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Connectivity of local tournaments

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Abstract

For a local tournament D with minimum out-degree δ^+, minimum in-degree δ^- and irregularity $i_g(D)$, we give a lower bound on the connectivity of D, namely $\kappa(D) \geq \lceil (2 \cdot \max\{\delta^+, \delta^-\} + 1 - i_g(D))/3 \rceil$ if there exists a minimum separating set S such that $D - S$ i is a tournament, and $\kappa(D) \geq \lceil (2 \cdot \max\{\delta^+, \delta^-\} + 2|\delta^+ - \delta^-| + 1 - 2i_g(D))/3 \rceil$ otherwise. This generalizes a result on tournaments presented by C. Thomassen [*J. Combin. Theory Ser. B* 28 (1980), 142–163]. An example shows the sharpness of this result.

1 Terminology and introduction

We consider finite digraphs without loops and multiple arcs. For any digraph D, the vertex set is denoted by $V(D)$ and the arc set by $A(D)$. By $n = n(D) = |V(D)|$ we refer to the *order* or D. For a vertex $x \in V(D)$ we denote by $N^+(x) = N^+_D(x)$ and $N^-(x) = N^-_D(x)$ the set of positive and negative neighbours of x in D, respectively. Furthermore, for a vertex set $X \subseteq V(D)$ the notation $N^+(X)$ refers to the vertex set $\left(\bigcup_{x \in X} N^+(x)\right) \setminus X$, and $N^-(X)$ accordingly. The *out-degree* $d^+(v) = d^+_D(v) = |N^+_D(v)|$ of a vertex v is the number of positive neighbours of v in D, and analogously, $d^-(v) = d^-_D(v) = |N^-_D(v)|$ denotes the *in-degree* of v. Furthermore, $\delta^+ = \delta^+(D) = \min\{d^+(v) \colon v \in V(D)\}$ denotes the *minimum out-degree* of D, and $\delta^- = \delta^-(D)$ the *minimum in-degree.* Also, we refer to the *minimum degree* $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$.

For two vertex sets $X, Y \subseteq V(D)$ we denote by $\overline{X}$ the vertex set $V(D) \setminus X$ and by (X, Y) the set of arcs from X to Y. Also, we define $[X, Y]$ as $[X, Y] = |(X, Y)|$. $D[X]$ is the digraph induced by X in D, and $D - X = D\left[\overline{X}\right]$. Let $x_1, \ldots, x_n$ be the vertices of D and $D_1, \ldots, D_n$ disjoint digraphs, then $H = D[D_1, \ldots, D_n]$ is

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defined by $V(H) = \bigcup_{i=1}^{n} V(D_i)$ and $A(H) = (\bigcup_{i=1}^{n} A(D_i)) \cup \{y_i y_j : y_i \in V(D_i), y_j \in V(D_j), x_i x_j \in A(D)\}$. If all arcs between two vertex sets X and Y are directed from X to Y, we write $X \to Y$, or $x \to Y$ in case $X = \{x\}$, and say X *dominates* Y. Let D_1 and D_2 be two digraphs, then D_1 *dominates* D_2, iff $V(D_1) \to V(D_2)$ and we write $D_1 \to D_2$.

A digraph which has at least one arc between every pair of distinct vertices is called *semicomplete digraph.* Orientations of complete graphs are called *tournament.* If for every vertex x of a digraph D the sets $N^+(x)$ and $N^-(x)$ both induce semicomplete digraphs, respectively, then D is called a *locally semicomplete digraph.* A local semicomplete digraph without cycles of length 2 is called *local tournament.*

A digraph is called *connected*, iff its underlying graph is connected, and *strongly connected* or *strong*, iff there exists a directed path from any vertex to any other vertex. By a *strong component* of a digraph that is not strong, we refer to a maximal strong induced subdigraph. We call a vertex set $S \subset V(D)$ a *separating set*, iff $D - S$ is not strong. A *minimal* separating set is minimal with respect to inclusion, and a *minimum* separating set is one of minimal cardinality. Furthermore, we define the *connectivity* of D as $\kappa(D) = \min\{|S| : S \text{ is a separating set of } D\}$. The *(global) irregularity* $i_g(D)$ of a digraph D is defined as $i_g(D) = \max\{\max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\} : x, y \in V(D)\}$. In case $i_g(D) = 0$ we call D a *regular* digraph.

Thomassen [6] studied the connectivity of tournaments according to their irregularity.

