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A note on the existence of edges in the $(1,2)$-step competition graph of a round digraph

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Abstract

Factor and Merz [*Discrete Appl. Math.* 159 (2011), 100–103] defined the $(1,2)$-step competition graph and studied the $(1,2)$-step competition graph of a tournament. In this paper, a sufficient and necessary condition for any two vertices to be adjacent in the $(1,2)$-step competition graph $C_{1,2}(D)$ of a round digraph D is given.

1 Terminology and Introduction

The competition graph of a digraph, introduced by Cohen to study "food web" models in 1968, has been extensively studied in connection to some biological models and some radio communication networks. For a comprehensive introduction to competition graphs, see [3, 11, 12]. Recent work in competition graph theory includes [8, 10]. In 1991, Hefner (Factor) et al. defined the (i,j) competition graph in [7]. In 2008, Hedetniemi et al. [6] introduced $(1,2)$-domination. This was followed by Factor and Langley's introduction of the $(1,2)$-domination graph in [4]. Because of the similarities in construction, the $(1,2)$-step competition graph was defined by Factor and Merz in 2011. In [5], they completely characterized the $(1,2)$-step competition graphs of tournaments and extended the results to the (i,k)-step competition graphs.

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It will be assumed that the reader is familiar with the concepts of graphs and digraphs. The other untouched terminology can be found in [1]. Let D be a digraph on n vertices. Then $V(D)$ and $A(D)$ denote its vertex and arc sets, respectively. If (x, y) is an arc of D, then we say that x *dominates* y and sometimes use the notation $x \to y$ to denote this arc. The *outset* of a vertex $x \in V(D)$ is the set $N^+(x) = \{y \mid (x, y) \in A(D)\}$. Similarly, $N^-(x) = \{y \mid (y, x) \in A(D)\}$ is the *inset* of x. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called the *outdegree* and *indegree* of x, respectively (if necessary, we write $d^+_D(x)$ and $d^-_D(x)$ instead of $d^+(x)$ and $d^-(x)$, respectively). Let H be a subgraph of D. If every arc of $A(D)$, which has both vertices in $V(H)$, is in $A(H)$, we say that H is *induced* by $X = V(H)$ (we write $H = D[X]$) and call H an *induced subdigraph* of D. In addition, $D - X = D[V(D) - X]$ for any $X \subseteq V(D)$.

Cycles and paths are always simple and directed. An arc (x, y) of a digraph D is *ordinary* if (y, x) is not in D. A cycle Q of a digraph D is *ordinary* if all arcs of Q are ordinary. If x and y are vertices of D then the *distance* from x to y in D, denoted $d_D(x, y) = d(x, y)$, is the minimum length of an (x, y)-path, if y is reachable from x, and otherwise $d(x, y) = \infty$. Let G be a graph. The vertex and edge sets are denoted by $V(G)$ and $E(G)$ respectively. Recall that the *competition graph* of a digraph of D is obtained by using vertex set $V(D)$ and adding edge xy whenever $N^+(x) \cap N^+(y) \neq \emptyset$. The $(1, 2)$-*step competition graph* of a digraph D, denoted $C_{1,2}(D)$, is a graph on $V(D)$ where $xy \in E(C_{1,2}(D))$ if and only if there exists a vertex $z \neq x, y$, such that either $d_{D-y}(x, z) = 1$ and $d_{D-x}(y, z) \leq 2$ or $d_{D-x}(y, z) = 1$ and $d_{D-y}(x, z) \leq 2$. A digraph on n vertices is called a *round digraph* if we can label its vertices $v_0, v_1, \ldots, v_{n-1}$ such that for each i, $N^+(v_i) = \{v_{i+1}, \ldots, v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-d^-(v_i)}, \ldots, v_{i-1}\}$, where the subscripts are taken modulo n. We will refer to the ordering $v_0, v_1, \ldots, v_{n-1}$ as a *round labelling* of D. For example, a round digraph on 5 vertices and its $(1, 2)$-step competition graph are shown in Fig. 1.

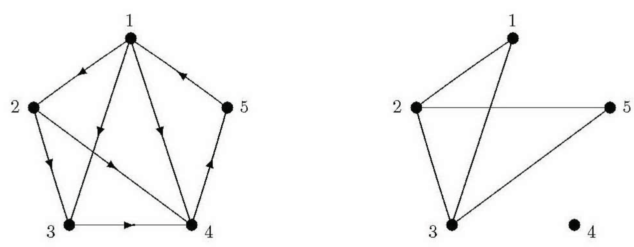

Figure 1: A round digraph on 5 vertices and its $(1, 2)$-step competition graph.

