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Total acquisition on grids*

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Abstract

On a weighted graph G, a *total acquisition move* transfers weight from a vertex u to a neighbor v provided that the weight on v is at least as much as the weight on u. Starting with all vertices having weight 1, the *total acquisition number of* G, denoted $a_t(G)$, is the minimum number of vertices with positive weight after a sequence of total acquisition moves. In [D. Lampert and P. Slater, *Congr. Numer.* 109 (1995), 203–210] it is shown that $a_t(G) \geq \lceil |V(G)|/2^{\Delta(G)} \rceil$ for all G, and $P_5 \Box P_5$ is given as an example where this bound is not sharp. In this paper, we determine $a_t(P_n \Box P_m)$ exactly when n and m are not 5 and give nontrivial upper and lower bounds on $a_t(P_n \Box P_5)$.

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1 Introduction

Consider a collection of military bases, some of which are joined by roads. If there are troops at each of the bases, we consider the following model for withdrawing the troops. If two bases, u and v are joined by a road, then the troops at u can move to v only if there are at least as many troops at v as there are at u. Furthermore, if troops move from u to v, then all of the troops at u must move to v simultaneously. The obvious goal in such a withdrawal model is to minimize the number of bases that have troops.

Let G be a graph with weights assigned to its vertices. A *total acquisition move* on G moves all of the weight from a vertex u to a vertex v provided that u and v are adjacent and the weight on v is at least the weight on u. The *total acquisition number* of G, denoted $a_t(G)$, is the minimum number of vertices with positive weight after a sequence of total acquisition moves, beginning with the weight assignment where all vertices have weight 1. A sequence of total acquisition moves on G that realizes $a_t(G)$ is *optimal.* More generally, an *acquisition move* on a vertex-weighted graph moves weight from u to a neighbor v provided that the weight on v is at least the weight on u. If the moves permitted allow only integer amounts of weight to move, they are called *unit acquisition moves*, and if any positive amount of weight is allowed to move, then they are called *fractional acquisition moves.* The *unit acquisition number* and *fractional acquisition number* of a graph are defined analogously to the total acquisition number.

Lampert and Slater introduced acquisition parameters in [1], in which they established a sharp upper bound on the total acquisition number of an n-vertex graph. LeSaulnier, et al. [2] obtained further results on total acquisition numbers, including bounds on the total acquisition number of trees based on their diameter, sufficient conditions for a graph to have total acquisition number 1, and bounds on the total acquisition numbers of graphs with diameter 2. LeSaulnier and West [3] then characterized the trees that realize the upper bound from [1]. Unit acquisition numbers are explored in [6]. Surprisingly, the case of fractional acquisition is much more tractable. Wenger [5] proved that every connected graph with maximum degree at least 3 has fractional acquisition number 1. In contrast, Slater and Wang [4] proved that for a given graph G, the question "Is $a_t(G) = 1$?" is NP-complete.

Lampert and Slater observed that the maximum weight that a vertex of degree d can acquire via total acquisition moves is 2^d. Consequently, $a_t(G) \geq \lceil |V(G)|/2^{\Delta(G)} \rceil$ for all graphs G, where $\Delta(G)$ denotes the maximum degree of G. As shown in [1], this bound is sharp on the 4×4 grid and not sharp on the 5×5 grid.

Let $G \Box H$ denote the Cartesian product of two graphs. Thus $P_n \Box P_m$ is the $n \times m$-grid. In this paper, we determine $a_t(P_n \Box P_m)$ when n and m are not 5. We also prove nontrivial upper and lower bounds on $a_t(P_n \Box P_5)$.

Throughout this paper we adopt the convention that $m \leq n$ in $P_n \Box P_m$. We represent $P_n \Box P_m$ as a portion of the integer lattice, with vertices lying at the points (x, y) satisfying $x \in \{1, \ldots, n\}$ and $y \in \{1, \ldots, m\}$; we let $v_{x,y}$ denote the vertex at the point (x, y). Two vertices are adjacent if their positions differ in exactly one coordinate by exactly 1. When convenient, we refer to the vertices $v_{i,j}$ with small values of i as the *left side* of $P_n \square P_m$ and the vertices $v_{i,j}$ with large values of i as the *right side* of $P_n \square P_m$. Similarly we refer to the set of vertices $\{v_{i,1} : 1 \leq i \leq n\}$ as the *bottom row* of $P_n \square P_m$ and the set of vertices $\{v_{i,m} : 1 \leq i \leq n\}$ as the *top row* of $P_n \square P_m$. The total acquisition move transferring weight from u to v is denoted by $u \to v$. Furthermore, since this paper is concerned only with total acquisition, we will refer to total acquisition moves simply as *acquisition moves*.

Given a graph G satisfying $a_t(G) = 1$, there may be several optimal sequences of acquisition moves on G. If $v \to u$ is the last acquisition move in an optimal sequence of acquisition moves on G, then we refer to u as a *terminal vertex of* G with *terminal edge* uv.

Section 2 contains preliminary results and lemmas that are used throughout the paper. The main results are in Section 3 with large grids discussed in Section 3.1, $P_n \square P_6$ discussed in Section 3.2, $P_n \square P_3$ discussed in Section 3.3, and $P_n \square P_5$ discussed in Section 3.4. Throughout, we follow the terminology and notation of [7].

2 Preliminary results, acquisition trees, and acquisition tilings

We begin with a formal statement of the bound on the maximum amount of weight that a vertex of degree d can acquire and the corresponding lower bound on the total acquisition number from [1].

Lemma 1 (Lampert and Slater). *If a vertex v has degree d, then the maximum weight that v can acquire is 2^d.*

Theorem 2 (Lampert and Slater). *For all graphs G,*

$$a_t(G) \geq \left\lceil \frac{|V(G)|}{2^{\Delta(G)}} \right\rceil .$$

It is clear that the set of edges used in a sequence of acquisition moves on a graph G corresponds to the edge set of a spanning forest of G. We call a tree an *acquisition tree* if it has total acquisition number 1. Thus, each component of the spanning forest of G corresponding to a sequence of total acquisition moves is an acquisition tree. It follows that $a_t(G)$ is equal to the minimum number of components in a spanning forest of G consisting only of acquisition trees.

Let T be an acquisition tree satisfying $\Delta(T) \leq 4$, and let T' be an embedding of T in the integer lattice. By definition, it is possible to move the weight from exactly those vertices in T' to a single vertex using acquisition moves. We choose to think of T' as a tile in the plane, and call T' an *acquisition tile*. An example is in Figure 1. We also say that the tile T' *covers* the vertices in the embedding of T. Throughout the paper, many different tiles are used, but it is straightforward to prove that each is an acquisition tile.

By studying acquisition tiles in the integer lattice, we are able to translate the problem of finding an optimal sequence of acquisition moves on $P_n \square P_m$ to that of finding a tiling of the $n \times m$ grid using acquisition tiles. We call such a tiling an

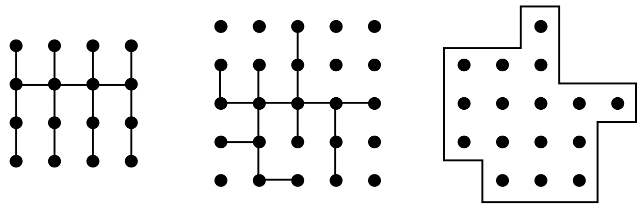

Figure 1: An acquisition tree, an embedding in the integer lattice, and the corresponding acquisition tile.

acquisition tiling. An acquisition tiling of $P_n \square P_m$ is *optimal* if it uses $a_t(P_n \square P_m)$ tiles, and *efficient* if every tile in the tiling covers 16 vertices.

Frequently it is informative to study acquisition parameters by keeping track of a particular unit of weight as it moves about the graph. We model this by considering each unit of weight to be a chip that is labeled by its initial vertex.

Lemma 3. *Let c_x be the chip corresponding to the initial unit of weight at vertex x in an acquisition tree T, and assume that a sequence of acquisition moves transfers c_x to the vertex u, with $x \neq u$. If $x = x_1, x_2, \ldots, x_n = u$ is the path that c_x moves along to reach the vertex u, then*

1. *the weight at u is at least 2^{n-1} when c_x reaches u, and*
2. *$d(x_i) \geq i$ for all $i \in \{1, \ldots, n-1\}$.*

A corollary of Lemma 3 is that if an acquisition tree has diameter 7, then it must contain at least 16 vertices.