Theorem 1.1 (Thomassen [6], 1980). *If T is a tournament with $i_g(T) \le k$, then*

$$\kappa(T) \ge \left\lceil \frac{|V(T)| - 2k}{3} \right\rceil. \tag{1}$$

He also characterized the tournaments for which (1) holds with equality. Lichiardopol [5] presented a generalization of this result for oriented graphs, i.e. digraphs which have at most one arc between two vertices.

Theorem 1.2 (Lichiardopol [5], 2008). *If T is an oriented graph, then*

$$\kappa(T) \ge \left\lceil \frac{2\delta^+(T) + 2\delta^-(T) + 2 - n(T)}{3} \right\rceil. \tag{2}$$

It is also shown that (2) implies (1) for tournaments and that for tournaments with $\delta^+ \ne \delta^-$ Theorem 1.2 is an improvement of Theorem 1.1. In this work we will prove two lower bounds on the connectivity of two classes of local tournaments. One of them implies Lichiardopol's bound for tournaments. Although local tournaments are oriented graphs, our bound gives a better approximation for the connectivity of local tournaments.

2 Local tournaments and locally semicomplete digraphs

Every tournament is also a local tournament, and every local tournament is also a locally semicomplete digraph. The structure of these digraphs has been studied by

Bang-Jensen [1] and Guo and Volkmann [4]. A collection of their results and proofs can be found in [3]. The following results will be helpful for the proof of our main result in the next section.

Lemma 2.1 (Bang-Jensen [1], 1990). *Let D be a strong locally semicomplete digraph and let S be a minimal separating set of D. Then $D - S$ is connected.*

According to this property, it is helpful to study the structure of connected locally semicomplete digraphs that are not strong. Since local tournaments are locally semicomplete digraphs, the following results hold for local tournaments as well.

Theorem 2.2 (Bang-Jensen [1], 1990). *Let D be a connected locally semicomplete digraph that is not strong. Then the following holds for D.*

1. *If A and B are distinct strong components of D with at least one arc between them, then either $A \to B$ or $B \to A$.*
2. *If A and B are strong components of D such that $A \to B$, then A and B are semicomplete digraphs.*
3. *The strong components of D can be ordered in a unique way $D_1, D_2, \ldots, D_p$ such that there are no arcs from D_j to D_i for $j > i$, and D_i dominates D_{i+1} for $i = 1, 2, \ldots, p-1$.*

For a digraph D fulfilling the condition of Theorem 2.2 the unique ordering of its strong components is called the *acyclic ordering* of the strong components of D.

Theorem 2.3 (Guo, Volkmann [4], 1994). *Let D be a connected locally semicomplete digraph that is not strong and let $D_1, \ldots, D_p$ be the acyclic ordering of the strong components of D. Then D can be decomposed into $r \geq 2$ induced subdigraphs $D'_1, D'_2, \ldots, D'_r$ which satisfy the following properties.*

1. *$D'_1 = D_p$ and D'_i consists of some strong components of D and is semicomplete for $i \geq 2$.*
2. *D'_{i+1} dominates the initial component of D'_i and there exists no arc from D'_i to D'_{i+1} for $i = 1, \ldots, r-1$.*
3. *If $r \geq 3$, then there is no arc between D'_i and D'_j for i, j satisfying $|i - j| \geq 2$.*

The unique sequence $D'_1, D'_2, \ldots, D'_r$ is called the semicomplete decomposition of D.

Finally, using the semicomplete decomposition defined in Theorem 2.3 the next lemma determines the structure of locally semicomplete digraphs that are not semicomplete.

Lemma 2.4 (Bang-Jensen, Guo, Gutin, Volkmann [2], 1997). *If a strong locally semicomplete digraph D is not semicomplete, then there exists a minimal separating set S such that $D - S$ is not semicomplete. Furthermore, if $D_1, D_2, \ldots, D_p$ is the acyclic ordering of the strong components of $D - S$ and $D'_1, D'_2, \ldots, D'_r$ is the semicomplete decomposition of $D - S$, then $r \geq 3$, $D[S]$ is semicomplete and we have $D_p \to S \to D_1$.*

3 Main result

The inequality (1) in Theorem 1.1 gives a lower bound on the connectivity of tournaments with respect to their order $|V(T)|$ and irregularity $i_g(T)$. In Theorem 1.2 Lichiardopol uses the minimum out-degree, the minimum in-degree and the order of an oriented graph for a lower bound on its connectivity. Considering local tournaments, the number of vertices becomes an unsuitable parameter for a lower bound on the connectivity. A simple example for this is the oriented cycle $\vec{C}_n$ with n vertices, which is a local tournament with connectivity $\kappa(\vec{C}_n) = 1$ for every $n \geq 3$. However, a more meaningful parameter for local tournaments seems to be the minimum degree. We note that if a digraph D is not strong, we have $\kappa(D) = 0$. The following result gives a lower bound on the connectivity of strong local tournaments.