A digraph D is *strong* if every vertex of D is reachable by a directed path from every other vertex of D. A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong. If $D_1, D_2, \ldots, D_t$ are the strong components of D, then we call $V(D_1), V(D_2), \ldots, V(D_t)$ the *strong decomposition* of D. The *underlying* *graph* of D, denoted by $UG(D)$, is the graph obtained by ignoring the orientations of arcs in D and deleting parallel edges. We say that D is *connected* if its underlying graph is connected. In this paper, we only consider connected digraphs.

A digraph D is *semicomplete* if, for any pair of vertices $x, y \in V(D)$, either $(x, y) \in A(D)$, or $(y, x) \in A(D)$, or both, i.e., $UG(D)$ is complete. A *tournament* is a semicomplete digraph without a cycle of length 2. A digraph D is *locally semicomplete* if $D[N^+(x)]$ and $D[N^-(x)]$ are both semicomplete for every vertex x of D. A digraph D is *transitive* if, for every pair of arcs (x, y) and (y, z) in D such that $x \neq z$, the arc (x, z) is also in D. It is easy to show that a tournament is transitive if and only if it is acyclic.

Proposition 1.1 (Huang [9]) *Every round digraph is locally semicomplete.*

Theorem 1.2 (Huang [9]) *A connected locally semicomplete digraph D is round if and only if the following holds for each vertex x of D:*

(a) $N^+(x) - N^-(x)$ and $N^-(x) - N^+(x)$ induce transitive tournaments; and

(b) $N^+(x) \cap N^-(x)$ induces a (semicomplete) subdigraph containing no ordinary cycle.

Theorem 1.3 (Bang-Jensen [2]) *Every strong locally semicomplete digraph is hamiltonian.*

2 Main Results

Lemma 2.1 *Let D be a round digraph and $v_0, v_1, \ldots, v_{n-1}$ be a round labelling of D. If v_i dominates v_j in D for $i \neq j$, then for all v_α, v_β such that $v_i, v_\alpha, v_\beta, v_j$ are in the order of the round labelling of D, v_α dominates v_β.*

Proof. Assume without loss of generality $0 \leq i < j \leq n-1$. For any $i < \alpha < \beta < j$, $v_\beta \in N^+(v_i)$ since $v_j \in N^+(v_i)$ and hence $v_\alpha \in N^-(v_\beta)$ since $v_i \in N^-(v_\beta)$ by Proposition 1.1 and Theorem 1.2. Thus v_α dominates v_β. ∎

Theorem 2.2 *Let D be a strong round digraph on n vertices and $v_0, v_1, \ldots, v_{n-1}$ be a round labelling of D. If G is the $(1, 2)$-step competition graph of D, then $v_iv_j \in E(G)$ if and only if one of the following conditions is satisfied:*

(a) $d(v_i, v_{j+1}) \leq 2$ and $v_{j-1} \to v_{j+1}$ or

(b) $d(v_j, v_{i+1}) \leq 2$ and $v_{i-1} \to v_{i+1}$,

where the subscripts are taken modulo n.

Proof. In the proof, we always assume that the subscripts are taken modulo n.

(Necessity) Let $A = \{v_{i+1}, v_{i+2}, \ldots, v_{j-1}\}$ and $B = \{v_{j+1}, v_{j+2}, \ldots, v_{i-1}\}$. Because $v_iv_j \in E(G)$, there is a vertex v_k such that either $d_{D-v_j}(v_i, v_k) = 1$ and $d_{D-v_i}(v_j, v_k) \leq 2$ or $d_{D-v_i}(v_j, v_k) = 1$ and $d_{D-v_j}(v_i, v_k) \leq 2$.

Claim $d(v_i, v_{j+1}) \leq 2$ or $d(v_j, v_{i+1}) \leq 2$.

Suppose that $d(v_i, v_{j+1}) \geq 3$ and $d(v_j, v_{i+1}) \geq 3$. According to the symmetry, we assume without loss of generality that $d_{D-v_j}(v_i, v_k) = 1$ and $d_{D-v_i}(v_j, v_k) \leq 2$. Thus, $v_i \in N^-(v_k)$.