Proof. We prove both results by induction on n. If $x_2 = u$, both results hold trivially. Now assume that $n \geq 3$. Since c_x reaches x_{n-1} along the path $x = x_1, x_2, \ldots, x_{n-1}$, we conclude that the move $x_{n-1} \to u$ transfers weight at least 2^{n-2}. Therefore, u has weight at least 2^{n-2} prior to the move $x_{n-1} \to u$, and thus u has weight at least 2^{n-1} when c_x reaches u. Since x_{n-1} transfers weight at least 2^{n-2} to u, it follows from Lemma 1 that x_{n-1} acquires weight from at least $n-2$ neighbors prior to the move $x_{n-1} \to u$. Hence $d(x_{n-1}) \geq n-1$. □

Lemma 4. *Let u be the terminal vertex of an acquisition tree T. If $d(u) = k$, then there is at most one vertex x in T such that $d(x, u) \geq k$.*

Proof. Since $d(u) = k$, the maximum amount of weight that u can acquire is 2^k. It follows from Lemma 3 that there is no vertex x in T such that $d(x, u) > k$. Suppose that there are two vertices x and y that are distance k from u. Let z be the first vertex that acquires the chips from x and y in an optimal sequence of acquisition moves on T. Since $d(x, u) = d(y, u)$, it follows that $d(x, z) = d(y, z)$. Therefore the chips from x and y reach z via acquisition moves on two distinct edges. Without loss of generality, assume that c_x reaches z before c_y. Letting $d(x, z) = d(y, z) = \ell$, it follows from Lemma 3, that z has weight at least $2^\ell + 2^{\ell-1}$ when c_y reaches z. Thus u will acquire weight at least $(2^\ell + 2^{\ell-1}) \cdot 2^{k-\ell} = 2^k + 2^{k-1}$, a contradiction. □

Lemma 5. *Let T be an acquisition tree with maximum degree 4 and 16 vertices, and let u be a terminal vertex of T with terminal edge uv. The following hold:*

1. *$d(u) = d(v) = 4$;*
2. *$diam(T) \leq 7$;*
3. *there is at most one pair of vertices $x, y \in V(T)$ such that $d(x, y) = 7$.*

Proof. 1) Since u acquires weight 16, it must have degree 4 and the terminal edge transfers weight 8 from v to u. Thus v acquires weight 8 without using the edge uv, and v must also have degree 4.

2) This statement follows immediately from Lemma 4.

3) If there are two vertices x and y in T such that $d(x, y) = 7$, then $d(x, u) = 4$, or $d(y, u) = 4$. Assume without loss of generality that $d(x, u) = 4$. Thus any pair of vertices at distance 7 in T includes x. Suppose that there are vertices y and z such that $d(x, y) = d(x, z) = 7$. Hence $d(u, y) = d(u, z) = 3$. Let x' be the neighbor of u on the unique u, x-path in T. Consequently, u must acquire the weight from y and z without using the edge ux'. However, this requires u to acquire weight from two vertices at distance 3 using at most three of its incident edges, contradicting Lemma 4. □

Lemma 6. *Let T be an acquisition tree with maximum degree 4, terminal vertex u, and terminal edge uv. If there is a vertex x in T such that $d(x, u) = 3$ and $d(x, v) = 4$, then $N(u) - v$ has degree sum at least 6.*

Proof. Let x' be the neighbor of u on the unique x, u-path in T. By Lemma 3, x' has degree at least 3. Furthermore, since the move $x' \to u$ transfers weight at least 4, it follows that u acquires weight at least 4 from the vertices in the set $N(u) \setminus \{x', v\}$. Since u has degree at most 4, it follows that $N(u) \setminus \{x', v\}$ contains two vertices and u acquires weight 1 from one of the vertices and 2 from the other. Thus one vertex in $N(u) \setminus \{x', v\}$ has degree at least 2. □

We conclude this section with the following result, which is included for completion.

Theorem 7. *For all positive integers n,*

$$a_t(P_n \square P_1) = a_t(P_n \square P_2) = a_t(P_n \square P_4) = \left\lceil \frac{n}{4} \right\rceil.$$

Proof. Theorem 2 implies that $a_t(P_n \square P_m) \geq \left\lceil \frac{n}{4} \right\rceil$ for all $m \in \{1, 2, 4\}$. Observe that $a_t(P_{n'} \square P_m) = 1$ for all $m \in \{1, 2, 4\}$ and $n' \in \{1, 2, 3, 4\}$. Equality is obtained for each $m \in \{1, 2, 4\}$ by covering the vertices of $P_n \square P_m$ with $\left\lfloor \frac{n}{4} \right\rfloor$ vertex-disjoint copies of $P_4 \square P_m$ and one copy of $P_{n'} \square P_m$ where $n' \equiv n \pmod 4$ and $0 \leq n' \leq 3$. □

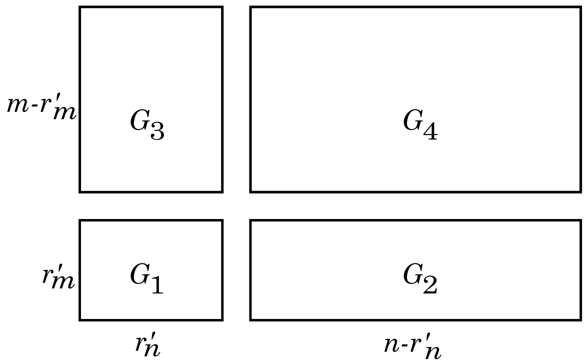

Figure 2: The partition of $P_n \square P_m$.

3 Main Results

3.1 Large grids

Applying Theorem 2 to grids whose dimensions are both at least 3 yields the bound $a_t(P_n \square P_m) \geq \lceil nm/16 \rceil$. In [1], $P_5 \square P_5$ is given as an example where this bound is not sharp. However, this bound is sharp for most grids.

Theorem 8. *If $n \geq 7$ and $m \geq 7$, then $a_t(P_n \square P_m) = \left\lceil \frac{nm}{16} \right\rceil$.*

Proof. By Theorem 2, it suffices to demonstrate an optimal acquisition tiling of $P_n \square P_m$. Let $G = P_n \square P_m$. We divide G into four subgraphs, G_1, G_2, G_3, and G_4, based on the congruence classes of n and m modulo 16. Let $n = 16q_n + r_n$ and $m = 16q_m + r_m$ where q_n, q_m, r_n, and r_m are integers and $0 \leq r_n \leq 15$ and $0 \leq r_m \leq 15$. Let

$$r'_n = \begin{cases} r_n & \text{if } r_n \in \{0,4,7,8,9,10,11,12,13,14,15\} \\ r_n + 16 & \text{if } r_n \in \{1,2,3,5,6\} \end{cases}.$$

Define r'_m similarly. Note that $r'_n \leq n$ and $r'_m \leq m$. Partition G as follows (see Figure 2):

$$G_1 = G[\{v_{i,j} : i \leq r'_n, j \leq r'_m\}];$$
$$G_2 = G[\{v_{i,j} : r'_n + 1 \leq i \leq n, j \leq r'_m\}];$$
$$G_3 = G[\{v_{i,j} : i \leq r'_n, r'_m + 1 \leq j \leq m\}];$$
$$G_4 = G[\{v_{i,j} : r'_n + 1 \leq i \leq n, r'_m + 1 \leq j \leq m\}].$$

Note that there are cases in which G_2, G_3, or G_4 will contain zero vertices.

We build an optimal acquisition tiling of G by exhibiting efficient tilings of G_2, G_3, and G_4, in addition to an optimal tiling of G_1. In these four tilings, there is at most one tile covering fewer than 16 vertices (such a tile would belong to the tiling of G_1), and thus together they form an optimal tiling of G.

Since the dimensions of G_4 are both divisible by 16, G_4 can be covered efficiently using the tile that covers $P_4 \square P_4$. Since $n - r'_n$ and $m - r'_m$ are both divisible by 16, there are efficient tilings of G_2 and G_3 provided there are efficient tilings of $P_{16} \times P_{r'_m}$ and $P_{r'_n} \times P_{16}$, respectively. Such tilings (up to symmetry) for all possible values of r'_n and r'_m can be found in Appendix 3.4. It remains to demonstrate an optimal tiling of $P_{r'_n} \square P_{r'_m}$; such tilings can be found in Appendix 3.4 for all values of r'_n and r'_m. □

3.2 $P_n \square P_6$

In this section we determine $a_t(P_n \square P_6)$. While there are cases when the bound from Theorem 2 is not sharp, it is asymptotically sharp.

