AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **57** (2013), Pages 251–270

Desargues, doily, dualities and exceptional isomorphisms

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Für Herrn Professor Dr. G. Pickert
aus Anlass seines 95. Geburtstags.

Abstract

We study various graphs associated with symmetric groups. This leads to a quick understanding of automorphisms, dualities and polarities both for the Desargues configuration and the smallest generalised quadrangle. The results are used to establish the existence of outer automorphisms of the symmetric group on six symbols, and the exceptional isomorphism between the alternating group on six symbols and the little projective group of the projective line over the field with nine elements. For the latter isomorphism, we present a new proof of uniqueness for the inversive plane of order 3.

1 Introduction

The Desargues configuration is an incidence structure with ten points and ten lines; it plays an important role in the foundations of geometry (see 1.3 below). It turns out that it is also of great interest of its own; this is due to its inherent symmetry. We give a sleek description of this configuration first:

1.1 Definition. Let F be a set of 5 elements. As point set, we take the set $\binom{F}{2}$ of all subsets of size 2 in F, the set $\binom{F}{3}$ of lines consists of all subsets of size 3, and incidence is inclusion: $\mathbb{D} := \left(\binom{F}{2}, \binom{F}{3}, \subset\right)$.

It seems that this labelling dates back to [12, IV, §1]. In the present notes, we will use it to give a simple proof of the (long known) fact that the automorphism group of $\mathbb{D}$ is isomorphic to the symmetric group S_5. We will also generalise the construction, so that the Petersen graph, the doily (viz., the smallest generalised quadrangle) and the inversive planes of orders 2 and 3 can be treated together with the Desargues configuration, in a certain uniform way. The study of dualities will shed light on the (also known) fact that the group S_6 admits automorphisms that are not inner ones. Finally, we use our observations to show that the alternating group A_6 is isomorphic to the little projective group $PSL(2, 9)$ of the projective line over the field with nine elements (which also acts on the inversive plane described by the field extension $\mathbb{F}_9/\mathbb{F}_3$). To this end, we establish an isomorphism between the incidence graph of the doily and the complement of the confluence graph of an inversive plane of order 3; see Section 6. Uniqueness of that inversive plane is shown in Section 7.

1.2 Desargues' configuration in the plane. Levi's labelling (as used in 1.1) does not give a good geometric intuition—unless you are used to looking at projective spaces in dimension 4, like Levi [12]. There are many ways to draw the Desargues configuration in the Euclidean plane, see Fig. 1.

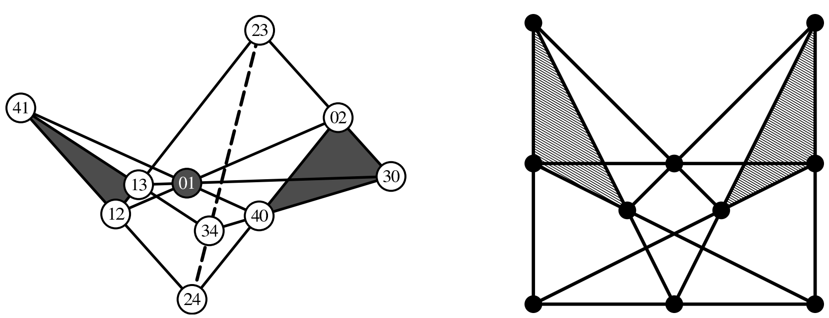

Figure 1: Different drawings of the Desargues configuration, using straight lines.

The shaded triangles in these drawings hint at an interpretation that is used frequently: We have two triangles that lie central with respect to some point (labelled 01 in the drawings on the left in Fig. 1 and in Fig. 2, respectively) and axial with respect to some line (containing the points with the labels 23, 34 and 24 in Figs. 1 and 2, respectively). Any such drawing will hide some of the inherent symmetry of the configuration (but see Fig. 3 below). However, if one is willing to accept that some of the lines are represented by curves rather than straight lines one can exhibit the dihedral group of order 10 as a subgroup of the group of all automorphisms. We show such a drawing in Fig. 2—and thank the anonymous referee for suggesting this picture.