Theorem 3.1. *Let D be a strong local tournament with irregularity $i_g(D)$, and let S be a minimum separating set of D. If $D - S$ is a tournament, then*

$$\kappa(D) \geq \left\lceil \frac{2 \cdot \max\{\delta^+, \delta^-\} + 1 - i_g(D)}{3} \right\rceil,$$

else

$$\kappa(D) \geq \left\lceil \frac{2 \cdot \max\{\delta^+, \delta^-\} + 2|\delta^+ - \delta^-| + 1 - 2 \cdot i_g(D)}{3} \right\rceil.$$

Proof. Let $S \subset V(D)$ be a minimum separating set and $s = |S|$, thus, $\kappa(D) = s$ and S is also a minimal separating set. If possible, we choose S such that $D - S$ is a tournament. Since D is a strong local tournament, according to Lemma 2.1 the digraph $D - S$ is connected but not strong. Thus, $D - S$ has the structure as described in Theorem 2.2. Let $D_1, D_2, \ldots, D_p$ be the acyclic ordering of the strong components of $D - S$ for some $p \geq 2$. Denote $n_i = |V(D_i)|$ for all $i = 1, \ldots, p$. Theorem 2.2 implies that D_p is a tournament. Thus there is a vertex $x^* \in V(D_p)$ dominating at most $(n_p - 1)/2$ vertices in D_p. By the acyclic ordering of the strong components of $D - S$ a vertex in $V(D_p)$ can only dominate vertices of D_p or S, leading to

$$\delta^+ \leq |N_D^+(x^*)| \leq d_{D_p}^+(x^*) + s \leq \frac{n_p - 1}{2} + s$$

which implies

$$n_p \geq 2(\delta^+ - s) + 1. \tag{3}$$

Similarly, every vertex in D_1 can only be dominated by vertices in D_1 or S. Therefore, an analogue deduction for a vertex of D_1 leads to

$$n_1 \geq 2(\delta^- - s) + 1. \tag{4}$$

We consider two cases, whether $D - S$ is a tournament or not.

Case 1. If $D - S$ is a tournament, then D_1 is a tournament and every vertex in D_1 dominates all elements of $V(D_p)$. We have $x^* \in V(D_1)$ with at least $(n_1 - 1)/2$

positive neighbours in D_1. Considering the vertices dominated by x^* in $D-S$ we have

$$d^+_{D_1}(x^*) + n_p \leq \delta + i_g(D).$$

This implies

$$n_p \leq \delta + i_g(D) - d^+_{D_1}(x^*) \leq \delta + i_g(D) - \frac{n_1 - 1}{2},$$

and with the use of (3) and (4) we deduce

$$2(\delta^+ - s) + 1 \leq n_p \leq \delta + i_g(D) - (\delta^- - s).$$

This implies

$$s \geq \frac{2\delta^+ + \delta^- - \delta + 1 - i_g(D)}{3}.$$

Considering a vertex $y^* \in V(D_p)$ with at least $(n_p - 1)/2$ negative neighbours in D_p, an analogue deduction leads to

$$s \geq \frac{2\delta^- + \delta^+ - \delta + 1 - i_g(D)}{3}.$$

Altogether we have

$$\begin{aligned} s &\geq \max\left\{\frac{2\delta^+ + \delta^- - \delta + 1 - i_g(D)}{3}, \ \frac{2\delta^- + \delta^+ - \delta + 1 - i_g(D)}{3}\right\} \\ &= \frac{\max\{\delta^+, \delta^-\} + \delta^+ + \delta^- - \delta + 1 - i_g(D)}{3}. \end{aligned}$$