Theorem 9. *If $n \geq 6$, then*

$$a_t(P_n \square P_6) = \begin{cases} \lceil \frac{6n}{16} \rceil & \text{if } n \equiv 1,3,4,6 \pmod 8 \\ \lceil \frac{6n}{16} \rceil + 1 & \text{if } n \equiv 0,2,5,7 \pmod 8). \end{cases}$$

Proof. To construct optimal tilings, we use an arrangement of three tiles that we collectively refer to as the *tI-tile* (see Figure 3). All three tiles in the tI-tile cover 16

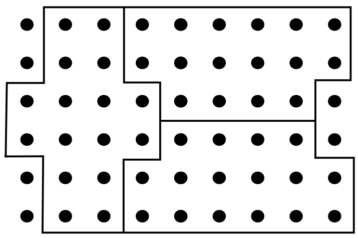

Figure 3: The tI-tile.

vertices; any portion of a tiling consisting of tI-tiles is efficient. The minimum tilings consist of a collection of tiles on the left side of $P_n \square P_6$, an appropriate number of tI-tiles, and then a collection of tiles on the right side of the graph. These tilings appear in Figure 4.

It remains to show that these tilings are optimal for all values of n. It suffices to show for each n that

$$a_t(P_n \square P_6) \geq \frac{6n+8}{16}.$$

This results from the following claim.

Claim 1. In a minimum tiling of $P_n \square P_6$, there is a set of k tiles whose union covers at most $16k - 8$ vertices.

The claim holds if there is any tile covering at most eight vertices; we assume that every tile covers at least nine vertices. Thus every tile contains a vertex of degree 4.

First assume that $v_{1,1}$ and $v_{1,6}$ lie in the same tile; call this tile T_1. Since the terminal vertex of T_1 has degree 4, it follows that $d_{T_1}(v_{1,1}, v_{1,6}) \geq 7$. Hence $|T_1| = 16$. By Lemmas 4 and 5, we conclude that the terminal vertex of T_1 is a vertex x of degree 4 in T_1 such that (without loss of generality) $d_{T_1}(x, v_{1,1}) \leq 4$ and $d_{T_1}(x, v_{1,6}) \leq 3$. The only such vertex is $v_{2,4}$; thus $v_{2,4}$ has degree 4 and is a terminal vertex of T_1. By

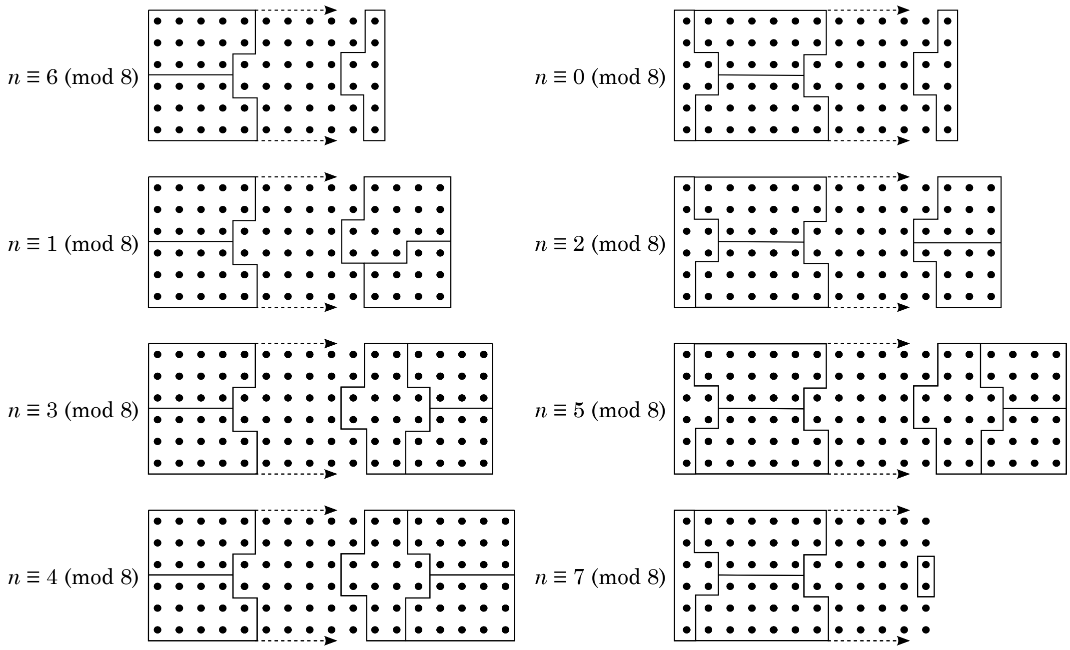

Figure 4: Minimum tilings of $P_6 \square P_n$. Arrows indicate the placement of tI-tiles.

Lemma 5, $v_{2,3}v_{2,4}$ is the terminal edge of T_1, and $v_{2,3}$ also has degree 4 in T_1. Since $v_{1,1}$ and $v_{1,6}$ form the unique pair of vertices in T_1 that are at distance 7, we conclude that $v_{i,j} \notin T_1$ if $i + j > 8$ or if $i > j + 1$. Finally, T_1 covers at least two vertices from the set $\{v_{2,1}, v_{3,2}, v_{4,3}\}$, as otherwise the neighbors of $v_{2,2}$ have degree sum at most 9, contradicting Lemma 6. Similarly T_1 covers at least two vertices from the set $\{v_{2,6}, v_{3,5}, v_{4,4}\}$.

We claim that $v_{3,1}$ and $v_{3,6}$, which are not in T_1, lie in distinct tiles. Since $v_{3,3}, v_{3,4} \in T_1$, it follows that $v_{4,3}$ and $v_{4,4}$ cannot have degree 4 in their respective tiles. Thus there is no vertex x that can have degree 4 in its tile and satisfy $d(x, v_{3,1}) \leq 4$ and $d(x, v_{3,6}) \leq 4$. Thus $v_{3,1}$ and $v_{3,6}$ lie in distinct tiles. Let T_2 be the tile covering $v_{3,1}$ and let T_3 be the tile covering $v_{3,6}$.

Since T_2 and T_3 both cover at least nine vertices, they both contain a vertex of degree 4. Furthermore, there is a vertex of degree 4 within distance 3 of $v_{3,1}$ in T_2. Since T_1 covers at least two vertices from $\{v_{2,1}, v_{3,2}, v_{4,3}\}$, it follows that $v_{4,2}$ cannot be a vertex of degree 4 in T_2. Thus $v_{5,2}$ has degree 4 in T_2. Similarly, $v_{5,5}$ has degree 4 in T_3.

If T_2 and T_3 each cover at most 12 vertices, then the claim holds. Thus we may assume without loss of generality that T_2 covers at least 13 vertices. Hence the terminal edge in T_2 joins two vertices of degree 4, so $v_{6,2}$ also has degree 4 in T_2 and $v_{5,2}v_{6,2}$ is the terminal edge. By Lemma 3, $v_{3,1}$ must be distance at most 2 from a vertex of degree 3 in T_2. Since $v_{5,1}$ is not adjacent to $v_{6,1}$ in T_2, it follows that $v_{5,1}$ cannot have degree 3 in T_2. Thus $v_{4,2}$ has degree 3, and $v_{1,4}v_{2,4} \in E(T_2)$ since v_4 can only have one neighbor in $\{v_{2,1}, v_{3,2}, v_{4,3}\}$. Hence $v_{5,1}$ has degree 1 in T_2. Furthermore, since $v_{5,4} \in T_3$, and the only vertex in $\{v_{2,1}, v_{3,2}, v_{4,3}\} \cap T_1$ is adjacent to $v_{4,2}$, it follows that $v_{5,3}$ also has degree 1 in T_2. However, this contradicts Lemma 6.

Thus T_2 and T_3 cover a total of at most 24 vertices and the claim holds.

Now we assume that $v_{1,1}$ and $v_{1,6}$ lie in distinct tiles. Using symmetry and the fact that $n \geq 6$, we actually may assume that $v_{1,1}$, $v_{1,6}$, $v_{n,1}$ and $v_{n,6}$ all lie in distinct tiles. Let T_1 and T_2 be the tiles that cover $v_{1,1}$ and $v_{1,6}$ respectively. We claim that either $T_1 \cup T_2$ covers at most 28 vertices, or there is a tile that covers at most eight vertices.