If $d_{D-v_i}(v_j, v_k) = 1$, then $v_j \in N^-(v_k)$. Hence, v_i is adjacent to v_j in D by Proposition 1.1. This implies that $d(v_i, v_{j+1}) \leq 2$ or $d(v_j, v_{i+1}) \leq 2$, a contradiction. So $d_{D-v_i}(v_j, v_k) = 2$. Then there is a vertex $v_t \in V(D) - \{v_i\}$ such that $v_j \rightarrow v_t \rightarrow v_k$. If $v_t \in A$, then by Lemma 2.1, $v_j \rightarrow v_{i+1}$ since $v_j \rightarrow v_t$, which contradicts $d(v_j, v_{i+1}) \geq 3$. So $v_t \in B$. If $v_k \in B$, then $v_i \rightarrow v_{j+1}$ since $v_i \rightarrow v_k$, which contradicts $d(v_i, v_{j+1}) \geq 3$. So $v_k \in A$. However, we have $v_j \rightarrow v_t \rightarrow v_{i+1}$ since $v_j \rightarrow v_t \rightarrow v_k$. Then $d(v_j, v_{i+1}) \leq 2$, a contradiction. The claim follows. □

Assume without loss of generality that $d(v_i, v_{j+1}) \leq 2$. We consider the following two cases.

Case 1: $d(v_i, v_{j+1}) \leq 2$ and $d(v_j, v_{i+1}) > 2$.

We will prove that $v_{j-1} \rightarrow v_{j+1}$. We claim that $v_k \in B$. Suppose not. We have $v_k \in A$. If $d_{D-v_i}(v_j, v_k) = 1$, then by Lemma 2.1, $v_j \rightarrow v_{i+1}$, which contradicts $d(v_j, v_{i+1}) > 2$. If $d_{D-v_i}(v_j, v_k) = 2$, there is $v_t \in V(D)$ with $v_j \rightarrow v_t \rightarrow v_k$. For the case $v_t \in A$, we have $v_j \rightarrow v_{i+1}$ since $v_j \rightarrow v_t$. For the case $v_t \in B$, we have $v_j \rightarrow v_t \rightarrow v_{i+1}$. In each case, we can get a contradiction. So $v_k \in B$. We consider the following two subcases.

Subcase 1.1: $d_{D-v_j}(v_i, v_k) = 1$ and $d_{D-v_i}(v_j, v_k) \leq 2$.

Since $v_k \in B$ and $v_i \rightarrow v_k$, we have $v_{j-1} \rightarrow v_{j+1}$ by Lemma 2.1.

Subcase 1.2: $d_{D-v_i}(v_j, v_k) = 1$ and $d_{D-v_j}(v_i, v_k) \leq 2$.

Similarly, if $d_{D-v_j}(v_i, v_k) = 1$, then $v_{j-1} \rightarrow v_{j+1}$. So $d_{D-v_j}(v_i, v_k) = 2$. Then there is a vertex v_t such that $v_i \rightarrow v_t \rightarrow v_k$. For $v_t \in B$, we see that $v_i \rightarrow v_t$ implies $v_{j-1} \rightarrow v_{j+1}$. For $v_t \in A$, we see that $v_t \rightarrow v_k$ implies $v_{j-1} \rightarrow v_{j+1}$.

Case 2: $d(v_i, v_{j+1}) \leq 2$ and $d(v_j, v_{i+1}) \leq 2$.

We will prove that either $v_{j-1} \rightarrow v_{j+1}$ or $v_{i-1} \rightarrow v_{i+1}$. Suppose to the contrary that $v_{j-1} \nrightarrow v_{j+1}$ and $v_{i-1} \nrightarrow v_{i+1}$. Because of symmetry, we may assume without loss of generality that $d_{D-v_j}(v_i, v_k) = 1$ and $d_{D-v_i}(v_j, v_k) \leq 2$.

If $v_k \in B$, then $v_i \rightarrow v_k$ implies that $v_{j-1} \rightarrow v_{j+1}$ by Lemma 2.1. So assume that

$v_k \in A$. If $d_{D-v_i}(v_j, v_k) = 1$, then $v_j \to v_k$ implies that $v_{i-1} \to v_{i+1}$. So assume that $d_{D-v_i}(v_j, v_k) = 2$. Then there is a vertex v_t such that $v_j \to v_t \to v_k$. For $v_t \in A$, we see that $v_j \to v_t$ implies $v_{i-1} \to v_{i+1}$. For $v_t \in B$, we see that $v_t \to v_k$ implies that $v_{i-1} \to v_{i+1}$.

