker products, and *quasi-Clifford* algebras to study systems of orthogonal designs [19, 20]. That work is related to the current investigation.

This paper describes a specific construction for Hadamard matrices, using real monomial representations of Clifford algebras, and its connection with graphs of amicability and anti-amicability. The aim of the paper is not to use the construction to find Hadamard matrices with previously unknown orders, but to better understand the relationships between amicability and anti-amicability for $\{-1, 1\}$ matrices.

The paper is organized as follows. Section 2 develops Kronecker product constructions for Hadamard matrices, by placing tighter and tighter restrictions on two n-tuples of matrices. Section 3 examines the relationship between the first n-tuple of matrices and canonical bases for the real representations of Clifford algebras. Section 4 investigates the second n-tuple of matrices in terms of graphs of amicability and anti-amicability. Section 5 places the construction in its historical context, and looks at prospects for further research.

2 Kronecker product constructions for Hadamard matrices

The construction considered in this paper is motivated by the Williamson construction [50] and by the properties of real monomial representations of the basis elements of Clifford algebras. Rather than presenting the construction at the outset, this section shows how the construction can be arrived at by specialization from a more general construction.

Our first generalization of the Williamson construction is the most general considered here. In this construction, we aim to find

$$A_k \in \{-1, 0, 1\}^{n \times n}, \quad B_k \in \{-1, 1\}^{b \times b}, \quad k \in \{1, \ldots, n\},$$

where the A_k are *monomial* matrices, and construct

$$H := \sum_{k=1}^{n} A_k \otimes B_k, \tag{H0}$$

such that

$$H \in \{-1, 1\}^{nb \times nb} \quad \text{and} \quad HH^T = nbI_{(nb)}, \tag{H1}$$

i.e. H is a Hadamard matrix of order nb.

Here we use monomial matrices, that is matrices with only one non-zero entry in each row and each column. This starting point could be generalized even further, but the use of n monomial matrices of order n here agrees with the Williamson construction, is sufficient for the constructions below, and simplifies the exposition.

Due to well-known and easily verified properties of the Kronecker product (e.g. [37, (2.8)],) if the order of the products in (H0) is reversed to yield the construction

$$G := \sum_{k=1}^{n} B_k \otimes A_k, \tag{G0}$$

we obtain the equivalent result

$$G \in \{-1, 1\}^{nb \times nb} \quad \text{and} \quad GG^T = nbI_{(nb)}. \tag{G1}$$

We now begin to specialize the construction. Since

$$HH^T = \sum_{j=1}^{n} A_j \otimes B_j \sum_{k=1}^{n} A_k^T \otimes B_k^T,$$

we will impose stronger conditions on the construction by making the non-zero contribution to HH^T come from the diagonal of this double sum, i.e.

$$\sum_{j=1}^{n} A_j A_j^T \otimes B_j B_j^T = nbI_{(nb)},$$
$$\sum_{j=1}^{n} \sum_{k=j+1}^{n} \left(A_j A_k^T \otimes B_j B_k^T + A_k A_j^T \otimes B_k B_j^T \right) = 0. \qquad \text{(H2)}$$

We also define the equivalent conditions (G2), with the Kronecker product reversed.

We now impose even stronger conditions by making each off-diagonal contribution separately sum to zero, i.e.

$$\sum_{k=1}^{n} A_k A_k^T \otimes B_k B_k^T = nbI_{(nb)},$$
$$A_j A_k^T \otimes B_j B_k^T + A_k A_j^T \otimes B_k B_j^T = 0 \quad (j \neq k). \qquad \text{(H3)}$$

We also define the equivalent conditions (G3), with the Kronecker product reversed.

Up until now, because we have retained the Kronecker product in our conditions, it is not clear how to find $2n$ matrices $(A_1, \ldots, A_n)$, $(B_1, \ldots, B_n)$, which simultaneously satisfy these conditions, other than by a brute force search. We therefore impose the still stronger conditions

$$\begin{aligned}
A_j * A_k = 0 \quad (j \neq k), \quad \sum_{k=1}^{n} A_k &\in \{-1, 1\}^{n \times n}, \\
A_k A_k^T &= I_{(n)}, \\
A_j A_k^T + \lambda_{j,k} A_k A_j^T &= 0 \quad (j \neq k), \\
B_j B_k^T - \lambda_{j,k} B_k B_j^T &= 0 \quad (j \neq k), \\
\lambda_{j,k} &\in \{-1, 1\}, \\
\sum_{k=1}^{n} B_k B_k^T &= nbI_{(b)},
\end{aligned} \qquad (4)$$

where $*$ is the Hadamard matrix product.

It is straightforward to check the following implications.

Theorem 1. *Conditions (4) on constructions (G0) and (H0) imply (G3) and (H3), which imply conditions (G2) and (H2), which, in turn, imply (G1) and (H1).*

The coupling between the A and B matrices is mediated by the λ parameters. If we find an n-tuple of A matrices satisfying conditions (4), we can then use the resulting λ values to search for an n-tuple of B matrices satisfying conditions (4), to complete the sums (G0) and (H0).

Example 1: Sylvester-like construction. For our first example, we set $n = 2$, $b = 2$. For the A matrices, we use two signed permutation matrices obtained from the 2×2 matrix used for the Sylvester construction,

$$A_1 = \begin{bmatrix} 1 & . \\ . & - \end{bmatrix}, \quad A_2 = \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix}.$$

Here, $\lambda_{1,2} = 1$. (Here and below, to reduce clutter in the display of matrices, we use the conventions '$-$' $= -1$, '.' $= 0$.)

To satisfy conditions (4), we need to find $B_1, B_2 \in \{-1, 1\}^{2\times 2}$ such that

$$B_1 B_1^T + B_2 B_2^T = 4I_{(2)}, \quad B_1 B_2^T - B_2 B_1^T = 0,$$

in other words, B_1 and B_2 must satisfy the Gram sum condition and be pairwise amicable. An amicable pair of Hadamard matrices of order b satisfies this requirement. For example, if we use the amicable pair

$$B_1 = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & - \\ 1 & 1 \end{bmatrix},$$

our constructions (G0) and (H0) yield

$$G = \begin{bmatrix} 1 & 1 & 1 & - \\ 1 & - & - & - \\ 1 & 1 & - & 1 \\ 1 & - & 1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 & 1 & - \\ 1 & - & 1 & 1 \\ 1 & - & - & - \\ 1 & 1 & - & 1 \end{bmatrix}.$$

Example 2: Anti-amicable construction. For our second example, we also have $n = 2$, $b = 2$, but we now want an example with $\lambda_{1,2} = -1$. For the A matrices, we use the two commuting permutation matrices,

$$A_1 = \begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix}$$

Since we now have $\lambda_{1,2} = -1$, to satisfy conditions (4), we need to find $B_1, B_2 \in \{-1, 1\}^{2\times 2}$ such that

$$B_1 B_1^T + B_2 B_2^T = 4I_{(2)}, \quad B_1 B_2^T + B_2 B_1^T = 0.$$

In other words, B_1 and B_2 must satisfy the Gram sum condition and be pairwise *anti*-amicable. For example, if we use

$$B_1 = \begin{bmatrix} - & 1 \\ 1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} - & - \\ - & 1 \end{bmatrix},$$

our constructions (G0) and (H0) yield

$$G = \begin{bmatrix} - & - & 1 & - \\ - & - & - & 1 \\ 1 & - & 1 & 1 \\ - & 1 & 1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} - & 1 & - & - \\ 1 & 1 & - & 1 \\ - & - & - & 1 \\ - & 1 & 1 & 1 \end{bmatrix}.$$

More examples. A Williamson-like construction has $n = 4$, and satisfies conditions (4) using 4 pairwise anti-amicable A matrices and 4 pairwise amicable B matrices. For example, we can use 4 A matrices such that

$$A_1 = I_{(4)}, \quad A_k^T = -A_k \quad (k > 1), \quad \lambda_{j,k} = 1 \quad (j \neq k).$$

An octonion-like construction has $n = 8$, and satisfies conditions (4) using 8 pairwise anti-amicable A matrices and 8 pairwise amicable B matrices. For example, we can use 8 A matrices such that

$$A_1 = I_{(8)}, \quad A_k^T = -A_k \quad (k > 1), \quad \lambda_{j,k} = 1 \quad (j \neq k).$$

Following from the work of Hurwitz [28], Radon [43], Taussky [49] and others, it is known that the only values of n for which an n-tuple of A matrices of order n with $\lambda_{j,k} = 1$ for all $(j \neq k)$ can be found are $n = 1, 2, 4, 8$ [21]. (The case $n = 1$ is vacuously true.) It is no coincidence that these are the dimensions of the real, complex, Quaternion and Octonion algebras over the real numbers.

To go further than $n = 8$ with our constructions (H0) and (G0), with our strongest conditions (4), we need to allow at least one case of $\lambda_{j,k}$ to equal -1. This leads us to consider the Clifford algebras.

To recap, we aim to find n-tuples $(A_1, \ldots, A_n)$, and $(B_1, \ldots, B_n)$, with

$$A_k \in \{-1, 0, 1\}^{n \times n}, \quad B_k \in \{-1, 1\}^{b \times b},$$

and all A_k monomial, satisfying conditions (4). For the A matrices, in the next section, we examine signed groups, cocycles and Clifford algebras. For the B matrices, in Section 4, we examine graphs of amicability and anti-amicability.

3 Signed groups and Clifford algebras

In this section, we examine in greater detail the properties of the n-tuples of A matrices that satisfy conditions (4). We then describe the real Clifford algebras and their underlying finite groups in terms of Craigen's signed groups, and Horadam and de Launey's cocycles. Finally, we show how the real monomial representations of Clifford algebras allow the construction of A matrices satisfying conditions (4).

Gastineau-Hills' systems of orthogonal designs. First, we note that an n-tuple $(A_1, \ldots, A_n)$ of matrices satisfying conditions (4) gives rise to a special case of a *regular n-system* of orthogonal designs, of order n, genus $(\delta_{j,k})$, type $(1; \ldots; 1)$, with $p_1 = \ldots = p_n = 1$, with $\lambda_{j,k} = (-1)^{(1+\delta_{j,k})}$, according to the definition and notation of Gastineau-Hills [19, Section (5.1), p. 36]. The special case arises because we require that $A_j * A_k = 0$ for all $(j \neq k)$, not just when $\lambda_{j,k} = 1$.

Let $(A_1, \ldots, A_n)$ satisfy conditions (4). For $j \neq k$ we define $E_{j,k} := A_j A_k^T = A_j A_k^{-1}$. Then

$$\begin{aligned} E_{j,k} \in \{-1, 0, 1\}^{n \times n}, \quad E_{j,k} = -\lambda_{j,k} E_{j,k}^T, \quad E_{j,k} E_{j,k}^T = I_{(n)}, \\ E_{j,k}^2 = -\lambda_{j,k} I_{(n)}, \quad E_{j,k} E_{k,l} = E_{j,l}, \quad E_{j,k} E_{k,j} = I_{(n)}. \end{aligned} \tag{5}$$

In other words, the E matrices are orthogonal $(-1, 0, 1)$ matrices, are either symmetric or skew, and the square of an E matrix is therefore either $I_{(n)}$ or $-I_{(n)}$. The condition $A_j * A_k = 0$ implies that $E_{j,k}$ always has zero diagonal.