Assume that T_1 and T_2 cover at least 29 vertices; hence T_1 and T_2 both cover at least 13 vertices. Thus the terminal edge of T_1 joins two vertices of degree 4 that are both distance at most 4 from $v_{1,1}$. Similarly, the terminal edge of T_2 joins two vertices of degree 4 that are both distance at most 4 from $v_{1,6}$. Depending on the choice of the terminal edges in T_1 and T_2, there are (up to symmetry) four cases to consider.

Case 1: $v_{2,2}v_{3,2}$ is the terminal edge in T_1 and $v_{2,5}v_{3,5}$ is the terminal edge in T_2. Observe that T_1 covers at most one vertex that is distance 4 from $v_{2,2}$, and T_2 covers at most one vertex that is distance 4 from $v_{2,5}$. Therefore $T_1 \cup T_2$ covers at most two vertices from the set $\{v_{5,1}, v_{6,2}, v_{5,3}, v_{5,4}, v_{6,5}, v_{5,6}\}$, and all other vertices $v_{i,j} \in T_1 \cup T_2$ satisfy $i \leq 5$. Therefore T_1 and T_2 cover at most 28 vertices.

Case 2: $v_{2,2}v_{3,2}$ is the terminal edge in T_1 and $v_{3,5}v_{4,5}$ is the terminal edge in T_2. In this case, Lemma 4 implies that $v_{1,6}$ is the unique vertex in T_2 that is distance 4 from $v_{4,5}$, and hence $v_{1,4} \notin T_2$. Thus $v_{1,4} \in T_1$ since otherwise $v_{1,4}$ lies in a tile covering at most four vertices. Therefore both T_1 and T_2 contain a vertex that is distance 3 from the terminal edge.

By Lemma 6, we conclude that the degree sum of $\{v_{2,1}, v_{1,2}, v_{2,3}, v_{3,4}, v_{2,5}, v_{3,6}\}$ in T_1 and T_2 is at least 12. Thus $N(\{v_{2,1}, v_{1,2}, v_{2,3}, v_{3,4}, v_{2,5}, v_{3,6}\})$ must contain at least six vertices from the set $\{v_{1,1}, v_{1,3}, v_{2,4}, v_{1,5}, v_{2,6}\}$, a contradiction.

Case 3: $v_{2,2}v_{2,3}$ is the terminal edge in T_1 and $v_{3,5}v_{4,5}$ is the terminal edge in T_2. As in Case 2, $v_{2,5}$ must have degree 3 in T_2. Therefore both $v_{3,4}$ and $v_{3,6}$ have degree 1, contradicting Lemma 4.

Case 4: $v_{3,2}v_{4,2}$ is the terminal edge in T_1 and $v_{3,5}v_{4,5}$ is the terminal edge in T_2. In this case, $v_{1,3} \notin T_1$ since $v_{1,1}$ is the unique vertex in T_1 that is distance 4 from $v_{4,2}$. Furthermore, $v_{1,3} \notin T_2$ since $v_{1,3}$ is distance 4 from the terminal edge of T_2. Thus the tile covering $v_{1,3}$ covers at most six vertices.

Under the assumption that all tiles cover at least nine vertices, we conclude that T_1 and T_2 cover a total of at most 28 vertices. Similarly, the distinct tiles that contain $v_{n,1}$ and $v_{n,6}$ also contain a total of at most 28 vertices. Therefore there is a set of four tiles in $\mathcal{T}$ whose union covers at most 56 vertices, and the claim holds. $\square$

3.3 $P_n \square P_3$

In this section we determine $a_t(P_n \square P_3)$, proving that the bound from Theorem 2 is not asymptotically sharp.

Theorem 10. $a_t(P_n \square P_3) = \left\lceil \frac{n}{4} \right\rceil$.

Proof. Following the argument from Theorem 7, we conclude that $a_t(P_n \square P_3) \leq \left\lceil \frac{n}{4} \right\rceil$,

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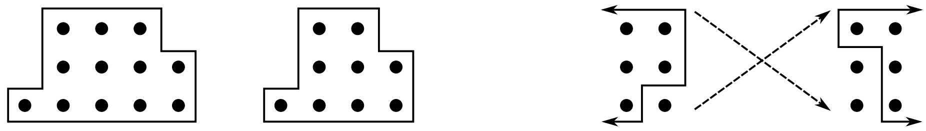

Figure 5: Two reducible tiles and their reduction.

To prove equality, we consider a minimum counterexample. Let m be the minimum value for which $a_t(P_n \Box P_3) < \lceil \frac{n}{4} \rceil$. Let $\mathcal{T}$ be a minimum tiling of $P_n \Box P_3$. If there is a tile $T \in \mathcal{T}$ such that $\{v_{i,1}, v_{i,2}, v_{i,3}\} \in T$ and $\{v_{i+1,1}, v_{i+1,2}, v_{i+1,3}\} \notin T$ for some $i \in \{1, \ldots, n-1\}$, then we say that $\mathcal{T}$ has a *vertical cut.* Observe that if $\mathcal{T}$ has a vertical cut, then $\mathcal{T}$ consists of a tiling of $P_i \Box P_3$ and $P_{n-i} \Box P_3$. However, $\lceil i/4 \rceil + \lceil (n-i)/4 \rceil \geq \lceil n/4 \rceil$, contradicting the minimality of n. Thus we assume that $\mathcal{T}$ does not have a vertical cut.

We now introduce a collection of tiles that we may assume do not appear in $\mathcal{T}$. A tile T is *reducible* if removing T from $\mathcal{T}$ yields two partial tilings of $P_n \Box P_3$ (one to the left of T and one to the right of T) that can be joined to form a tiling of $P_{n-\ell} \Box P_3$ for some positive integer ℓ.

First, for $\ell \in \{3, 4\}$, any tile covering exactly ℓ vertices in each row is reducible. Removing such a tile and joining the two partial tilings yields a tiling of $P_{n-\ell} \Box P_3$ using fewer than $\lceil (n-\ell)/4 \rceil$ tiles, contradicting the minimality of n.

The 12-vertex tile covering $\{v_{i,1}\} \cup \{v_{i+\ell,j} | 1 \leq \ell \leq 3, 1 \leq j \leq 3\} \cup \{v_{i+4,1}, v_{i+4,2}\}$ and the 9-vertex tile covering $\{v_{i,1}\} \cup \{v_{i+\ell,j} | 1 \leq \ell \leq 2, 1 \leq j \leq 3\} \cup \{v_{i+3,1}, v_{i+3,2}\}$ (see Figure 5) are both reducible. If such a tile is removed and one of the remaining partial tilings is reflected vertically, then the partial tilings may be joined to form a tiling of $P_{n-4} \Box P_3$ or $P_{n-3} \Box P_3$. However, these tilings use fewer than $\lceil (n-4)/4 \rceil$ or $\lceil (n-3)/4 \rceil$ tiles, respectively, contradicting the minimality of n.

Suppose that a tile covers the vertices $v_{i,2}$ and $v_{i+1,2}$, but does not contain $v_{i,1}$, $v_{i+1,1}$, $v_{i+1,3}$, and $v_{i+2,2}$. It follows that $v_{i+1,2}$ is a leaf in T and is joined to $v_{i,2}$. Similarly, either $v_{i,1}$ is a leaf in its tile and its neighbor is $v_{i+1,1}$ or $v_{i,1}$ is in a tile of order 1. Therefore we may exchange $v_{i,1}$ and $v_{i+1,2}$ in their respective tiles to obtain a new tiling. We refer to this process as a *tab exchange.* We may assume that $\mathcal{T}$ has no tiles on which we can perform a tab exchange.

Now we consider the possible shapes of tiles covering at least 13 vertices that may be in $\mathcal{T}$. Let T be a tile in $\mathcal{T}$ that covers at least 13 vertices. Since the terminal move in such a tile moves weight at lest 5, it follows that the terminal edge in T joins two vertices of degree 4, say $v_{i-1,2}$ and $v_{i,2}$. Therefore T contains an acquisition tree T' with root $v_{i,2}$ that contains a vertex in T if an only if the first index of the vertex is at least i. Furthermore, T' contains at least five vertices and no more than eight vertices.

Operating under the assumption that there are no tiles in $\mathcal{T}$ that permit a tab exchange, there are (up to symmetry) four possible shapes for T'; these shapes are in Figure 6. By combining two of the possible T' tiles, we generate all possibilities for T. In particular, up to symmetry, for $i \in \{16, 14\}$ there are two possible tiles covering i vertices, there is one tile covering 15 vertices, and there are three possible

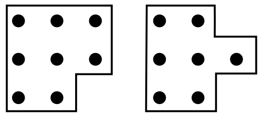

Figure 6: Possible shapes of T'.