(Sufficiency) Assume without loss of generality that $d(v_i, v_{j+1}) \leq 2$ and $v_{j-1} \to v_{j+1}$. It is not difficult to check that $d_{D-v_j}(v_i, v_{j+1}) \leq 2$. Since $d_{D-v_i}(v_j, v_{j+1}) = 1$, we have $v_iv_j \in E(G)$ according to the definition of the $(1, 2)$-step competition graph.

This completes the proof of this theorem. ∎

Proposition 2.3 *Let D be a connected and non-strong round digraph on n vertices. If $V(D_1) \cup V(D_2) \cup \cdots \cup V(D_t)$ is the strong decomposition of D, then $|V(D_i)| = 1$ for $i = 1, 2, \ldots, t$.*

Proof. By Proposition 1.1, D is a locally semicomplete digraph. Suppose there is an integer $r \in \{1, 2, \ldots, t\}$ such that $|V(D_r)| > 1$. Since $D[V(D_r)]$ is a strong locally semicomplete digraph, there is a hamiltonian cycle in $D[V(D_r)]$ by Theorem 1.3. Let $C = v_1v_2 \ldots v_mv_1$ be a hamiltonian cycle of $D[V(D_r)]$. Since D is connected, there is a vertex $x \in V(D - D_r)$ such that x dominates or is dominated by some $v_i \in V(D_r)$. Say $v_1 \to x$. Note that $v_1 \to v_2$. Hence x is adjacent to v_2 and then $v_2 \to x$. Similarly, since $v_2 \to v_3$, $v_3 \to x$. Continuing this way, we see that $V(D_r) \subseteq N^-(x)$. Obviously, $v_i \notin N^+(x)$ for any $v_i \in V(D_r)$. Hence $V(D_r) \subseteq N^-(x) - N^+(x)$, which contradicts Theorem 1.2. Thus $V(D_i) = 1$ for $i = 1, 2, \ldots, t$. ∎

Theorem 2.4 *Let D be a connected and non-strong round digraph on n vertices and $\{v_0\} \cup \{v_1\} \cup \cdots \cup \{v_{n-1}\}$ is the strong decomposition of D. If G is the $(1, 2)$-step competition graph of D, then $v_iv_j \in E(G)$ for $i < j$ if and only if $j < n - 1$, $d(v_i, v_{j+1}) \leq 2$ and $v_{j-1} \to v_{j+1}$.*

Proof. (Necessity) Let $P = v_0v_1v_2 \ldots v_{n-1}$ be a hamiltonian path of D. Clearly, $v_iv_j \notin E(G)$ for $j = n - 1$. Hence $j < n - 1$.

Since $v_iv_j \in E(G)$, there is a vertex v_k such that either $d_{D-v_j}(v_i, v_k) = 1$ and $d_{D-v_i}(v_j, v_k) \leq 2$ or $d_{D-v_i}(v_j, v_k) = 1$ and $d_{D-v_j}(v_i, v_k) \leq 2$. If $k < j$, then there are two integers $\alpha, \beta \in \{0, 1, \ldots, n - 1\}$ such that $v_\alpha \to v_\beta$ with $\alpha > \beta$. So there is a strong component containing at least two vertices, a contradiction. So $k > j$. By Lemma 2.1, $d(v_i, v_{j+1}) \leq 2$ since $d(v_i, v_k) \leq 2$.

Suppose $v_{j-1} \nrightarrow v_{j+1}$. If $d_{D-v_j}(v_i, v_k) = 1$, then we see that $v_{j-1} \to v_{j+1}$ since $k > j$. So $d_{D-v_j}(v_i, v_k) = 2$. Thus there is a vertex v_t such that $v_i \to v_t \to v_k$. For $t < j$, we have $v_{j-1} \to v_{j+1}$ since $v_t \to v_k$. For $t > j$, we have $v_{j-1} \to v_{j+1}$ since $v_i \to v_t$. In any case, we get a contradiction. Thus $v_{j-1} \to v_{j+1}$.

(Sufficiency) It is easy to check that $d_{D-v_j}(v_i, v_{j+1}) \leq 2$ since $d(v_i, v_{j+1}) \leq 2$ and $v_{j-1} \to v_{j+1}$. Since $d_{D-v_i}(v_j, v_{j+1}) = 1$, we have $v_iv_j \in E(G)$ according to the definition of the $(1, 2)$-step competition graph. ∎

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