At this point, we could go on to follow Gastineau-Hills [19, Section (7), pp. 58-61], and examine *quasi-Clifford* algebras, but there is a point of distinction between between the analysis there and what is needed in our case. The E matrices defined and examined by Gastineau-Hills [19, p. 59] may obey more relations than are listed at [19, (7.4), p. 60], and consequently, the set of generators listed there may not be minimal.

Signed groups and cocycles. Rather than pursuing Gastineau-Hills' construction of quasi-Clifford algebras any further, we now briefly examine signed groups, and go on to look at the canonical generation of a specific class of signed groups, leading a construction for the real representation of certain the real Clifford algebras.

A *signed group* [10] is a finite group E of even order containing a distinguished element of order 2 in its centre. This distinguished element is called -1.

The group E can be considered to be a central extension of the abelian group $C := \{-1, 1\} \equiv \mathbb{Z}_2$ by some group G, such that the elements of E can be written as ordered pairs, $(s, \mathbf{g})$,with $s \in C$, and $\mathbf{g} \in G$ [11] [13, Chapter 12]. This is easy to see: given the group E, the set C forms a normal subgroup of E. Take a transversal G of C in E. The set G is not yet a group, but we can define a multiplication as follows. Each pair of elements $\mathbf{g}, \mathbf{h} \in G$ yields the element $\mathbf{gh} \in E$ under the multiplication of E. Define the multiplication in G by $(\mathbf{gh})_G := s\mathbf{gh}$ if $s\mathbf{gh} \in G$, where $s \in S$.

Given a group G, the multiplication in the extension E is determined by a sign function $\psi : G \times G \to S$ such that

$$(s, \mathbf{g})(t, \mathbf{h}) = \big(st\ \psi(\mathbf{g}, \mathbf{h}), \mathbf{gh}\big).$$

Here the multiplications are in S and in G respectively.

Remark. *Many authors (e.g. Isaacs [30]) use the opposite convention, and say that E is a central extension of G by C.*

Remark. *It has been noted by Craigen [11] that since multiplication in E is associative, we automatically have*

$$\begin{aligned}(r, \mathbf{f})\big((s, \mathbf{g})(t, \mathbf{h})\big) &= (rst\ \psi(\mathbf{f}, \mathbf{gh})\psi(\mathbf{g}, \mathbf{h}), \mathbf{fgh})\\ =\big((r, \mathbf{f})(s, \mathbf{g})\big)(t, \mathbf{h}) &= (rst\ \psi(\mathbf{f}, \mathbf{g})\psi(\mathbf{fg}, \mathbf{h}), \mathbf{fgh}).\end{aligned}$$

So ψ is a cocycle in the sense of Horadam, de Launey and Flannery [27] [26, Chapter 6] [13, Chapter 12].

For much more on the relationship between central extensions of $\mathbb{Z}_2$ and cocycles, see de Launey and Smith [15, Section 2], and de Launey and Kharaghani [14, Section 2.2]. Chapter 12 of de Launey and Flannery [13] treats central extensions and cocycles in more generality.

Signed groups yielding the real Clifford algebras. We now construct the signed groups relevant to the real Clifford algebras. The signed group $\mathbb{G}_{p,q}$ of order 2^{1+p+q} is extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2^{p+q}$, defined by the signed group presentation

$$\begin{aligned}\mathbb{G}_{p,q} := \Big\langle\ & \mathbf{e}_{\{k\}}\ (k \in S_{p,q})\ \mid \\ & \mathbf{e}_{\{k\}}^2 = -1\ (k < 0), \quad \mathbf{e}_{\{k\}}^2 = 1\ (k > 0), \\ & \mathbf{e}_{\{j\}}\mathbf{e}_{\{k\}} = -\mathbf{e}_{\{k\}}\mathbf{e}_{\{j\}}\ (j \neq k) \Big\rangle,\end{aligned}$$

where $S_{p,q} := \{-q, \ldots, -1, 1, \ldots, p\}$.

The groups $\mathbb{G}_{p,q}$ for all non-negative integer values p, q, have been studied extensively by Braden [5], Lam and Smith [33], and others, but there is no generally accepted collective name for them. In [34], these groups are called *frame groups.*

The papers on asymptotic existence of cocyclic Hadamard matrices, by de Launey and Smith [15], and de Launey and Kharaghani [14], as well as Chapters 22 and 23 of de Launey and Flannery [13], treat these groups in some detail as central extensions of the group $\mathbb{Z}_2$ by $\mathbb{Z}_2^{p+q}$ with some differences in notation from that of Braden or Lam and Smith.

Multiplication in $\mathbb{Z}_2^{p+q}$ is isomorphic to the exclusive or (XOR) of bit vectors, or the symmetric set difference of subsets of $S_{p,q}$, so elements of $\mathbb{G}_{p,q}$ can be written as $\pm\mathbf{e}_T$, $T \subset S_{p,q}$, with $\mathbf{e}_\varnothing = 1$. The 2^{p+q} subsets $\{\pm\mathbf{e}_T\}$ are the cosets of $\{\pm 1\} \equiv \mathbb{Z}_2$ in $\mathbb{G}_{p,q}$. These cosets can be enumerated by using a canonical indexing, using the indices $-q$ up to p (excluding 0) of the bits of each bit vector in $\mathbb{Z}^{p+q}$. The interpretation of each bit vector as the binary representation of a number in $\mathbb{Z}_{2^{p+q}}$ then gives a canonical ordering of the cosets. For example, in $\mathbb{G}_{2,1}$

$$\begin{aligned} 0 &\leftrightarrow 000 \leftrightarrow \varnothing \leftrightarrow \{\pm 1\}, \\ 1 &\leftrightarrow 001 \leftrightarrow \{-1\} \leftrightarrow \{\pm\mathbf{e}_{\{-1\}}\}, \\ & \qquad \ldots \\ 3 &\leftrightarrow 011 \leftrightarrow \{-1, 1\} \leftrightarrow \{\pm\mathbf{e}_{\{-1,1\}}\}, \\ & \qquad \ldots \\ 7 &\leftrightarrow 111 \leftrightarrow \{-1, 1, 2\} \leftrightarrow \{\pm\mathbf{e}_{\{-1,1,2\}}\}. \end{aligned}$$

If we take a transversal of $\mathbb{Z}_2$ in $\mathbb{G}_{p,q}$, in particular, if we use the element $\mathbf{e}_T$ from each coset, we obtain a *canonical basis* in $\mathbb{G}_{p,q}$.

The group $\mathbb{G}_{p,q}$ extends to the universal real Clifford algebra $\mathbb{R}_{p,q}$, of dimension 2^{p+q}, by expressing each element $\mathbf{x} \in \mathbb{R}_{p,q}$ as a linear combination of the 2^{p+q} basis elements $\mathbf{e}_T$,

$$\mathbf{x} = \sum_{T \subset S_{p,q}} x_T \mathbf{e}_T.$$

The real Clifford algebra $\mathbb{R}_{p,q}$ is the quotient of the real group algebra $\mathbb{R}\mathbb{G}_{p,q}$ by the ideal $\langle \mathbf{e}_\emptyset + (-\mathbf{e}_\emptyset) \rangle$. That is, $-\mathbf{e}_\emptyset$ in $\mathbb{G}_{p,q}$ is identified with -1 in $\mathbb{R}$ [33, pp. 778-779] [36, Section 14.3] [34]. If, instead of the field $\mathbb{R}$, we use the ring of integers $\mathbb{Z}$, we obtain the *signed group ring* $\mathbb{Z}[\mathbb{G}_{p,q}]$ [10, p. 244].

Real monomial representations of real Clifford algebras. In this paper, we construct canonical real monomial representations $P(\mathbb{G}_{p,q})$ and $P(\mathbb{R}_{p,q})$ via sets of generating matrices [10, p. 243]. The key theorem in this construction is (paraphrasing Porteous [42, Prop. 13.17, p. 247])

Theorem 2. *If the set of matrices $S \subset \{-1, 0, 1\}^{n\times n}$ generates $P(\mathbb{G}_{p,q}) \equiv \mathbb{G}_{p,q}$, then the set of matrices*

$$\left\{ \begin{bmatrix} 1 & . \\ . & - \end{bmatrix} \otimes E \middle| E \in S \right\} \cup \left\{ \begin{bmatrix} . & - \\ 1 & . \end{bmatrix} \otimes I_{(n)}, \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix} \otimes I_{(n)} \right\} \subset \{-1, 0, 1\}^{2n\times 2n}$$

generates $P(\mathbb{G}_{p+1,q+1}) \equiv \mathbb{G}_{p+1,q+1}$.

The group $P(\mathbb{G}_{0,0}) \equiv \mathbb{G}_{0,0} \equiv \mathbb{Z}_2$ is generated by the of 1×1 matrix $[-1]$, so that Theorem 2 yields the generating set

$$\left\{ \begin{bmatrix} - & . \\ . & 1 \end{bmatrix}, \begin{bmatrix} . & - \\ 1 & . \end{bmatrix}, \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix} \right\}$$

for $P(\mathbb{G}_{1,1})$. Note that this set is redundant and that, in particular, the last two elements listed also generate $P(\mathbb{G}_{1,1})$.

Real monomial representations for $\mathbb{G}_{m,m}$ and $\mathbb{R}_{m,m}$ can be generated by extending this process. These representations are *faithful*: $P(\mathbb{R}_{m,m})$ is isomorphic to $\mathbb{R}^{2^m\times 2^m}$ [42, Prop 13.27] [36, Section 16.4]. Note well that the order of the matrices here is 2^m, in contrast to the order of 4^m needed for the regular representation of the group Z_2^{2m}.

An alternative construction giving the representation $P(\mathbb{G}_{m,m})$ and the group $\mathbb{Z}_2^{2m}$. There is a second, equivalent construction of the real monomial representation $P(\mathbb{G}_{m,m})$ of the group $\mathbb{G}_{m,m}$, which gives a different ordering of the cosets of $\{\pm I\}$ from the one given above. This construction is more useful for the purposes of this paper.

The 2×2 orthogonal matrices

$$\mathrm{E}_1 := \begin{bmatrix} . & - \\ 1 & . \end{bmatrix}, \quad \mathrm{E}_2 := \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix}$$

generate $P(\mathbb{G}_{1,1})$, the real monomial representation of group $\mathbb{G}_{1,1}$. The cosets of $\{\pm I\} \equiv \mathbb{Z}_2$ in $P(\mathbb{G}_{1,1})$ are ordered using a pair of bits, as follows.

$$\begin{aligned}
0 &\leftrightarrow 00 \leftrightarrow \{\pm I\}, \\
1 &\leftrightarrow 00 \leftrightarrow \{\pm \mathrm{E}_1\}, \\
2 &\leftrightarrow 10 \leftrightarrow \{\pm \mathrm{E}_2\}, \\
3 &\leftrightarrow 11 \leftrightarrow \{\pm \mathrm{E}_1\mathrm{E}_2\}.
\end{aligned}$$

For $m > 1$, the real monomial representation $P(\mathbb{G}_{m,m})$ of the group $\mathbb{G}_{m,m}$ consists of matrices of the form $G_1 \otimes G_{m-1}$ with G_1 in $P(\mathbb{G}_{1,1})$ and G_{m-1} in $P(\mathbb{G}_{m-1,m-1})$.