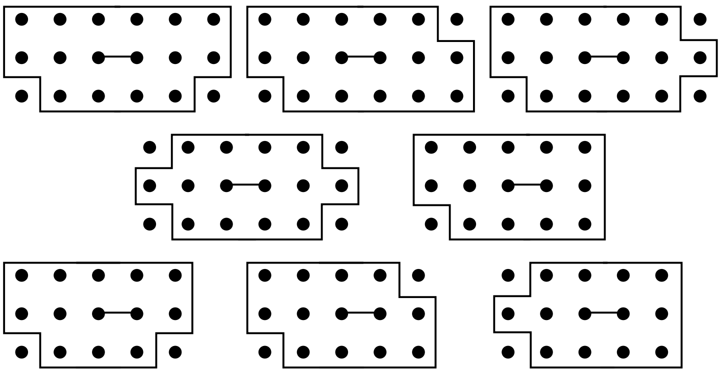

Figure 7: Possible tiles containing at least 13 vertices with terminal edges shown.

tiles covering 13 vertices. These tiles appear in Figure 7.

Label the tiles in $\mathcal{T}$ in order by their leftmost point (if multiple tiles have an equal leftmost coordinate, label them in order from bottom to top). Consider the sum $\sum_{i=1}^{k} |T_i|$. Since $a_t(P_n \square P_3) < \lceil \frac{n}{4} \rceil$, there is a minimum value k such that $\sum_{i=1}^{k} |T_i| > 12k$ and $\sum_{i=1}^{k'} |T_i| > 12k'$ for all $k' > k$. It follows that $-3 \leq \sum_{i=1}^{k-1} |T_i| - 12(k-1) \leq 0$ and T_k contains at least 13 vertices. Furthermore $\sum_{i=1}^{k} |T_i| - 12k \leq 4$.

If $k = a_t(P_n \square P_3)$, then T_k covers $v_{n,1}$, $v_{n,2}$, and $v_{n,3}$. It follows that T_k contains either 13 or 14 vertices (see Figure 7). Consequently $|T_k| + \sum_{i=1}^{k-1} |T_i| \leq 12k+2$. Since $|T_k| + \sum_{i=1}^{k-1} |T_i| = 3n$, it follows that $|T_k| + \sum_{i=1}^{k-1} |T_i| = 12k$ and $k = a_t(P_n \square P_3) = \lceil n/4 \rceil$, a contradiction.

Henceforth we assume that $k \neq a_t(P_n \square P_3)$, and therefore T_{k+1} exists. By assumption, the right edge of T_k is not a vertical cut. Furthermore, T_{k+1} contains at least 9 vertices. Inspection of the tiles in Figure 7 shows that no two tiles covering at least 13 vertices can appear consecutively in $\mathcal{T}$; hence $|T_{k+1}| \leq 12$. Therefore, up to vertical reflection, there are two possibilities for the right edge of T_k. We show in both cases that $\mathcal{T}$ covers a reducible tile.

Case 1: $v_{i-1,1}, v_{i,2}, v_{i-1,3} \in T_k$ and $v_{i,1}, v_{i+1,2}, v_{i,3} \notin T_k$. In this case, $v_{1,1} \in T_{k+1}$. Since T_{k+1} covers at least 9 vertices, the terminal vertex has degree 4. As $|T_{k+1}| \leq 12$, the terminal vertex of T_{k+1} is within distance 3 of $v_{i,1}$. Thus $v_{i+2,2}$ is the terminal vertex of T_{k+1}. If $v_{i,3} \notin T_{k+1}$, then $|T_{k+2}| \leq 2$, and $\sum_{i=1}^{k+2} |T_i| \leq 12(k+2)$, a contradiction. Therefore $v_{i,3} \in T_{k+1}$. It follows from Lemma 4 that if an acquisition tree T contains two vertices that are distance 3 from the terminal vertex, then $|T| \geq 12$. Thus $|T_{k+1}| = 12$. Lemma 4 also implies that a 12-vertex acquisition tree contains at most two vertices that are distance 3 from the terminal vertex. Thus $v_{i+4,1}, v_{i+5,2}, v_{i+4,3} \notin T_{k+1}$. Consequently T_{k+1} contains four vertices from each row and is a reducible tile.

Case 2: $v_{i-1,1}, v_{i,2}, v_{i,3} \in T_k$ and $v_{i,1}, v_{i+1,2}, v_{i+1,3} \notin T_k$. In this case, $v_{i+2,2}$ is the terminal vertex of T_{k+1}. If $v_{i+1,3} \notin T_{k+1}$, then $|T_{k+2}| = 1$ and $\sum_{i=1}^{k+2} |T_i| \leq 12(k+2)$, a contradiction. Therefore $v_{i+1,3} \in T_{k+1}$. Furthermore, T_{k+1} covers at most one vertex from $\{v_{i+4,1}, v_{i+5,2}, v_{i+4,3}\}$ since since $v_{i,1}$ is distance 3 from $v_{i+2,2}$. If $|T_{k+1}| \leq 11$, then T_{k+1} cannot cover any vertex from $\{v_{i+4,1}, v_{i+5,2}, v_{i+4,3}\}$. Finally, T_{k+1} covers no vertex that is distance 4 from $v_{i+2,2}$.

We break this case into subcases depending on the order of T_{k+1}. We classify T_{k+1} using an ordered triple (a_1, a_2, a_3) where a_i indicates the number of vertices in T_{k+1} that lie in row i.

Case 2.1: $|T_{k+1}| = 12$. In this case, T_{k+1} has one of the following three forms: $(5, 4, 3)$, $(4, 5, 3)$, or $(4, 4, 4)$. If T_{k+1} has the form $(4, 5, 3)$, T_{k+1} permits a tab exchange, a contradiction. If T_{k+1} has either the form $(5, 4, 3)$ or the form $(4, 4, 4)$, then T_{k+1} is reducible.

Case 2.2: $|T_{k+1}| = 11$. In this case, T_{k+1} has the form $(4, 4, 3)$. It follows that T_{k+1} is not the final tile in $\mathcal{T}$, and the analysis of T_{k+2} follows Case 1, and therefore T_{k+2} is reducible.

Case 2.3: $|T_{k+1}| = 10$. In this case, T_{k+1} has one of the following three forms: $(4, 4, 2)$, $(4, 3, 3)$, or $(3, 4, 3)$. Observe that $|T_k| \geq 15$, as otherwise $\sum_{i=1}^{k+1} |T_i| \leq 12(k + 1)$. Thus $25 \leq |T_k| + |T_{k+1}| \leq 26$. If T_{k+1} is of the form $(4, 3, 3)$, then it is possible that T_{k+1} is the final tile in $\mathcal{T}$. If T_{k+1} is the final tile, then $\sum_{i=1}^{k+1} |T_i| \equiv 0$ mod 3. Since $12(k - 1) - 3 \leq \sum_{i=1}^{k-1} |T_i| \leq 12(k - 1)$, there are two cases to consider: $\sum_{i=1}^{k-1} |T_i| = 12(k - 1) - 2$ and $|T_k| = 16$, or $\sum_{i=1}^{k-1} |T_i| = 12(k - 1) - 1$ and $|T_k| = 15$. In both cases, $3n = 12(k - 1) + 24$, and $a_t(P_n \square P_3) = k + 1 = \lceil n/4 \rceil$.

Now we assume that T_{k+1} is not the final tile in $\mathcal{T}$. In this case, if T_{k+1} is of the form $(4, 3, 3)$, then the right edge of T_{k+1} is a vertical cut, a contradiction. Otherwise, T_{k+1} is of the form $(4, 4, 2)$ or $(3, 4, 3)$, both of which permit a tab exchange, a contradiction.

Case 2.4: $|T_{k+1}| = 9$. In this case, T_{k+1} has one of the following three forms: $(4, 3, 2)$, $(3, 4, 2)$, or $(3, 3, 3)$. If T_{k+1} has the form $(3, 4, 2)$, then it permits a tab exchange, a contradiction. Otherwise, T_{k+1} is reducible. □

3.4 $P_n \square P_5$

In this section, we provide nontrivial upper and lower bounds on $a_t(P_n \square P_5)$.