Taking into account that $\delta^+ + \delta^- - \delta = \max\{\delta^+, \delta^-\}$ we have

$$s \geq \frac{2 \cdot \max\{\delta^+, \delta^-\} + 1 - i_g(D)}{3}.$$

Case 2. If $D - S$ is not a tournament, then by Lemma 2.4 we have $D_p \to S$. Therefore,

$$[D_p, S] = n_p \cdot s.$$

Also, $D[S]$ is a tournament and thus $|A(D[S])| = \frac{1}{2}s(s-1)$. For $w \in S$ we have

$$\delta + i_g(D) \geq d^-_{D-(S\setminus\{w\})}(w) + d^-_{D[S]}(w).$$

For the number of arcs from D_p to S we now deduce

$$\begin{aligned} n_p \cdot s = [D_p, S] &\leq \sum_{w\in S} d^-_{D-(S\setminus\{w\})}(w) \\ &\leq \sum_{w \in S} \Big(\delta + i_g(D) - d^-_{D[S]}(w)\Big) \\ &= s(\delta + i_g(D)) - |A(D[S])| \\ &= s(\delta + i_g(D)) - \frac{1}{2}s(s-1), \end{aligned}$$

which implies

$$0 \leq s(\delta + i_g(D) - n_p) - \frac{1}{2}s(s-1) = s\left(\delta + i_g(D) - n_p - \frac{s}{2} + \frac{1}{2}\right).$$

Since $s \geq 1$, we have

$$n_p \leq \delta + i_g(D) - \frac{s}{2} + \frac{1}{2}. \tag{5}$$

Combining (5) with (3) yields

$$2(\delta^+ - s) + 1 \leq n_p \leq \delta + i_g(D) + \frac{1}{2} - \frac{s}{2},$$

and thus

$$s \geq \frac{2(2\delta^+ - \delta) + 1 - 2 \cdot i_g(D)}{3}.$$

By Lemma 2.4 we also have $S \to D_1$ and in analogy to (5) we deduce

$$n_1 \leq \delta + i_g(D) - \frac{s}{2} + \frac{1}{2}.$$

In combination with (4) we conclude

$$2(\delta^- - s) + 1 \leq n_1 \leq \delta + i_g(D) + \frac{1}{2} - \frac{s}{2},$$

which implies

$$s \geq \frac{2(2\delta^- - \delta) + 1 - 2 \cdot i_g(D)}{3}.$$

Altogether we have

$$\begin{aligned} s &\geq \max\left\{\frac{2(2\delta^+ - \delta) + 1 - 2 \cdot i_g(D)}{3}, \ \frac{2(2\delta^- - \delta) + 1 - 2 \cdot i_g(D)}{3}\right\} \\ &= \frac{4 \cdot \max\{\delta^+, \delta^-\} - 2\delta + 1 - 2 \cdot i_g(D)}{3}. \end{aligned}$$

Since $\max\{\delta^+, \delta^-\} - \delta = |\delta^+ - \delta^-|$, we arrive at

$$s \geq \frac{2 \cdot \max\{\delta^+, \delta^-\} + 2|\delta^+ - \delta^-| + 1 - 2 \cdot i_g(D)}{3}.$$

□

Since Theorem 3.1 uses the irregularity instead of the order of a digraph, it is not included in Theorem 1.2. To see this, we recognize e.g. that for the oriented cycle $\vec{C}_n$ with $n \geq 6$ our lower bound implies $\kappa(\vec{C}_n) \geq 1$, while the inequality (2) becomes trivial. In fact, for strong regular local tournaments T of order n and degree r the new bound is an improvement on the one given by Lichardopol [5]:

Theorem 3.1 implies

$$\kappa(T) \geq \left\lceil \frac{2r+1}{3} \right\rceil,$$

while Theorem 1.2 implies

$$\kappa(T) \geq \left\lceil \frac{4r+2-n}{3} \right\rceil,$$

and it is easy to see that $2r+1 \geq 4r+2-n$.

However, when considering tournaments the bounds of Theorem 3.1 and Theorem 1.2 coincide.

Corollary 3.2. *Theorems 3.1 and 1.2 imply the same lower bound on the connectivity of tournaments.*

Proof. Let T be a tournament with irregularity $i_g(T)$, then T has exactly $n = 2\delta(T)+1+i_g(T)$ vertices. We have

$$\left\lceil \frac{2\delta^+(T)+2\delta^-(T)+2-n}{3} \right\rceil = \left\lceil \frac{2\delta^+(T)+2\delta^-(T)+1-2\delta(T)-i_g(T)}{3} \right\rceil = \left\lceil \frac{2\cdot\max\{\delta^+(T),\delta^-(T)\}+1-i_g(T)}{3} \right\rceil.$$

Therefore, our bound and the one given by Lichiardopol coincide for tournaments. □

In the following example we present local tournaments for which the bound of Theorem 1.2 exceeds the bound given in Theorem 3.1.