The cosets of $\{\pm I\} \equiv \mathbb{Z}_2$ in $P(\mathbb{G}_{m,m})$ are ordered by concatenation of pairs of bits, where each pair of bits uses the ordering as per $P(\mathbb{G}_{1,1})$, and the pairs are ordered as follows.

$$
\begin{aligned}
0 &\leftrightarrow 00\ldots 00 \leftrightarrow \{\pm I\},\\
1 &\leftrightarrow 00\ldots 01 \leftrightarrow \{\pm I_{(2)}^{\otimes(m-1)} \otimes \mathrm{E}_1\},\\
2 &\leftrightarrow 00\ldots 10 \leftrightarrow \{\pm I_{(2)}^{\otimes(m-1)} \otimes \mathrm{E}_2\},\\
&\ldots\\
2^{2m}-1 &\leftrightarrow 11\ldots 11 \leftrightarrow \{\pm(\mathrm{E}_1\mathrm{E}_2)^{\otimes m}\}.
\end{aligned}
$$

(Here $I_{(2)}$ is used to distinguish this 2×2 unit matrix from the $2^m \times 2^m$ unit matrix I.) In this paper, this ordering is called the *Kronecker product ordering* of the cosets of $\{\pm I\}$ in $P(\mathbb{G}_{m,m})$.

The Kronecker product ordering of the canonical basis matrices of $P(\mathbb{R}_{m,m})$ the real monomial representation of the Clifford algebra $\mathbb{R}_{m,m}$ is given by an ordered transversal of $\{\pm I\} \equiv \mathbb{Z}_2$ in $P(\mathbb{G}_{m,m})$, using the Kronecker product ordering. For example, $(I, \mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_1\mathrm{E}_2)$ is the Kronecker product ordering of the canonical basis matrices of $P(\mathbb{R}_{1,1})$.

Definition 1. *For some transversal of $\mathbb{Z}_2$ in $P(\mathbb{G}_{m,m})$, in the Kronecker product ordering, we define a function $\gamma_m : \mathbb{Z}_{2^{2m}} \to P(\mathbb{G}_{m,m})$ to choose the corresponding canonical basis matrix for $P(\mathbb{R}_{m,m})$. The Kronecker product ordering then defines a corresponding function on $\mathbb{Z}_2^{2m}$, which we also call γ_m. For example, $\gamma_1(1) = \gamma_1(01) :=$ E_1.*

Properties of the representation $P(\mathbb{G}_{m,m})$. We collect here a number of well-known and easily proved properties of the representation $P(\mathbb{G}_{m,m})$.

Lemma 3. *The group $\mathbb{G}_{m,m}$ and its real monomial representation $P(\mathbb{G}_{m,m})$ satisfy the following properties.*

1. *Pairs of elements of $\mathbb{G}_{m,m}$ (and therefore $P(\mathbb{G}_{m,m})$) either commute or anti-commute: for $g, h \in \mathbb{G}_{m,m}$, either $hg = gh$ or $hg = -gh$.*

2. *The matrices $E \in P(\mathbb{G}_{m,m})$ are orthogonal: $EE^T = E^TE = I$.*

3. *The matrices $E \in P(\mathbb{G}_{m,m})$ are either symmetric and square to give I or skew and square to give $-I$: either $E^T = E$ and $E^2 = I$ or $E^T = -E$ and $E^2 = -I$.*

The following properties of the diagonal elements of $P(\mathbb{G}_{m,m})$ are not so well-known, but are also easily proven by induction using the alternative construction given above.

Lemma 4. *The set of diagonal matrices $D_m \subset P(\mathbb{G}_{m,m})$ forms a subgroup of order 2^{m+1} of $P(\mathbb{G}_{m,m})$, consisting of the union of the following cosets of $\{\pm I\}$, listed in*

Kronecker product order.

$$\begin{aligned}
00\ldots00 &\leftrightarrow \{\pm I\},\\
00\ldots11 &\leftrightarrow \{\pm I_{(2)}^{\otimes(m-1)} \otimes \mathrm{E}_1E_2\},\\
&\ldots\\
11\ldots1100 &\leftrightarrow \{\pm(\mathrm{E}_1\mathrm{E}_2)^{\otimes(m-1)} \otimes I_{(2)}\},\\
11\ldots11 &\leftrightarrow \{\pm(\mathrm{E}_1\mathrm{E}_2)^{\otimes m}\}.
\end{aligned}$$

Each coset of D_m in $P(\mathbb{G}_{m,m})$ consists of a set of 2^{m+1} monomial matrices, all of which have the same support – i.e. the same set of non-zero indices.

Application to the construction of Section 2. We see that the Clifford algebra $\mathbb{R}^{2^m\times 2^m}$ has a canonical basis consisting of 4^m real monomial matrices, corresponding to the basis of the algebra $\mathbb{R}_{m,m}$. From Lemma 3 it is seen that these 4^m monomial *canonical basis matrices* have the following properties:

Pairs of basis matrices either commute or anticommute. Basis matrices are either symmetric or skew, and so the basis matrices A_j, A_k satisfy

$$A_kA_k^T = I_{(2^m)}, \quad A_jA_k^T + \lambda_{j,k}A_kA_j^T = 0 \quad (j \neq k), \quad \lambda_{j,k} \in \{-1, 1\}.$$

From Lemma 4 we see that we can choose a *transversal* of the cosets of D_m, consisting of $n = 2^m$ canonical basis matrices such that

$$A_j * A_k = 0 \quad (j \neq k), \quad \sum_{k=1}^{n} A_k \in \{-1, 1\}^{n\times n}.$$

This satisfies conditions (4) for A matrices. Thus, if n is a power of 2, an n-tuple of A matrices satisfying conditions (4) can always be found. In Section 4, the following theorem is proven, completing the construction.

Theorem 5. *If n is a power of 2, the constructions (G0) and (H0) with conditions (4) can always be completed, in the following sense. If an n-tuple of A matrices which produce a particular λ is obtained by taking a transversal of canonical basis matrices of the Clifford algebra $\mathbb{R}_{m,m}$, an of n-tuple of B matrices with a matching λ can always be found.*

Example: $\mathbb{R}_{2,2}$. The real Clifford algebra $\mathbb{R}_{2,2}$ is isomorphic to the real matrix algebra $\mathbb{R}^{4\times4}$. The corresponding frame group $\mathbb{G}_{2,2}$ is generated as a signed group by the four matrices

$$\begin{bmatrix} 1 & . \\ . & - \end{bmatrix} \otimes \begin{bmatrix} . & - \\ 1 & . \end{bmatrix}, \quad \begin{bmatrix} . & - \\ 1 & . \end{bmatrix} \otimes \begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix},$$
$$\begin{bmatrix} 1 & . \\ . & - \end{bmatrix} \otimes \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix}, \quad \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix} \otimes \begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix}.$$

The group has 32 elements and the canonical basis of $\mathbb{R}_{2,2}$ has 16 elements. As matrices, these canonical basis matrices form 4 equivalence classes of 4 elements each, where a pair of basis matrices is equivalent if they have the same support, i.e. the same sparsity pattern. To form a 4-tuple of canonical basis matrices satisfying (4), we simply take a transversal, that is, we choose one basis matrix from each class. For example,

$$A_1 := \begin{bmatrix} - & . & . & . \\ . & 1 & . & . \\ . & . & - & . \\ . & . & . & 1 \end{bmatrix}, \quad A_2 := \begin{bmatrix} . & 1 & . & . \\ 1 & . & . & . \\ . & . & . & - \\ . & . & - & . \end{bmatrix},$$
$$A_3 := \begin{bmatrix} . & . & 1 & . \\ . & . & . & - \\ - & . & . & . \\ . & 1 & . & . \end{bmatrix}, \quad A_4 := \begin{bmatrix} . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{bmatrix}.$$

In this case, $\lambda_{1,2} = \lambda_{1,3} = \lambda_{1,4} = \lambda_{2,3} = \lambda_{2,4} = \lambda_{3,4} = 1$.

An exhaustive enumeration of the $4^4 = 256$ different transversals, each consisting of a 4-tuple of 4×4 canonical basis matrices, yields 256 graphs, here called *transversal graphs*.

Each transversal graph is a graph giving the amicability / anti-amicability relationship of the 4 matrices defining the vertices. Each such graph is a complete graph on 4 vertices, with two edge colours. Each edge of the graph has one of two colours, -1 ("red") and 1 ("blue"). Matrices A_j and A_k are connected by a red edge if they have disjoint support and are anti-amicable, i.e. $\lambda_{jk} = 1$. Matrices A_j and A_k are connected by a blue edge if they have disjoint support and are amicable, i.e. $\lambda_{jk} = -1$.

When collected into equivalence classes by graph isomorphism, the set of 256 transversal graphs yields the six classes shown in Figure 1 (plotted using the *Graphviz dot* program [18]).

In each box of Figure 1, a red edge, corresponding to anti-amicability, is given a solid line, and a blue edge, corresponding to amicability is given a dashed line. The name 'Aabcd' in each box corresponds to the degree sequence with respect to red edges.

Colour-complementary graphs of A matrices. Graphs A0000 , A3333 are complementary with respect to the exchange of red and blue edges, as are graphs A1111 , A2222, and graphs A2110 , A3221.

This phenomenon is also observed in the cases of order 2 and order 8. In each of these three cases it is caused by the existence of a permutation π of the basis matrices of the real representation of the corresponding Clifford algebra, with the following property:

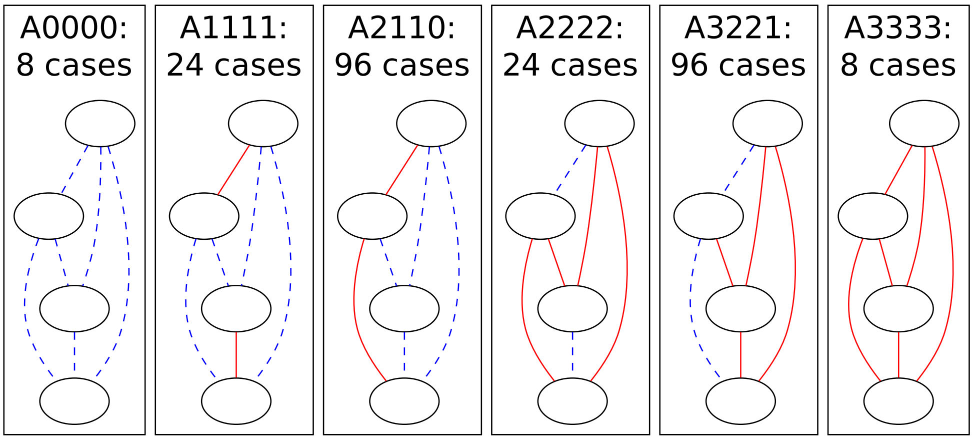

Figure 1: 4-tuples of A matrices of order 4, dashed: $\lambda_{j,k} = -1$, solid: $\lambda_{j,k} = 1$.

Property 1. *For canonical basis matrices $A_j, A_k \in \{-1, 0, 1\}^{n \times n}$,*

$$\text{if } A_j * A_k = 0 \text{ and } A_j A_k^T + \lambda_{j,k} A_k A_j^T = 0,$$
$$\text{then } \pi(A_j) * \pi(A_k) = 0 \text{ and } \pi(A_j)\ \pi(A_k)^T - \lambda_{j,k} \pi(A_k)\ \pi(A_j)^T = 0.$$

In other words, for pairs of basis matrices A_j, A_k, the permutation π sends an amicable pair with disjoint support to an anti-amicable pair, and vice-versa.