Theorem 11. $\lceil n/3 \rceil \leq a_t(P_n \square P_5) \leq 11 \lfloor n/32 \rfloor + 16.$

Proof. To establish the upper bound, we demonstrate a tiling of $P_{32} \square P_5$ that uses 11 tiles (see Figure 8). If $a \equiv n \mod 32$ and $0 \leq a \leq 31$, then

$$\begin{aligned} a_t(P_n \square P_5) &\leq \lfloor n/32 \rfloor \, a_t(P_{32} \square P_5) + a_t(P_a \square P_5) \\ &\leq \lfloor n/32 \rfloor \, a_t(P_{32} \square P_5) + a_t(P_a \square P_3) + a_t(P_a \square P_2) \\ &\leq 11 \lfloor n/32 \rfloor + 16. \end{aligned}$$

To prove the lower bound, we consider the number of vertices from the top and bottom rows of $P_n \square P_5$ that may be covered by a single acquisition tile.

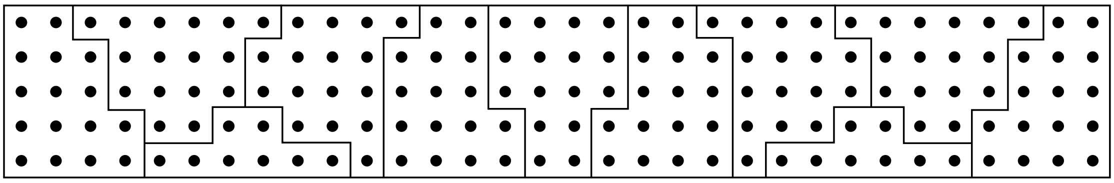

Figure 8: An 11-tile tiling of $P_{32}\Box P_5$.

Let T be an acquisition tile in $P_n\Box P_5$ and let $v_{i,1} \in T$. If $v_{i+6,1} \in T$, then the path between $v_{i,1}$ and $v_{i+6,1}$ in T must contain a vertex of degree 4. Therefore $v_{i+6,1}$ is distance at least 8 from $v_{i,1}$ in T, a contradiction. Thus an acquisition tile contains at most six vertices from the bottom (or, by symmetry, top) row of $P_n\Box P_5$.

Assume that T covers $v_{i+j,1}$ for all $j \in \{0, 1, 2, 3, 4, 5\}$. If u is a vertex in the top row, then either $d(u, v_{i,1}) > 7$, $d(u, v_{i+5,1}) > 7$, or $d(u, v_{i,1}) = d(u, v_{i+5,1}) = 7$. By Lemma 5, $u \notin T$ and consequently T contains only six vertices from the top or bottom rows.

Now assume that T covers $v_{i+j,1}$ for all $j \in \{0, 1, 2, 3, 4\}$. If T covers two vertices from the top row, then since T contains at most one pair of vertices of distance 7, we may assume without loss of generality that $v_{i+2,5}, v_{i+3,5} \in T$. Since $d(v_{i,1}, v_{i+3,5}) = 7$, it follows that $|T| = 16$. Thus we may assume that the terminal vertex of T is a vertex of degree 4 that is distance at most 4 from at most one of $v_{i,1}$, $v_{i+4,1}$, $v_{i+2,5}$, and $v_{i+3,5}$. It follows that the only possible terminal vertex of T is $v_{i+2,2}$. However, since $|T| = 16$, there must be two possible terminal vertices in T, a contradiction.

Now assume that T covers $v_{i+j,1}$ for all $j \in \{0, 1, 2, 3\}$ and three vertices in the top row. Since T contains at most one pair of vertices of distance 7, we may assume without loss of generality that $v_{i+1,5}, v_{i+2,5}, v_{i+3,5} \in T$. Again $|T| = 16$ and T must contain two possible terminal vertices. Each terminal vertex is distance 4 from at most one of $v_{i,1}$, $v_{i+3,1}$, $v_{i+1,5}$, and $v_{i+3,5}$. However, $v_{i+2,3}$ is the only such vertex, a contradiction.

We now conclude that each tile covers at most six vertices from the top and bottom rows of $P_n\Box P_5$. Since there are $2n$ such vertices, it follows that $a_t(P_n\Box P_5) \geq \lceil n/3 \rceil$. □

An argument similar to that in the proof of Theorem 10 could show that in fact the upper bound in Theorem 11 is asymptotically sharp. However, the potential shapes of tiles increases dramatically when considering $P_n\Box P_5$ rather than $P_n\Box P_3$, indicating that such a proof would be very long and complicated without providing much insight. We leave the following conjecture that the upper bound of Theorem 11 is asymptotically sharp.

Conjecture 12. *There is a constant c such that*

$$11 \lfloor n/32 \rfloor - c \leq a_t(P_n\Box P_5).$$

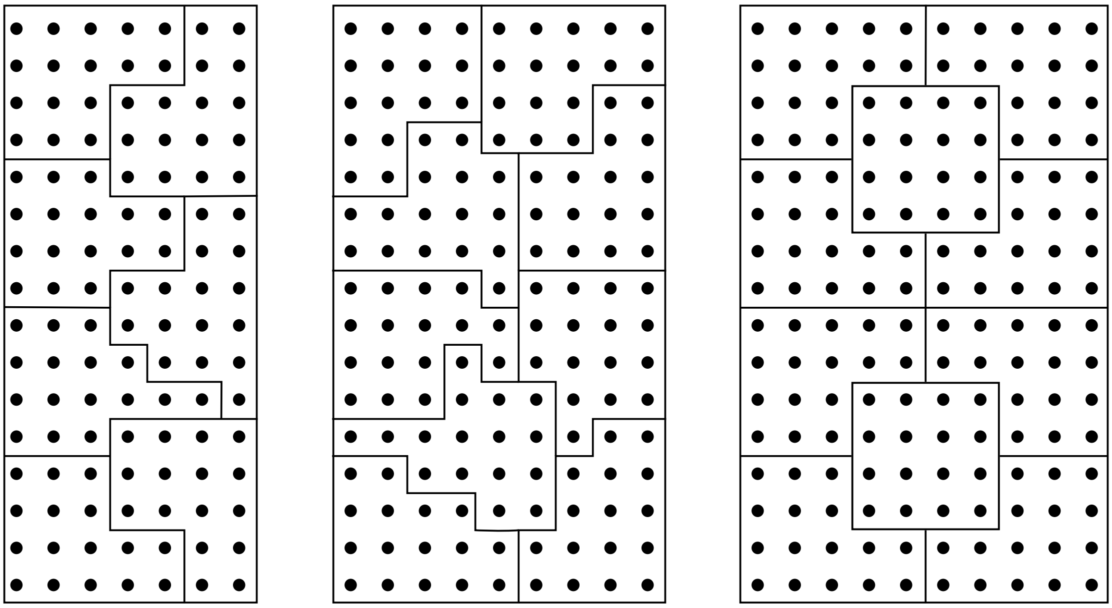

Figure 9: Efficient tilings of $P_7 \square P_{16}$, $P_9 \square P_{16}$, and $P_{10} \square P_{16}$.

Acknowledgments

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Appendix: Efficient tilings of $P_{r'_n} \square P_{16}$ and optimal tilings of $P_{r'_n} \times P_{r'_m}$

To construct an efficient tiling of $P_{r'_n} \square P_{16}$ (and by symmetry, $P_{16} \square P_{r'_m}$), first note that $P_{r'_n} \square P_{16}$ may be efficiently tiled using $P_4 \square P_4$ whenever r'_n is a multiple of 4. For all remaining values of r'_n, it suffices to use an efficient tiling of $P_\ell \square P_{16}$ where ℓ is an appropriate multiple of 4 and an efficient tiling of $P_7 \square P_{16}$, $P_9 \square P_{16}$, or $P_{10} \square P_{16}$ depending on the value of r'_n modulo 4. Efficient tilings of $P_7 \square P_{16}$, $P_9 \square P_{16}$, and $P_{10} \square P_{16}$ are in Figure 9.

Table 1 indicates the number of tiles in an optimal tiling of $P_{r'_n} \square P_{r'_m}$. Without loss of generality, we assume that $r'_n \geq r'_m$. In many cases, an optimal tiling of $P_{r'_n} \times P_{r'_m}$ can be constructed using efficient tilings of $P_k \square P_{r'_m}$ and $P_\ell \square P_{r'_m}$ where $k + \ell = r'_n$. Those instances are listed in Tables 2 and 3, where the notation $P_k || P_\ell$ indicates a construction of an optimal tiling of $P_{r'_n} \times P_{r'_m}$ using optimal tilings of $P_k \square P_{r'_m}$ and $P_\ell \square P_{r'_m}$. The necessary specific constructions can be found in Figures 10 through 17.