Example 3.3. *From the oriented* $(18l+6)$*-cycle* $v_1v_2...v_{18l+6}$ *we obtain the* $(6l+2)$*-regular local tournament* T_l *by adding arcs from every vertex to its* $6l+2$ *successors on the cycle (without multiple arcs). Furthermore, from* T_l *we obtain the local tournament* T_l^* *by adding the arc set* $\{v_iv_j\colon 1 \leq i \leq 3l+1, 6l+3+i \leq j \leq 9l+4\}$.

Obviously, we have $\delta^+(T_l^*) = \delta^-(T_l^*) = \delta(T_l^*) = 6l+2$, $n(T_l^*) = 18l+6$ and $i_g(T_l^*) = 3l+1$. The bound given in Theorem 3.1 leads to $\kappa(T_l^*) \geq 2l+1$, while Theorem 1.2 implies $\kappa(T_l^*) \geq 2l+2$. However, it is easy to see that in fact it is $\kappa(T_l^*) = 6l+2$.

In [5] it has been shown that Theorem 1.1 follows from Theorem 1.2. By Corollary 3.2 we see that Theorem 1.1 also follows from Theorem 3.1. If T is a tournament with $i_g(T) \leq k$, then $|V(T)| \leq 2\delta+k+1$, and if S is a minimum separating set, then $T-S$ is a tournament as well. By Theorem 3.1 we obtain

$$\kappa(T) \geq \left\lceil \frac{2\cdot\max\{\delta^+,\delta^-\}+1-i_g(T)}{3} \right\rceil \geq \left\lceil \frac{2\delta+2|\delta^+-\delta^-|+1-k}{3} \right\rceil \geq \left\lceil \frac{|V(T)|-2k+2|\delta^+-\delta^-|}{3} \right\rceil \geq \left\lceil \frac{|V(T)|-2k}{3} \right\rceil.$$

In case $\delta^+ \neq \delta^-$ we have an improvement of the lower bound in (1) by Thomassen.

For tournaments T the examples given by Thomassen [6] also show the sharpness of Theorem 3.1 for the case that T has a minimum separating set S such that $T - S$ is a tournament. In the other case, the following examples confirm that the second bound presented in Theorem 3.1 is best possible as well.

Example 3.4. *Let T_0 be a single vertex and T_r an r-regular tournament for $r \geq 1$. $\vec{C}_{2i}$ denotes the directed cycle of length $2i$. For integers $l, m \geq 0$ and $i \geq 2$ we define the digraph H by*

$$H = \vec{C}_{2i}[T_l, T_{l+m}, T_l, T_{l+m}, \ldots, T_l, T_{l+m}].$$

Obviously, H is a strong local tournament, where every subdigraph of the form T_r has $2r+1$ vertices. Therefore, we have $\delta^+(H) = \delta^-(H) = \delta(H) = 3l + m + 1$ and $i_g(H) = m$. The vertex set of every subdigraph T_l is a separating set, thus, according to Theorem 3.1 we have

$$2l + 1 = |V(T_l)| \geq \kappa(H) \geq \frac{2\delta(H) - 2m + 1}{3} = 2l + 1.$$

Finally, we notice that a result similar to Theorem 3.1 can not be obtained for locally semicomplete digraphs. The following examples show that the gap between the minimum degree and the connectivity of local semicomplete digraphs can be arbitrarily large.

Example 3.5. *Let $\overleftrightarrow{K}_r$ be the complete digraph on $r \geq 1$ vertices. For integers $l, m \geq 1$ and $i \geq 2$ we define the digraph F by*

$$F = \vec{C}_{2i}[\overleftrightarrow{K}_l, \overleftrightarrow{K}_{l+m}, \overleftrightarrow{K}_l, \overleftrightarrow{K}_{l+m}, \ldots, \overleftrightarrow{K}_l, \overleftrightarrow{K}_{l+m}].$$

According to this definition F is a strong locally semicomplete digraph, which is $(2l + m - 1)$-regular. The vertex set of every subdigraph $\overleftrightarrow{K}_l$ is a separating set, thus, it is easy to see that $\kappa(F) = l$.

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