Let Δ_m be the graph whose vertices are the $n^2 = 4^m$ canonical basis matrices of the real representation of the Clifford algebra $\mathbb{R}_{m,m}$, with each edge having one of two colours, -1 (red) and 1 (blue):

- Matrices A_j and A_k are connected by a red edge if they have disjoint support and are anti-amicable.
- Matrices A_j and A_k are connected by a blue edge if they have disjoint support and are amicable.
- Otherwise there is no edge between A_j and A_k.

We call this graph the *restricted amicability / anti-amicability graph* of the Clifford algebra $\mathbb{R}_{m,m}$, the restriction being the requirement that an edge only exists for pairs of matrices with disjoint support.

We now introduce some notation that is used in the remainder of this paper.

Definition 2. *For a graph Γ with edges coloured by -1 (red) and 1 (blue), $\Gamma[-1]$ denotes the* red subgraph *of Γ, the graph containing all of the vertices of Γ, and all of the red (-1) coloured edges. Similarly, $\Gamma[1]$ denotes the* blue subgraph *of Γ.*

The existence of a permutation π with Property 1 is equivalent to the graph Δ_m having the following property.

Property 1a. *The graph Δ_m is self-edge-colour-complementary. That is, there exists a permutation of the vertices which takes every red edge to a blue edge and vice-versa. (This permutation is π itself.)*

Property 1a (and therefore Property 1) was verified for $m = 0, 1, 2, 3$ via the Python interface to the *igraph* network research package [12]. For each m, the graph Δ_m was formed from the relevant coloured adjacency matrix, and the routine `get_isomorphism_vf2` was called to find all isomorphisms between the graph Δ_m and the same graph with the complementary colouring. The Python code used, and the pickled Python output are available via the author's web page [35]. The results are listed in Table 1.

m	$n = 2^m$	$\lvert\Delta_m\rvert = 4^m$	degrees	isomorphisms
0	1	1	$(0, 0)$	1
1	2	4	$(1, 1)$	4
2	4	16	$(6, 6)$	192
3	8	64	$(28, 28)$	86016
4	16	256	$(120, 120)$	?

Table 1: Isomorphisms of Δ_m with its edge-colour-complement.

One key result is that each graph Δ_m is a regular two-edge-coloured graph on 4^m vertices. The fourth column of Table 1 gives the number of red and blue edges from each vertex. For example, the graph Δ_2 with 16 vertices has 6 red edges and 6 blue edges meeting each vertex. The total number of edges meeting each vertex is $4^m - 2^m$, since an edge only exists for a pair of matrices with disjoint support.

The fifth column gives the number of isomorphisms found by the *igraph* library function `get_isomorphism_vf2`. No attempt was made to further identify the groups. Also, since the algorithm used by `get_isomorphism_vf2` is exponential in the number of vertices, and since the case of Δ_3 took about 8 hours on a 2 GHz CPU, no attempt has yet been made to obtain the isomorphisms for Δ_4.

If we take every subset S of the vertices of Δ_m, of size $n = 2^m$ such that each pair of matrices has disjoint support, then, as in our example for $n = 4$ above, each such subset yields a subgraph that gives a two-edge-colouring to the edges of the complete graph on n vertices. The permutation π then induces a map $S \mapsto \pi S$ such that the corresponding subgraph maps to a edge-colour-complementary subgraph. Thus the existence of the permutation π implies the weaker property:

Property 2. *For the Clifford algebra $\mathbb{R}_{m,m}$, the subset of transversal graphs that are not self-edge-colour-complementary can be arranged into a set of pairs of graphs with each member of the pair being edge-colour-complementary to the other member.*

This, in turn implies the even weaker property:

Property 3. *For the Clifford algebra $\mathbb{R}_{m,m}$, if a graph T exists amongst the transversal graphs, then so does at least one graph with edge colours complementary to those of T.*

Since Property 1 is true for the three cases $m = 1, n = 2$, $m = 2, n = 4$, and $m = 3, n = 8$, then so are Properties 1a, 2 and 3.

These properties may continue for larger values of m, and so it is worth making the relevant conjectures:

Conjecture 1. *Property 1 holds for all $m \geqslant 0$. In other words, for all $m \geqslant 0$ there is a permutation π of the set of 4^m canonical basis matrices, that sends an amicable pair of basis matrices with disjoint support to an anti-amicable pair, and vice-versa.*

Conjecture 2. *Property 2 holds for all $m \geqslant 0$. In other words, for all $m \geqslant 0$, for the Clifford algebra $\mathbb{R}_{m,m}$, the subset of transversal graphs that are not self-edge-colour-complementary can be arranged into a set of pairs of graphs with each member of the pair being edge-colour-complementary to the other member.*

Conjecture 3. *Property 2 holds for all $m \geqslant 0$. In other words, for all $m \geqslant 0$, for the Clifford algebra $\mathbb{R}_{m,m}$, if a graph T exists amongst the transversal graphs, then so does at least one graph with edge colours complementary to those of T.*

As is shown in Section 4, these conjectures are also relevant to the $\{-1, 1\}$ matrices.

The full amicability / anti-amicability graphs of $\mathbb{R}_{m,m}$ and $\mathbb{G}_{p,q}$.

The full amicability / anti-amicability graph $\Lambda_{m,m}$ of the canonical basis matrices for the real representation of the Clifford algebra $\mathbb{R}_{m,m}$ can be obtained from the restricted graph Δ_m by recalling that, as a result of Lemma 4, two canonical basis matrices A_1 and A_2 have common support if and only if $A_2 = SA_1$, where $S \in D_m$ is a diagonal signed permutation matrix. We then have

$$A_1 A_2^T = A_1 A_1^T S^T = S^T = S, \quad A_2 A_1^T = S A_1 A_1^T = S.$$

So A_1 and A_2 are amicable. Thus the graph $\Lambda_{m,m}$ is a complete graph on 4^m vertices, with a self-loop on each vertex, and two-edge-coloured so that each vertex has $(4^m - 2^m)/2$ red edges and the remaining edges, including the self-loops, are coloured blue.

The full amicability / anti-amicability graph $\Gamma_{m,m}$ of the group $\mathbb{G}_{m,m}$ is obtained from $\Lambda_{m,m}$ by including the negatives of all of the 4^m canonical basis matrices that are the vertices of $\Lambda_{m,m}$. Thus $\Gamma_{m,m}$ has $|\mathbb{G}_{m,m}| = 2 \times 4^m$ vertices. Every canonical basis matrix A_j is amicable with $-A_j$, and if A_j is amicable with A_k, then $-A_j, -A_k, A_j$, and A_k are all pairwise amicable. If A_j is anti-amicable with A_k, this yields the subgraph shown in Figure 2 (plotted using the *Graphviz circo* program [18]).

Thus the number of red edges on each vertex of $\Gamma_{m,m}$ is twice that of $\Lambda_{m,m}$. (We say that the red subgraph $\Gamma_{m,m}[-1]$ is the *double graph* of the red subgraph

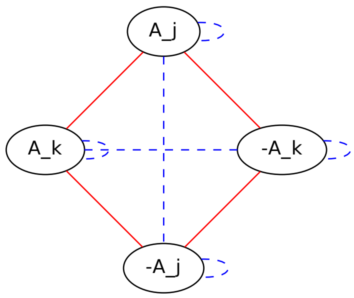

Figure 2: Anti-amicable matrices A_j, A_k and their negatives, solid: anti-amicable, dashed: amicable.

$\mathit{\Lambda}_{m,m}[-1]$ [29].) So $\mathit{\Gamma}_{m,m}$ is the complete graph on 2×4^m vertices, with self-loops on each vertex, with a two-edge-colouring such that the vertices and the red edges form a regular graph of degree $4^m - 2^m$.

Examples: $\mathit{\Gamma}_{1,1}$ has 8 vertices and 2 red edges on each vertex. $\mathit{\Gamma}_{2,2}$ has 32 vertices and 12 red edges on each vertex. These examples occur again in Section 4.

It is clear from the presentation of $\mathbb{G}_{p,q}$ that $\mathbb{G}_{p,q}$ is isomorphic to a subgroup of $\mathbb{G}_{P,Q}$ whenever $p \leqslant P$ and $q \leqslant Q$. Thus the full amicability / anti-amicability graph $\mathit{\Gamma}_{p,q}$ is a subgraph of $\mathit{\Gamma}_{m,m}$ whenever $p \leqslant m$ and $q \leqslant m$.

The anti-amicability graph of $\mathbb{R}_{m,m}$. Let Φ_m be the graph whose vertices are the $n^2 = 4^m$ canonical basis matrices of the real representation of the Clifford algebra $\mathbb{R}_{m,m}$, with matrices A_j and A_k connected by an edge if and only if they have disjoint support and are anti-amicable. We call this graph the *anti-amicability graph* of the Clifford algebra $\mathbb{R}_{m,m}$. This graph is isomorphic to the red subgraph $\Delta_m[-1]$ of the restricted graph Δ_m described above, and is also isomorphic to the red subgraph $\mathit{\Lambda}_{m,m}[-1]$ of the graph $\mathit{\Lambda}_{m,m}$.

Recall the following.

Definition 3. *[4, 7][8, Chapter 9]. A simple graph Γ of order v is* strongly regular *with parameters (v, k, λ, μ) if*

- *each vertex has degree k,*
- *each adjacent pair of vertices has λ common neighbours, and*
- *each nonadjacent pair of vertices has μ common neighbours.*

It was verified, using *igraph* and the Python *networkx* package [24], that Φ_m is a strongly regular graph for m from 1 to 5. In particular, the *networkx* package was used to verify that the graph Φ_2 is isomorphic to the lattice graph $L(4)$, and not the Shrikhande graph [9, p. 92] [47]. The graph parameters are listed in Table 2.

m	v	k	$\lambda = \mu$
1	4	1	0
2	16	6	2
3	64	28	12
4	256	120	56
5	1024	496	240
m	4^m	$2^{2m-1} - 2^{m-1}$	$2^{2m-2} - 2^{m-1}$

Table 2: Strongly regular graph parameters of Φ_m.

The last line of Table 2 gives a general formula for the parameters in terms of m, suggesting the following.

Theorem 6. *For all $m \geqslant 1$, the graph Φ_m is strongly regular, with parameters $v(m) = 4^m$, $k(m) = 2^{2m-1} - 2^{m-1}$, $\lambda(m) = \mu(m) = 2^{2m-2} - 2^{m-1}$.*

A proof of this theorem is given in the rest of this section.

Hadamard difference sets and bent functions. We first review some well known properties of Hadamard difference sets and bent functions.

Definition 4. *[16, pp. 10, 13].*
The k-element set D is a (v, k, λ, n) difference set *in an abelian group G of order v if for every non-zero element g in G, the equation $g = d_i - d_j$ has exactly λ solutions (d_i, d_j) with d_i, d_j in D. The parameter $n := k - \lambda$. A (v, k, λ, n) difference set with $v = 4n$ is called a* Hadamard difference set.