$r_{n'} \backslash r_{m'}$	7	8	9	10	11	12	13	14	15	17	18	19	21	22
7	4	4	4	5	5	6	6	7	7	8	8	9	10	10
8	4	4	5	5	6	6	7	7	8	9	9	10	11	11
9	4	5	6	6	7	7	8	8	9	10	11	11	12	13
10	5	5	6	7	7	8	9	9	10	11	12	12	14	14
11	5	6	7	7	8	9	9	10	11	12	13	14	15	16
12	6	6	7	8	9	9	10	11	12	13	14	15	16	17
13	6	7	8	9	9	10	11	12	13	14	15	16	18	18
14	7	7	8	9	10	11	12	13	14	15	16	17	19	20
15	7	8	9	10	11	12	13	14	15	16	17	18	20	21
17	8	9	10	11	12	13	14	15	16	19	20	21	23	24
18	8	9	11	12	13	14	15	16	17	20	21	22	24	25
19	9	10	11	12	14	15	16	17	18	21	22	23	25	27
21	10	11	12	14	15	16	18	19	20	23	24	25	28	29
22	10	11	13	14	16	17	18	20	21	24	25	27	29	31

Table 1: Number of tiles in an optimal tiling of $P_{r_{n'}} \square P_{r_{m'}}$.

$r_{n'} \backslash r_{m'}$	7	8	9	10	11	12	13	14
7	Fig. 10							
8	Fig. 10	$P_4 \Vert P_4$						
9	Fig. 10	Fig. 11	Fig. 12					
10	Fig. 10	Fig. 11	Fig. 12	Fig. 13				
11	Fig. 10	$P_4 \Vert P_7$	Fig. 12	Fig. 13	$P_4 \Vert P_7$			
12	Fig. 10	$P_4 \Vert P_8$	Fig. 12	$P_4 \Vert P_8$	$P_4 \Vert P_8$	$P_4 \Vert P_8$		
13	$P_4 \Vert P_9$	$P_4 \Vert P_9$	Fig. 12	$P_4 \Vert P_9$	Fig. 14	$P_4 \Vert P_9$	Fig. 15	
14	$P_4 \Vert P_{10}$	$P_4 \Vert P_{10}$	$P_7 \Vert P_7$	Fig. 13	$P_4 \Vert P_{10}$	$P_4 \Vert P_{10}$	$P_7 \Vert P_7$	$P_4 \Vert P_{10}$
15	$P_4 \Vert P_{11}$	$P_4 \Vert P_{11}$	$P_7 \Vert P_8$	$P_8 \Vert P_7$	$P_4 \Vert P_{11}$	$P_4 \Vert P_{11}$	$P_4 \Vert P_{11}$	$P_4 \Vert P_{11}$
17	$P_4 \Vert P_{13}$	$P_4 \Vert P_{13}$	$P_7 \Vert P_{10}$	$P_8 \Vert P_9$	$P_4 \Vert P_{13}$	$P_4 \Vert P_{13}$	Fig. 15	$P_8 \Vert P_9$
18	$P_9 \Vert P_9$	$P_4 \Vert P_{14}$	$P_7 \Vert P_{11}$	$P_8 \Vert P_{10}$	$P_4 \Vert P_{14}$	$P_4 \Vert P_{14}$	$P_7 \Vert P_{11}$	$P_8 \Vert P_{10}$
19	$P_4 \Vert P_{15}$	$P_4 \Vert P_{15}$	$P_7 \Vert P_{12}$	$P_8 \Vert P_{11}$	$P_4 \Vert P_{15}$	$P_4 \Vert P_{15}$	$P_7 \Vert P_{12}$	$P_8 \Vert P_{11}$
21	$P_4 \Vert P_{17}$	$P_4 \Vert P_{17}$	$P_7 \Vert P_{14}$	$P_8 \Vert P_{13}$	$P_4 \Vert P_{17}$	$P_4 \Vert P_{17}$	$P_7 \Vert P_{14}$	$P_8 \Vert P_{13}$
22	$P_4 \Vert P_{18}$	$P_4 \Vert P_{18}$	$P_7 \Vert P_{15}$	$P_8 \Vert P_{14}$	$P_4 \Vert P_{18}$	$P_4 \Vert P_{18}$	$P_{11} \Vert P_{11}$	$P_8 \Vert P_{14}$

Table 2: Constructions of efficient tilings of $P_{r_{n'}} \square P_{r_{m'}}$ for $r_{m'} \leq 14$. The notation $P_k||P_\ell$ indicates use of optimal tilings of $P_k \square P_{r_{m'}}$ and $P_\ell \square P_{r_{m'}}$.

$r_{n'} \backslash r_{m'}$	15	17	18	19	21	22
15	$P_4\|\|P_{11}$					
17	Fig. 16	$P_4\|\|P_{13}$				
18	Fig. 16	$P_4\|\|P_{14}$	$P_8\|\|P_{10}$			
19	Fig. 16	$P_4\|\|P_{15}$	$P_8\|\|P_{11}$	$P_4\|\|P_{15}$		
21	$P_4\|\|P_{17}$	$P_7\|\|P_{14}$	$P_8\|\|P_{13}$	Fig. 17	$P_9\|\|P_{12}$	
22	$P_4\|\|P_{18}$	$P_7\|\|P_{15}$	$P_8\|\|P_{14}$	$P_4\|\|P_{18}$	Fig. 17	$P_8\|\|P_{14}$

Table 3: Constructions of efficient tilings of $P_{r_{n'}} \square P_{r_{m'}}$ for $r_{m'} \geq 15$. The notation $P_k||P_\ell$ indicates use of optimal tilings of $P_k \square P_{r_{m'}}$ and $P_\ell \square P_{r_{m'}}$.

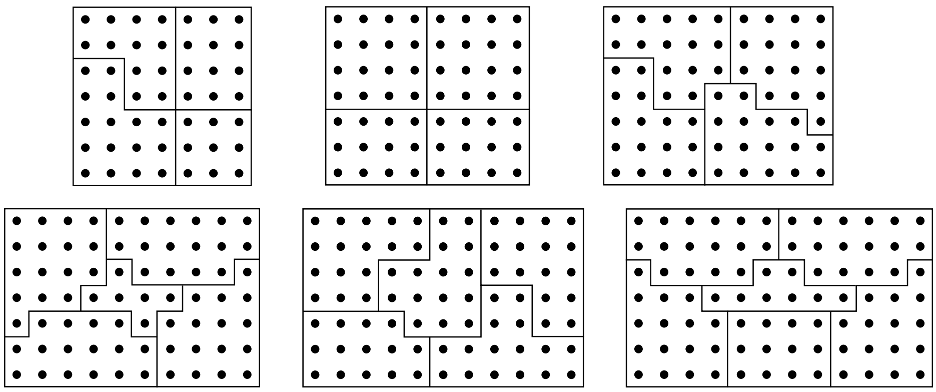

Figure 10: Optimal tilings of $P_7 \square P_7$, $P_8 \square P_7$, $P_9 \square P_7$, $P_{10} \square P_7$, $P_{11} \square P_7$, and $P_{12} \square P_7$.

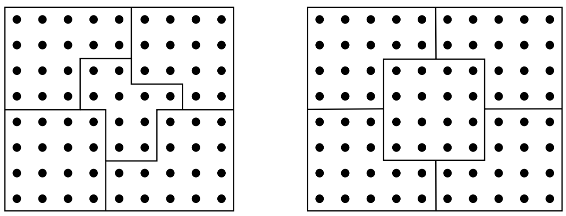

Figure 11: Optimal tilings of $P_9 \square P_8$ and $P_{10} \square P_8$.