Remark. *[38] [16, Remark 2.2.7] [45].*
A Hadamard difference set has parameters of the form

$$\begin{aligned}(v, k, \lambda, n) &= (4N^2, 2N^2 - N, N^2 - N, N^2)\\ \textit{or} \quad &(4N^2, 2N^2 + N, N^2 + N, N^2).\end{aligned}$$

Definition 5. *[16, p. 74].*
A Boolean function $f : \mathbb{Z}_2^m \to \mathbb{Z}_2$ is bent *if its Hadamard transform has constant magnitude.*
Specifically:

1. *The Sylvester Hadamard matrix H_m, of order 2^m, is defined by*

$$H_1 := \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix},$$
$$H_m := H_{m-1} \otimes H_1, \quad \textit{for} \quad m > 1.$$

2. *For a Boolean function $f : \mathbb{Z}_2^m \to \mathbb{Z}_2$, define the vector $\underline{f}$ by*

$$\underline{f} := [(-1)^{f[0]}, (-1)^{f[1]}, \ldots, (-1)^{f[2^m-1]}]^T,$$

where the value of $f[i], i \in \mathbb{Z}_{2^m}$ is given by the value of f on the binary digits of i.

3. *In terms of these two definitions, the Boolean function $f : \mathbb{Z}_2^m \to \mathbb{Z}_2$ is bent if*

$$\left|H_m \underline{f}\right| = C[1, \ldots, 1]^T.$$

for some constant C.

Remark. *[16, Theorem 6.2.2]*
The Boolean function $f : \mathbb{Z}_2^m \to \mathbb{Z}_2$ is bent if and only if $D := f^{-1}(1)$ is a Hadamard difference set.

Remark. *[16, Remark 6.2.4]*
Bent functions exist on $\mathbb{Z}_2^m$ only when m is even.

The sign-of-square function σ_m on $\mathbb{Z}_{2^{2m}}$ and $\mathbb{Z}_2^{2m}$. We use the basis element selection function γ_m of Definition 1 to define the sign-of-square function $\sigma_m : \mathbb{Z}_2^{2m} \to \mathbb{Z}_2$ as

$$\sigma_m(i) := \begin{cases} 1 \leftrightarrow \gamma_m(i)^2 = -I \\ 0 \leftrightarrow \gamma_m(i)^2 = I, \end{cases}$$

for all i in $\mathbb{Z}_2^{2m}$. Using the vector notation from Definition 5, we see that $\underline{\sigma_1} = [1, -1, 1, 1]^T$. If we define $\odot : \mathbb{Z}_2 \times \mathbb{Z}_2^{2m-2} \to \mathbb{Z}_2^{2m}$ as concatenation of bit vectors, e.g. $01 \odot 1111 := 011111$, it is easy to verify that

$$\sigma_m(i_1 \odot i_{m-1}) = \sigma_1(i_1) + \sigma_{m-1}(i_{m-1})$$

for all i_1 in $\mathbb{Z}_2$ and i_{m-1} in $\mathbb{Z}_2^{2m-2}$, and therefore $\underline{\sigma_m} = \underline{\sigma_1} \otimes \underline{\sigma_{m-1}}$. Also, since each $\gamma_m(i)$ is orthogonal (from Lemma 3), $\sigma_m(i) = 1$ if and only if $\gamma_m(i)$ is skew.

We are now in a position to prove the following.

Lemma 7. *The function σ_m is a bent function on $\mathbb{Z}_2^{2m}$.*

Proof. Recall that $\underline{\sigma_1} = [1, -1, 1, 1]^T$.

We show that σ_1 is bent by forming

$$H_2[\sigma_1] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix} \begin{bmatrix} 1 \\ - \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix}.$$

Recall that for $m > 1$, $H_{2m} = H_2 \otimes H_{2m-2}$ and $\underline{\sigma_m} = \underline{\sigma_1} \otimes \underline{\sigma_{m-1}}$. Therefore

$$H_{2m}\underline{\sigma_m} = H_2\underline{\sigma_1} \otimes H_{2m-2}\underline{\sigma_{m-1}} = (H_2\underline{\sigma_1})^{(\otimes m)},$$

which has constant absolute value. $\square$

Bent functions and strongly regular graphs Now that we have established that the sign-of-square function σ_m is bent, we complete the proof of Theorem 6 by using a result of Bernasconi and Codenotti [2] on the relationship between bent functions and strongly regular graphs.

First we recall a special case of the definition of a Cayley graph.

Definition 6. *The* Cayley graph *of a binary function $f : \mathbb{Z}_2^m \to \mathbb{Z}_2$ is the undirected graph with adjacency matrix F given by $F_{i,j} = f(g_i+g_j)$, for some ordering $(g_1, g_2, \ldots)$ of $\mathbb{Z}_2^m$.*

The relevant result is the following.

Lemma 8. *[2, Lemma 12]. The Cayley graph of a bent function on Z_2^m is a strongly regular graph with $\lambda = \mu$.*

Remark. *[3, Theorem 3]. Bent functions are the only binary functions on Z_2^m whose Cayley graph is a strongly regular graph with $\lambda = \mu$.*

Proof of Theorem 6. Lemma 7 says that the function σ_m on the canonical basis matrices of $\mathbb{R}_{m,m}$ such that $\sigma_m(A) = 0$ when the matrix A is symmetric and $\sigma_m(A) = 1$ when the matrix A is skew, is a bent function. Lemma 8 then implies that the Cayley graph Θ_m corresponding to this bent function f_m is strongly regular. But this Cayley graph Θ_m is isomorphic to Φ_m, since

$$(\gamma(i)\gamma(j)^{-1})^2 = (\gamma(i)\gamma(j)^T)^2 = \gamma(i+j)^2$$

for all $i, j \in \mathbb{Z}_2^{2m}$. □

4 Amicability / anti-amicability of $\{-1, 1\}$ matrices

Given an n-tuple of A matrices, this fixes $\lambda_{j,k}$. We now must find an n-tuple of $\{-1, 1\}$ B matrices with a complementary graph of amicability and anti-amicability.

We start with a theoretical result which may help our search a little.

Theorem 9. *For anti-amicable pairs of matrices in $\{-1, 1\}^{b \times b}$,*

$$B_1 B_2^T + B_2 B_1^T = 0,$$

therefore $B_1 B_2^T$ is skew, so b must be even.

As a result of this theorem, our interest in odd b is restricted to the cases where $n = 2, 4, 8$, as remarked in Section 2.

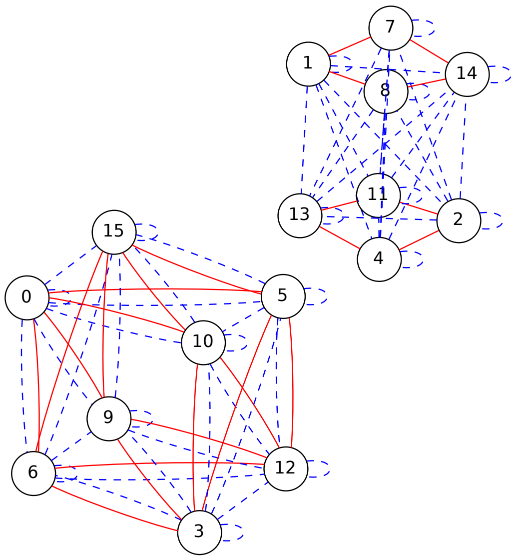

Figure 3: $\{-1, 1\}$ matrices of order 2, solid: anti-amicable, dashed: amicable.

Example: 2×2 matrices. An exhaustive search over the $16 \times 15 = 240$ distinct multisets of 2 matrices chosen from the 16 matrices of the form $\{-1, 1\}^{2 \times 2}$, with no constraints on the sum of the Gram matrices, reveals the amicability / anti-amicability relationships seen in the graph of Figure 3 (plotted using the *Graphviz neato* program [18]).

Here, each of the 16 matrices is given a numbered vertex, from 0 to 15. The map from a number a to a matrix A is obtained via numbering the element positions of the matrix as

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}.$$

The number a is then written in binary, with each position in the matrix A being given the value $(-1)^b$ where b is the corresponding bit. For example, the number 4 yields the matrix

$$\begin{bmatrix} 1 & 1 \\ - & 1 \end{bmatrix}.$$

In Figure 3, a dashed edge between vertices corresponds to an amicable pair. A solid edge corresponds to an anti-amicable pair. Each matrix is amicable with itself, so each vertex has a dashed loop attached. The graph has two connected components. One component is the complete graph K_8 with self-loops, with two edge-colours. This represents a set of 8 matrices that are pairwise either amicable or anti-amicable. The other component includes a number of double edges, representing pairs of matrices that are both amicable and anti-amicable. An example of such a pair is

$$M := \begin{bmatrix} - & 1 \\ 1 & - \end{bmatrix}, \quad N := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

where $MN^T = 0$.

In Figure 4 (plotted using the *Graphviz neato* program [18]) the edges are restricted to those pairs of matrices where $B_j B_j^T + B_k B_k^T = 4I_{(2)}$.

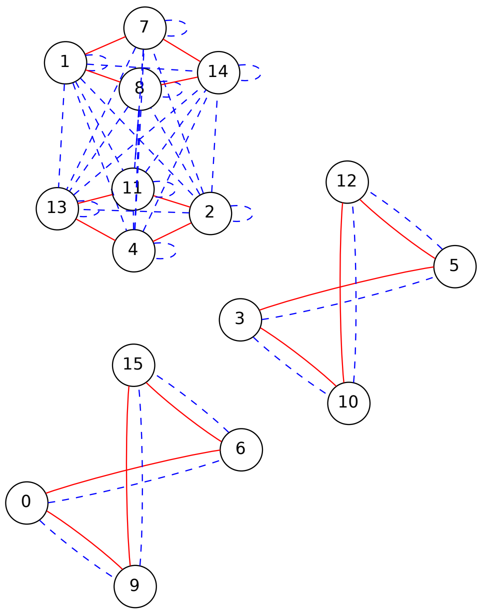

Figure 4: $\{-1, 1\}$ matrices of order 2, $B_j B_j^T + B_k B_k^T = 4I_{(2)}$.

The two-edge-coloured K_8 still appears, but the other component is now split into two. These two new components each consist of 4 matrices, where each matrix is both amicable and anti-amicable with two other matrices. Our previous example is also an example here, since

$$MM^T = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, \quad NN^T = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix},$$

so

$$MM^T + NN^T = 4I_{(2)}.$$

The vertices of the K_8 still have self-loops, indicating that the corresponding 8 matrices are Hadamard. This K_8 of Hadamard matrices is remarkable, but it occurs for a simple reason, given by Theorem 10 below. The vertices of the other two components do not have self-loops.

Multisets of 4 matrices. An exhaustive search over the $\binom{19}{4} = 3876$ distinct multisets of 4 matrices of type $\{-1, 1\}^{2\times 2}$ for multisets where each pair is either amicable or anti-amicable, and where the sum of the 4 Gram matrices is $8I_{(2)}$, results in 618 qualifying multisets, yielding seven isomorphism classes of graphs, as shown in Figure 5 (plotted using the *Graphviz dot* program [18]).

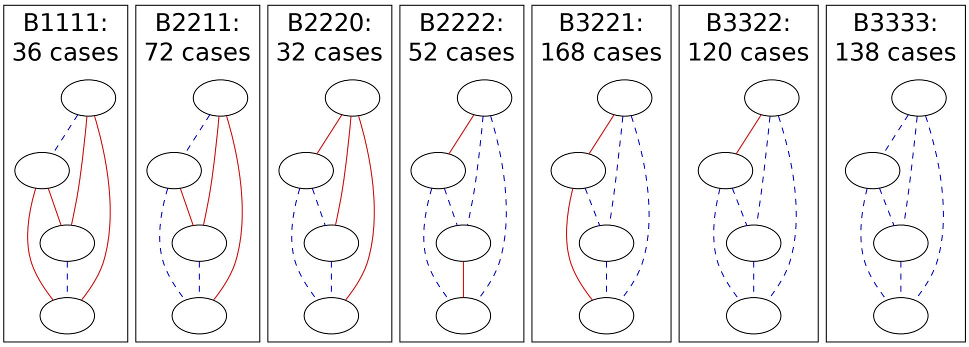

Figure 5: 4-multisets of B matrices of order 2, solid: $\lambda_{j,k} = -1$, dashed: $\lambda_{j,k} = 1$.

A naive algorithm was used to obtain the graphs in Figure 5. First, a canonical ordering was given for both the 4 vertices and the 6 edges of the graph K_4, implying an ordering of the 64 possible two-edge-colourings of this graph. Then, a canonical ordering was given for the 16 matrices of type $\{-1, 1\}^{2\times 2}$, implying an ordering for the 3876 possible multisets. For each of the 64 possible two-edge-colourings, each of the 3876 possible multisets was checked: firstly that the amicability and anti-amicability relations of the four matrices matched the two-edge-colouring, and secondly, that the Gram sum was $8I_{(2)}$. The number of matches was counted for each of the 64 possible two-edge-colourings. The two-edge-colourings were then combined into the isomorphism classes shown in Figure 5, by examining the corresponding degree sequences. The code to implement the algorithm was written in Octave, and is available from the author's web page [35]. It takes less than a second to run on an Intel® Core™ i7 870 CPU at 2.93 GHz.

In each box of Figure 5, the complete graph K_4 is given two colours. The vertices correspond to 4 B matrices. If $\lambda_{j,k} = -1$, then the edge between the vertices corresponding to matrices B_j and B_k is coloured red and is given a solid line. If $\lambda_{j,k} = 1$, then the edge between the vertices corresponding to matrices B_j and B_k is coloured blue and is given a dashed line. The name 'Babcd' in each box corresponds to the degree sequence with respect to dashed edges.

Graphs B1111 and B2222 are dual with respect to the exchange of solid and dashed edges, and graph B2211 is self-dual.

Note that only graphs B1111, B2222, B3221 and B3333 are colour complementary to graphs of A matrices in Figure 1. The pairs of graphs (A1111,B1111), (A2222,B2222), (A3221,B3221), and (A3333,B3333) result in Hadamard matrices of order 8.

The existence of complete two-edge-coloured subgraphs. As was noted above, the amicability / anti-amicability graph of the matrices in $\{-1,1\}^{2\times 2}$ contains a complete two-edge-coloured subgraph containing 8 Hadamard matrices. This is a general phenomenon, as shown by the following theorem.

Theorem 10. *If b is a power of 2, $b = 2^m$, $m \geqslant 0$, the amicability / anti-amicability graph P_b of the matrices $\{-1,1\}^{b\times b}$ has the following properties.*

1. *The graph P_b contains a complete two-edge-coloured graph on $2b^2$ vertices with each vertex being a Hadamard matrix. This graph is isomorphic to $\mathit{\Gamma}_{m,m}$, the amicability / anti-amicability graph of the group $\mathbb{G}_{m,m}$.*

2. *Call two Hadamard matrices, H and H' of order b,* Hadamard-row-equivalent *if there exists a signed permutation matrix S of order b such that $H' = SH$. If $r(b)$ is the number of Hadamard-row-equivalence classes of Hadamard matrices of order b, and $s(b)$ is the order of the group of signed permutation matrices of order b, then the graph P_b contains at least $r(b)s(b)/(2b^2)$ isomorphic copies of the graph $\mathit{\Gamma}_{m,m}$.*

Proof. First, some notation. Let $\mathbb{S}_b$ be the group of signed permutation matrices of order b. The real representation of the Clifford algebra $\mathbb{R}_{m,m}$ has a canonical basis consisting of b^2 matrices in $\mathbb{S}_b$. These matrices and their negatives form the group $\mathbb{G}_{m,m}$, as a subgroup of $\mathbb{S}_b$.

We first prove statement 1. Since $b = 2^m$, it is well-known that b is a Hadamard order. Choose a Hadamard matrix H of order b and any signed permutation matrix $S \in \mathbb{S}_b$. Since S is orthogonal, we have

$$SH(SH)^T = SHH^TS^T = SpI_{(b)}S^T = pI_{(b)},$$

so SH is a Hadamard matrix. This is well known.

Now take A_1, A_2 to be members of the canonical matrix basis of the real representation of the Clifford algebra $\mathbb{R}_{m,m}$. Thus A_1 and A_2 are elements of $\mathbb{S}_b$. If

$$A_1A_2^T + \lambda_{1,2}A_2A_1^T = 0,$$

then

$$(A_1H)(A_2H)^T = pA_1A_2^T = -\lambda_{1,2}pA_2A_1^T = -\lambda_{1,2}(A_2H)(A_2H)^T,$$

that is, the Hadamard matrices A_1H and A_2H have the same amicability relationship as the matrices A_1 and A_2. The same argument applies to any combination of $\pm A_1$ and $\pm A_2$. Thus the set of $2b^2$ Hadamard matrices

$$\mathbb{G}_{m,m}H := \{AH \mid A \in \mathbb{G}_{m,m}\}$$

has an amicability / anti-amicability graph isomorphic to $\mathit{\Gamma}_{m,m}$.

Now for statement 2. The group $\mathbb{G}_{m,m}$ has $s(b)/(2b^2)$ disjoint cosets within $\mathbb{S}_b$. For $S, T \in \mathbb{S}_b$ with disjoint cosets $\mathbb{G}_{m,m}S$, $\mathbb{G}_{m,m}T$, and for some Hadamard matrix H of order b, consider the two sets

$$\mathbb{G}_{m,m}SH := \{ASH \mid A \in \mathbb{G}_{m,m}\},$$
$$\mathbb{G}_{m,m}TH := \{ATH \mid A \in \mathbb{G}_{m,m}\}.$$

These two sets are disjoint, since the corresponding cosets are disjoint and H is invertible. Using the argument from the proof of statement 1, we see that each set yields an amicability / anti-amicability graph isomorphic to the graph $\mathit{\Gamma}_{m,m}$.

The union of all of these $s(b)/(2b^2)$ disjoint cosets is the group $\mathbb{S}_b$ itself, and the set $\mathbb{S}_bH$ is the Hadamard-row-equivalence class containing H. Now repeat the argument for representatives of each of the $r(b)$ equivalence classes. Each class yields $s(b)/(2b^2)$ disjoint cosets, giving a total of $r(b)s(b)/(2b^2)$ disjoint sets of Hadamard matrices, each of which yields a graph isomorphic to $\mathit{\Gamma}_{m,m}$. □

Corollary 11. *For $b = 2^m$, the red subgraph $P_b[-1]$ contains at least $r(b)s(b)/(2b^2)$ isomorphic copies of the graph $\mathit{\Gamma}_{m,m}[-1]$, the double graph of the graph Φ_m.*

In the light of Theorem 10, we now re-examine the case $b = 2$. A matrix in $\{-1, 1\}^{2\times2}$ has only two rows and is singular if the second row is $\pm$ the first row. There are 4 possible assignments of -1 and 1 to the first row, and thus 8 of the 16 matrices of $\{-1, 1\}^{2\times2}$ are singular. The remaining 8 matrices are Hadamard, of the form SH, where S is one of the $2^2 \times 2! = 8$ signed permutation matrices of $\mathbb{S}_2$, and H is the representative matrix

$$H := \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}.$$

These 8 matrices form a Hadamard row equivalence class. All of this is well-known.

Theorem 10 says that the graph P_2 contains at least $r(2)s(2)/(2 \times 2^2)$ isomorphic copies of the amicability / anti-amicability graph $\mathit{\Gamma}_{1,1}$. Here, $r(2) = 1$ is the number of Hadamard row equivalence classes, and $s(2) = 8$ is the order of $\mathbb{S}_2$. Thus the theorem says that the graph P_2 contains at least one isomorphic copy of $\mathit{\Gamma}_{1,1}$. Our exhaustive search has found the only such copy.

Example: 4×4 matrices. In the case $b = 4$, we can form representatives of two distinct Hadamard row equivalence classes as follows.

$$H := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & - & 1 \\ 1 & - & 1 & - \end{bmatrix}, \quad H' := \begin{bmatrix} - & 1 & 1 & 1 \\ - & 1 & - & - \\ - & - & - & 1 \\ - & - & 1 & - \end{bmatrix}.$$

It is clear that there is no signed permutation matrix $S \in \mathbb{S}_4$ such that $SH = H'$, because the number of -1 entries in each row of H is even, the number in each row of H' is odd, and the total number of $\{-1, 1\}$ entries in each row of each matrix is even. Therefore, with respect to Theorem 10, $r(4)$ is at least 2, and $s(4)$, the order of $\mathbb{S}_4$ is $2^4 \times 4! = 16 \times 24 = 384$. Theorem 10 therefore says that the amicability / anti-amicability graph P_4 contains at least $2 \times 384/32 = 24$ isomorphic copies of the graph $\mathit{\Gamma}_{2,2}$, a complete two-edge-coloured graph on 32 vertices. Corollary 11 says that the anti-amicability graph $P_4[-1]$ contains at least 24 copies of the double graph of Φ_2.

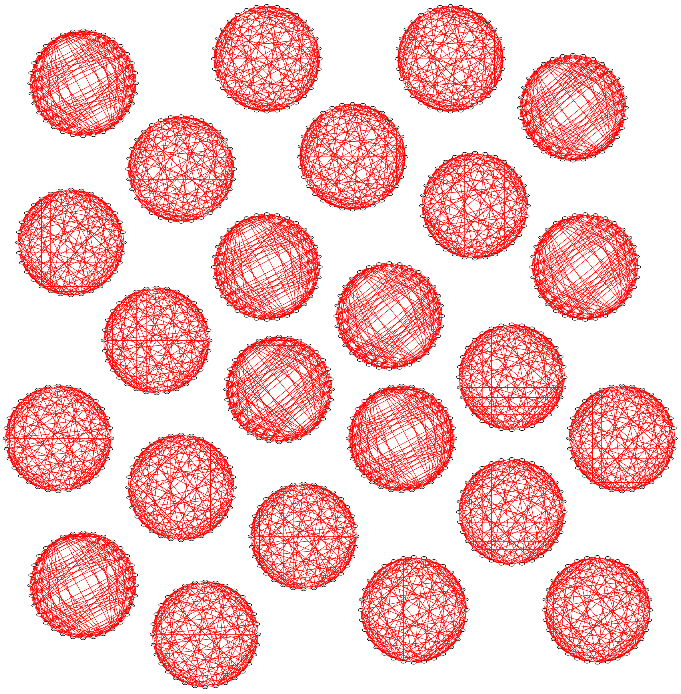

Figure 6: Hadamard matrices of order 4, edges denote anti-amicability.