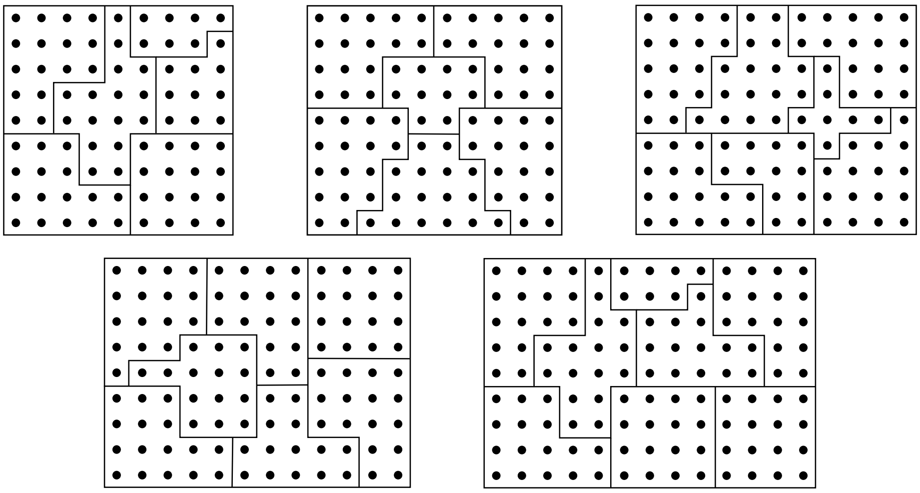

Figure 12: Optimal tilings of $P_9 \square P_9$, $P_{10} \square P_9$, $P_{11} \square P_9$, $P_{12} \square P_9$, and $P_{13} \square P_9$.

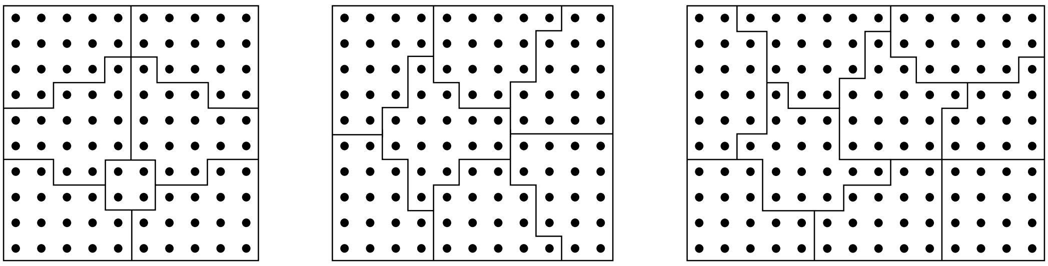

Figure 13: Optimal tilings of $P_{10} \square P_{10}$, $P_{11} \square P_{10}$, and $P_{14} \square P_{10}$.

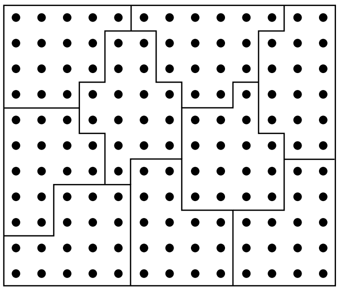

Figure 14: Optimal tiling of $P_{13} \square P_{11}$.

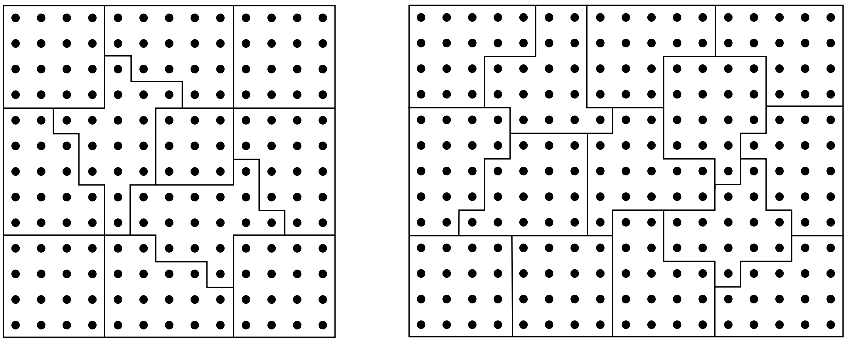

Figure 15: Optimal tilings of $P_{13} \square P_{13}$ and $P_{17} \square P_{13}$.

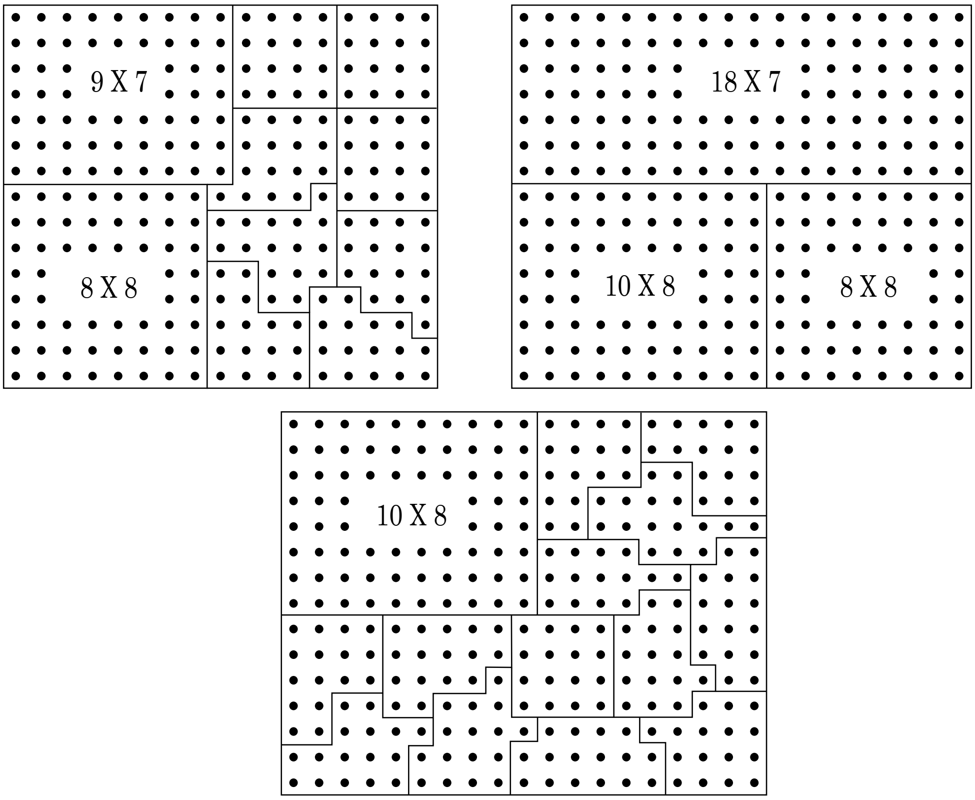

Figure 16: Optimal tilings of $P_{17} \square P_{15}$, $P_{18} \square P_{15}$, and $P_{19} \square P_{15}$. Here larger tiles are used to simplify the tiling.

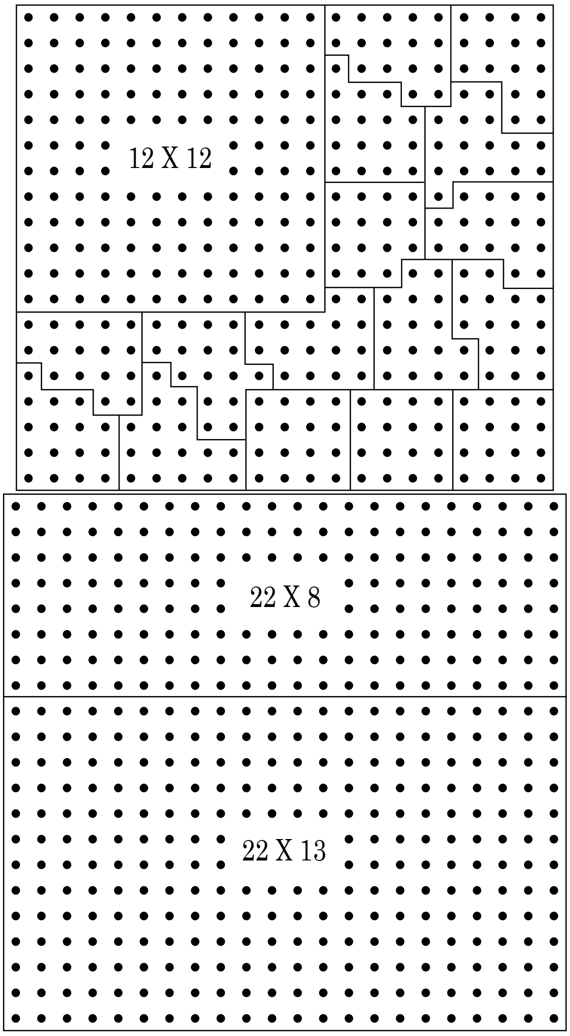

Figure 17: Optimal tilings of $P_{21}\square P_{19}$ and $P_{22}\square P_{21}$. Here larger tiles are used to simplify the tiling.

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