An exhaustive search over the $65\,536 \times 65\,535 = 4\,294\,901\,760$ distinct 2-multisets of matrices of the form $\{-1, 1\}^{4\times 4}$ was undertaken to find anti-amicable pairs of Hadamard matrices of order 4. The algorithm used to obtain these 2-multisets was essentially the same as that used in the 2×2 case above. The search program was a modified version of the Octave program used in the 2×2 case, run as 16 parallel jobs on a cluster of AMD Opteron 2356 CPUs, each running at 2.3 GHz, taking a total of about 260 CPU hours. Figure 6 (plotted using the *Graphviz circo* program [18]) shows the pairwise anti-amicability relationships between these Hadamard matrices. The graph, a subgraph of the graph $P_2[-1]$, contains 24 connected components, each of which is 12-regular on 32 vertices, in agreement with Corollary 11 and the properties of the graph $\Gamma_{2,2}$ described in Section 3. Specifically, as a consequence of Corollary 11, each component is the double graph of the graph Φ_2, which was identified using *networkx* as the lattice graph $L(4)$.

The search criterion was then relaxed to look for anti-amicable 2-multisets of matrices B_j, B_k of the form $\{-1, 1\}^{4\times 4}$ where $B_j B_j^T + B_k B_k^T = 4I_{(2)}$.

The resulting graph was then examined using the open source *Gephi* package

[1]. This package reported that the graph contains 20 352 vertices and 36 864 edges, with 3552 connected components, comprising 2304 components with 4 vertices each (isomorphic to $K_{2,2}$), 1152 with 8 vertices (isomorphic to $K_{4,4}$), 72 with 16 vertices (isomorphic to $K_{8,8}$), and 24 components with 32 vertices. These last 24 components are the 24 copies of the double graph of $\Phi_2 \equiv L(4)$ seen in Figure 6.

Proof of Theorem 5. As a result of Theorem 10, we know that if $n = 2^m$, for every n-tuple $(A_1, \ldots, A_n)$ of matrices given by a transversal of the canonical matrix basis of the Clifford algebra $\mathbb{R}_{m,m}$, there is an n-tuple $(B_1, \ldots, B_n)$ of distinct Hadamard matrices of order n, with an amicability / anti-amicability graph isomorphic to that of $(A_1, \ldots, A_n)$. The conditions (4) require instead that an n-tuple of B matrices be found with an amicability / anti-amicability graph *edge-colour-complementary* to that of $(A_1, \ldots, A_n)$. One way to do this stems from the following result.

Lemma 12. *(See also Gastineau-Hills [20, Theorem 3.4].)*
Given an n-tuple of $\{-1,1\}^{b\times b}$ matrices $(B_1, \ldots, B_n)$ satisfying

$$B_j B_k^T = \lambda_{j,k} B_k B_j^T \quad (j \neq k), \qquad \sum_{k=1}^{n} B_k B_k^T = nbI_{(b)},$$

and an n-tuple of Hadamard matrices $(C_1, \ldots, C_n)$ of order c, satisfying

$$C_j C_k^T = \mu_{j,k} C_k C_j^T \quad (j \neq k),$$

the n-tuple of matrices $(B_1 \otimes C_1, \ldots, B_n \otimes C_n)$ satisfies

$$(B_j \otimes C_j)(B_k \otimes C_k)^T = \lambda_{j,k}\mu_{j,k}(B_k \otimes C_k)(B_j \otimes C_j)^T \quad (j \neq k),$$
$$\sum_{k=1}^{n}(B_k \otimes C_k)(B_k \otimes C_k)^T = nbcI_{(bc)}.$$

Proof. For $(j \neq k)$ we have

$$\begin{aligned}(B_j \otimes C_j)(B_k \otimes C_k)^T &= (B_j B_k^T) \otimes (C_j C_k^T)\\ &= (\lambda_{j,k} B_k B_j^T) \otimes (\mu_{j,k} C_k C_j^T)\\ &= \lambda_{j,k}\mu_{j,k}(B_k \otimes C_k)(B_j \otimes C_j)^T.\end{aligned}$$

Also,

$$\begin{aligned}\sum_{k=1}^{n}(B_k \otimes C_k)(B_k \otimes C_k)^T &= \sum_{k=1}^{n}(B_k B_k^T) \otimes (C_k \otimes C_k)^T\\ &= \sum_{k=1}^{n}(B_k B_k^T) \otimes cI_{(c)}\\ &= nbI_{(b)} \otimes cI_{(c)} = nbcI_{(bc)}.\end{aligned}$$

□

In particular, if the n-tuple of Hadamard matrices $(C_1, \ldots, C_n)$ are mutually anti-amicable then $\mu_{j,k} = -1$ and the matrices $(B_1 \otimes C_1, \ldots, B_n \otimes C_n)$ have an amicability / anti-amicability graph that is edge-colour-complementary to that of $(B_1, \ldots, B_n)$. All that is left is to find n mutually anti-amicable Hadamard matrices. To do this, we use the argument given in the proof of Theorem 10: given a Hadamard matrix H of order c, and n mutually anti-amicable signed permutation matrices $(S_1, \ldots, S_n)$ of the same order, $(S_1 H, \ldots, S_n H)$ is an n-tuple of mutually anti-amicable Hadamard matrices.

We now use the following observation:

Lemma 13. *In the frame group $\mathbb{G}_{0,n-1}$, the identity and the $n-1$ generators $\mathbf{e}_{\{1-n\}}, \ldots, \mathbf{e}_{\{-1\}}$ are mutually anti-amicable.*

Proof. Recall that a real monomial representation of a signed group consists of signed permutation matrices, which are orthogonal. Thus the anti-amicability relationship is expressed in the context of the group $\mathbb{G}_{0,n-1}$ as

$$a_j a_k^{-1} = -a_k a_j^{-1}, \quad \text{equivalently,} \quad (a_j a_k^{-1})^2 = -1.$$

If we set $a_n = 1$, and $a_k = \mathbf{e}_{\{-k\}}$ for k from 1 to $n-1$, then

$$\begin{aligned}
(a_n a_k^{-1})^2 &= (\mathbf{e}_{\{-k\}}^{-1})^2 = \mathbf{e}_{\{-k\}}^2 = -1. \\
(a_j a_k^{-1})^2 &= \mathbf{e}_{\{-j\}} \mathbf{e}_{\{-k\}}^{-1} \mathbf{e}_{\{-j\}} \mathbf{e}_{\{-k\}}^{-1} \\
&= \mathbf{e}_{\{-j\}} \mathbf{e}_{\{-k\}} \mathbf{e}_{\{-j\}} \mathbf{e}_{\{-k\}} = -\mathbf{e}_{\{-j\}} \mathbf{e}_{\{-j\}} \mathbf{e}_{\{-k\}} \mathbf{e}_{\{-k\}} = -1.
\end{aligned}$$

□

We now recall the following result from [34]. See that paper for the proof.

Lemma 14. *[34, Theorem 4.3] Define*

$$M(p,q) := \begin{cases} \lceil \frac{p+q}{2} \rceil + 1, & \textit{if } q - p \equiv 2, 3, 4 \pmod 8, \\ \lceil \frac{p+q}{2} \rceil & \textit{otherwise.} \end{cases}$$

There is a faithful real monomial representation of the Clifford algebra $\mathbb{R}_{p,q}$ where the matrices have order $2^{M(p,q)}$.

This gives us the result we need.

Corollary 15. *The set of $-1, 1$ matrices of order $2^{M(0,n-1)}$ contains an n-tuple of mutually anti-amicable Hadamard matrices.*

Thus, if n is a power of 2, $n = 2^m$, and an n-tuple $(A_1, \ldots, A_n)$ of $\{-1, 0, 1\}^{n \times n}$ matrices is obtained by taking a transversal of the canonical basis matrices for $\mathbb{R}_{m,m}$, there is an algorithm to construct an n-tuple of matrices $(B_1, \ldots, B_n)$ in $\{-1, 1\}^{b \times b}$ with matching λ:

1. Find a Hadamard matrix H of order n. Since n is a power of 2, the Sylvester Hadamard matrix will do.

2. Form the n-tuple $(A_1H, \ldots, A_nH)$. This has the same amicability / anti-amicability graph as $(A_1, \ldots, A_n)$.

3. Form the n-tuple $(C_1, \ldots, C_n)$ of matrices in $\{-1, 0, 1\}^{c \times c}$, where $c = 2^{M(0,n-1)}$, the matrices $C_1, \ldots, C_{n-1}$ are the canonical signed permutation matrices corresponding to the Clifford algebra generators $\mathbf{e}_{\{-1\}}, \ldots, \mathbf{e}_{\{1-n\}}$, and the matrix $C_n = I_{(c)}$. By Lemma 13, these n matrices are mutually anti-amicable.

4. By Lemma 12 the n-tuple of Hadamard matrices $(B_1, \ldots, B_n) = ((A_1H) \otimes C_1, \ldots, (A_nH) \otimes C_n)$ of order nc matches the λ values of $(A_1, \ldots, A_n)$, satisfying conditions (4), and completes the constructions (G0) and (H0).

This completes the proof.

5 Discussion

Historical context. Much of the credit for the following historical discussion goes to an anonymous reviewer of an early draft of this paper.

The current paper describes and investigates one of a long line of plug-in constructions for Hadamard matrices, extending at least as far back as Williamson [50]. The review paper by Seberry and Yamada [46] describes many more of these constructions, especially Williamson-type constructions [46, p. 445 and Sections 8 and 9]. While it is a long and comprehensive review, with a special focus on orthogonal designs and amicability, the paper of Seberry and Yamada does not discuss anti-amicability, or mention the work of Gastineau-Hills [19, 20].

The constructions (G0) and (H0) with conditions (4) can be viewed as a generalization of a plug-in construction using an orthogonal design of the form $OD(n; 1, \ldots, 1)$ with n suitable mutually amicable $\{-1, 1\}$ matrices. The difference between that construction and the one in the current paper is that the matrices used to define the orthogonal design are mutually anti-amicable, but conditions (4) use the parameters $\lambda_{j,k}$ to allow each pair of matrices to be either amicable or anti-amicable.

In his paper of 1982 Gastineau-Hills describes Kronecker products of systems of orthogonal designs [20, Theorem 3.4]. This is essentially the published version of the concepts and results of his Ph.D. thesis [19, see especially Theorem 6.3, p. 47]. The constructions (G0) and (H0) with conditions (4) can be viewed as being similar to a special case of the Kronecker product construction of Gastineau-Hills. Specifically, in Section 3 it is mentioned that conditions (4) make the n-tuple of A matrices into a special case of a regular n-system of orthogonal designs, of order n, genus $(\delta_{j,k})$, type $(1; \ldots; 1)$, with $p_1 = \ldots = p_n = 1$, with $\lambda_{j,k} = (-1)^{(1+\delta_{j,k})}$, in the notation of Gastineau-Hills [19]. The 1982 paper [20] uses this same notation, and Theorem 3.4 in that paper gives a Kronecker product construction for systems of orthogonal designs that can be made into a special case of the constructions (G0) and (H0) with conditions (4). In particular, if we also use an n-system of orthogonal designs of order b, genus $(1 - \delta_{j,k})$, type $(b; \ldots; b)$, with $p_1 = \ldots = p_n = 1$, and set all of the variables $x_{i,1}$ to 1, then we obtain an n-tuple of Hadamard matrices $(B_1, \ldots, B_n)$

with λ matching that of our n-tuple of A matrices. We then use [20, Theorem 3.4] with $r = 2$ to complete the construction and obtain a Hadamard matrix.

The differences between the construction of [20, Theorem 3.4] and constructions (G0) and (H0) with conditions (4) are:

1. Conditions (4) ensure that the A matrices have disjoint support. This is stronger than just being a regular n-system.

2. Conditions (4) impose a constraint on the Gram sum of the B matrices rather than constraining them to be Hadamard matrices. This is weaker than the n-system constraint of Gastineau-Hills [20].

Gastineau-Hills' paper [20] cites a construction by Robinson of *product designs* [44] as an example of the more general construction of Theorem 3.4.

Part II of Gastineau-Hills' thesis [19] consists of a thorough classification of *quasi-Clifford* algebras, each of which is essentially a direct sum of 2^k copies of a Clifford algebra for some $k \geqslant 0$. In Section 3 we remark that the papers by de Launey and Smith [15], and de Launey and Kharaghani [14], as well as Chapters 22 and 23 of de Launey and Flannery [13], examine the finite groups underlying the Clifford algebras in some detail. In these papers there is described the set of finite groups $R(Q)$. The structure, and in particular, the power-commutator presentations of these groups suggest that these are the groups underlying Gastineau-Hills' quasi-Clifford algebras. A deeper examination of the relationship between the $R(Q)$ groups and the quasi-Clifford algebras has not yet been undertaken.

A 1981 paper by Hammer and Seberry [25] mentions anti-amicability and produces the following isolated example.

> If X and Y are anti-amicable and
>
> $$XX^T + 5YY^T = 6nI_{(n)}$$
>
> then
>
> $$Z := \begin{bmatrix} X & Y & Y & Y & Y & Y \\ Y & X & Y & -Y & -Y & Y \\ Y & Y & X & Y & -Y & -Y \\ Y & -Y & Y & X & Y & -Y \\ Y & -Y & -Y & Y & X & Y \\ Y & Y & -Y & -Y & Y & X \end{bmatrix}$$
>
> satisfies
>
> $$ZZ^T = 6nI_{(6n)}.$$

– Hammer and Seberry [25, p. 183].

This particular construction can be seen as a generalization of constructions (G0) and (H0) with conditions (4). Here the A matrices are $A_1 := I_{(6)}$ and

$$A_2 := \begin{bmatrix} . & 1 & 1 & 1 & 1 & 1 \\ 1 & . & 1 & - & - & 1 \\ 1 & 1 & . & 1 & - & - \\ 1 & - & 1 & . & 1 & - \\ 1 & - & - & 1 & . & 1 \\ 1 & 1 & - & - & 1 & . \end{bmatrix}$$

so that A_2 is not monomial. The relevant generalization of this construction is

$$G := \sum_{k=1}^{r} B_k \otimes A_k, \tag{G0'}$$

with conditions

$$\begin{aligned} A_j * A_k = 0 \quad (j \neq k), \quad \sum_{k=1}^{r} A_k &\in \{-1, 1\}^{n \times n}, \\ A_k A_k^T = a_k I_{(n)}, \quad \sum_{k=1}^{r} a_k &= n, \\ A_j A_k^T + \lambda_{j,k} A_k A_j^T &= 0 \quad (j \neq k), \\ B_j B_k^T - \lambda_{j,k} B_k B_j^T &= 0 \quad (j \neq k), \\ \lambda_{j,k} &\in \{-1, 1\}, \\ \sum_{k=1}^{n} B_k B_k^T &= nbI_{(b)}, \end{aligned} \tag{4'}$$

which is a generalization of the construction (G0) with conditions (4). The paper [25] does not mention the work of Gastineau-Hills, and does not examine n-tuples of anti-amicable matrices for n larger than 2. This provides some incentive to re-do the analysis of the current paper in the context of this, more general construction.

Questions. We have shown that for the constructions (G0) and (H0) with conditions (4), to construct the A matrices, it is sufficient that n is a power of 2, and that the real monomial representations of Clifford algebras can be used in this case. The following related questions arise.

Question 1. *By Theorem 5, if n is a power of 2 and an n-tuple of A matrices is obtained by taking a transversal of the canonical basis matrices for $\mathbb{R}_{m.m}$ an n-tuple of B matrices can always be found to complete the constructions (G0) and (H0), but the order of the B matrices constructed in the proof of the theorem is quite large. Can this order be improved?*

Recall that, for $n = 2^m$, by Theorem 10, the set $\{-1, 1\}^{n\times n}$ contains at least one isomorphic copy of the whole graph $\Gamma_{m,m}$, where each vertex is a Hadamard matrix. For the constructions (G0) and (H0) to work for some particular n-tuple $(A_1, \ldots, A_n)$, all that is needed is that the full amicability / anti-amicability graph $\Gamma_{m,m}$ of the group $\mathbb{G}_{m,m}$ contains a subgraph on n vertices that is edge-colour-complementary to that of $(A_1, \ldots, A_n)$. This would then imply that there was at least one isomorphic copy of this subgraph, whose vertices are Hadamard matrices of order n. These vertices would be the n-tuple of B matrices needed to complete the constructions (G0) and (H0).

Now recall that in the cases where $m = 1, 2, 3$ we found a permutation of the canonical matrix basis of the Clifford algebra $\mathbb{R}_{m,m}$ that mapped each such transversal graph onto its edge-colour-complement (Property 1 in Section 3 above). This implies the weaker Property 3: "For the Clifford algebra $\mathbb{R}_{m,m}$, if a graph T exists amongst the transversal graphs, then so does at least one graph with edge colours complementary to those of T." If Property 3 is true for all $m \geq 1$, this is sufficient to complete the constructions (G0) and (H0) with an n-tuple of B matrices of order n. This provides some motivation for the following.

Conjecture. *If n is a power of 2, the constructions (G0) and (H0) with conditions (4) can always be completed, in the following sense. If an n-tuple of A matrices which produce a particular λ is obtained by taking a transversal of canonical basis matrices of the Clifford algebra $\mathbb{R}_{m,m}$, an of n-tuple of B matrices* of order n *with a matching λ can always be found.*

Question 2. *For the constructions (G0) and (H0) with conditions (4), is it* necessary *that n is a power of 2 [19, Chapters 16, 17]?*

We recap conditions (4), splitting these into sub-conditions for closer examination.

$$A_j * A_k = 0 \quad (j \neq k), \quad \sum_{k=1}^{n} A_k \in \{-1, 1\}^{n\times n}, \tag{4a}$$

$$A_k A_k^T = I_{(n)}, \tag{4b}$$

$$A_j A_k^T + \lambda_{jk} A_k A_j^T = 0 \quad (j \neq k), \tag{4c}$$

So, each A_k is a signed permutation matrix. If we multiply each A_k on the left by some fixed signed permutation matrix S, we permute and change the signs of the all the corresponding rows of each A_k, so (4a) is still satisfied. Since $SS^T = I_{(n)}$, (4b) and (4c) are also satisfied, and in particular, multiplication by S does not affect the values of $\lambda_{j,k}$ in (4c). Similarly, if we multiply each A_k on the right by S. We therefore have an equivalence class of n-tuples under these two types of transformation, and without loss of generality, can set $A_1 = I_{(n)}$. In this representative case, each of the other $A_k, k > 1$ must be symmetric or skew, with zero diagonal.

If we now take a linear combination of the corresponding permutation matrices $P_k = |A_k|$, we have a *symmetric Latin square with constant diagonal.* This type of

Latin square must have even order. Sequence A003191 in Sloane's Online Encyclopedia of Integer Sequences [48] lists the number of such Latin squares for each even order. The entire listed sequence is

$$1, 1, 6, 5972, 1\,225\,533\,120,$$

corresponding to orders 2, 4, 6, 8 and 10, respectively. The sole examples of orders 2 and 4 can be obtained via the Clifford algebra representation, as per Figure 7.

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \qquad \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}$$

Figure 7: Symmetric Latin squares with constant diagonal: orders 2 and 4.

The 6 cases of order 6 are as per Figure 8.

$$\begin{bmatrix} a & b & c & d & e & f \\ b & a & f & e & c & d \\ c & f & a & b & d & e \\ d & e & b & a & f & c \\ e & c & d & f & a & b \\ f & d & e & c & b & a \end{bmatrix} \begin{bmatrix} a & b & c & d & e & f \\ b & a & f & c & d & e \\ c & f & a & e & b & d \\ d & c & e & a & f & b \\ e & d & b & f & a & c \\ f & e & d & b & c & a \end{bmatrix} \begin{bmatrix} a & b & c & d & e & f \\ b & a & e & c & f & d \\ c & e & a & f & d & b \\ d & c & f & a & b & e \\ e & f & d & b & a & c \\ f & d & b & e & c & a \end{bmatrix}$$
$$\begin{bmatrix} a & b & c & d & e & f \\ b & a & e & f & d & c \\ c & e & a & b & f & d \\ d & f & b & a & c & e \\ e & d & f & c & a & b \\ f & c & d & e & b & a \end{bmatrix} \begin{bmatrix} a & b & c & d & e & f \\ b & a & d & e & f & c \\ c & d & a & f & b & e \\ d & e & f & a & c & b \\ e & f & b & c & a & d \\ f & c & e & b & d & a \end{bmatrix} \begin{bmatrix} a & b & c & d & e & f \\ b & a & d & f & c & e \\ c & d & a & e & f & b \\ d & f & e & a & b & c \\ e & c & f & b & a & d \\ f & e & b & c & d & a \end{bmatrix}$$

Figure 8: Symmetric Latin squares with constant diagonal: order 6.

Recalling condition (4c),

$$A_j A_k^T + \lambda_{jk} A_k A_j^T = 0 \quad (j \neq k),$$

we see that $A_j A_k^T$ must either be symmetric or skew, and so each corresponding product of permutation matrices $P_j P_k^T$ for our representative case must be symmetric, for each pair $j, k > 1$. If we enumerate all six cases of symmetric Latin squares of order 6 with constant diagonal, we find that *none* of these cases yields permutation matrices P_2, P_3 with $P_2 P_3^T$ symmetric.

For general even order n, we see that there must be a set of $n-1$ permutation matrices which each represent a fixed-point-free involution on the set of n symbols, and that all $n-1$ of these involutions must commute. Further, each product $P_j P_k =$

$P_j P_k^T$ must again be a fixed-point-free involution, because the supports of P_j and P_k are disjoint.

A deeper analysis of the general case is yet to be performed, although it is quite obvious that the more general construction (G0') with conditions (4') does allow $n = 6$ as per the example of Hammer and Seberry [25].

Question 3. *Based on the tables listed in the Masters and PhD theses of Ó Catháin [39, 40], the frame groups $\mathbb{G}_{m,m}$ we use to construct our A matrices are not Hadamard groups in the sense of Ito [31, 17], yet these frame groups arise naturally in the work of de Launey and Smith [15, Section 2], de Launey and Kharaghani [14, Section 2.2], and de Launey and Flannery [13]. What is the reason for this seeming discrepancy, and are there cases where the construction described in the current paper does not give a matrix equivalent to a cocyclic Hadamard matrix?*

This question is yet to be addressed.

Prospects. The matrices of $\{-1, 1\}^{2\times 2}$ were investigated via an exhaustive search using naive methods. A search for pairs of $\{-1, 1\}^{4\times 4}$ matrices was also conducted, but no attempt was made to obtain larger n-tuples. To investigate higher orders b and larger n-tuples, a more sophisticated strategy is needed. Perhaps the way to proceed is to first find multisets of size n of B matrices obeying the Gram constraint of (4), and then examine the multiset for pairwise amicability and anti-amicability. The methods of Osborn [41] and some of the software techniques of Brent [6] could be used as the basis for such a search. Surely, more work is needed before the graphs of amicability and anti-amicability can be truly said to be well understood.

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