The following fact has long been known, see [9, § 19] for a proof:

1.3 Desargues' Theorem. *If two triangles* (a_1, a_2, a_3) *and* (b_1, b_2, b_3) *in a projective space of dimension at least* 3 *are in central position (i.e., if the lines* $a_j \vee b_j$ *are confluent) then they are in axial position, as well (i.e., the points* $(a_j \vee a_k) \wedge (b_j \vee b_k)$ *are collinear).*

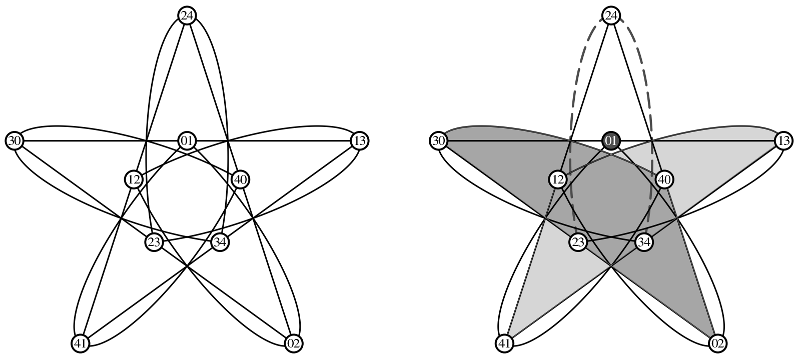

Figure 2: A more symmetric drawing, with bent lines (left), and with triangles (right).

The two triangles, the centre and the points on the axis form a Desargues configuration.

There are, however, many projective planes containing triangles in central position that are not in axial position. The reader may consult Hilbert's classic [8, V § 30] for the role of the Desargues configuration in the foundations of geometry, or get an impression of how the Desargues configuration is used to understand the wealth of projective planes from Chapter 3 of [17] or Sections IV, V and VI in [10]. The early history of non-Desarguesian plane geometries is traced in [22].

2 Some graphs associated with symmetric groups

Let F be any set (finite or not), and let n be a natural number. By $\binom{F}{n}$ we denote the set of all subsets of size n in F. In order to save some brackets, we denote the sets $\{i,j\}$ and $\{i,j,k\}$ by ij and ijk, respectively.

We need some terminology: an *incidence structure* (with two types) is a triplet $\mathbb{P} = (P, B, I)$ consisting of two sets P and B together with the incidence relation $I \subseteq P \times B$. Often, one calls the elements of P *points* and those of B *blocks* (sometimes *lines*, *edges*, or *circles*), the elements of I are also called *flags*. The *dual* $\mathbb{P}^* = (B, P, I^*)$ of $\mathbb{P}$ is obtained by interchanging the roles of points and blocks; with the incidence relation $I^* := \{(b,p) \mid (p,b) \in I\}$. If each block $b \in B$ is determined by its *point row* $P_b := \{p \in P \mid (p,b) \in I\}$ then we may replace B by the set $\{P_b \mid b \in B\}$; the incidence relation is then just "$\in$". The set of blocks through a point p is denoted by B_p.

A *graph*[1] is an incidence structure $\mathbb{G} = (V, E, \in)$ where $E \subseteq \binom{V}{2}$. Traditionally, the points are called *vertices*, the blocks are called *edges*, and two vertices are called *adjacent* if they form an edge. The *complement graph* $\mathbb{G}^c := \left(V, \binom{V}{2} \setminus E, \in\right)$ has the same vertex set, but two vertices are adjacent in $\mathbb{G}^c$ precisely if they are not adjacent

[1] We only consider unordered graphs without loops here.

in $\mathbb{G}$.

Two different graphs are naturally defined on the set $\binom{F}{2}$:

2.1 Definitions. The graph $\mathbb{E}_F$ has $\binom{F}{2}$ as vertex set, and two vertices (i.e., two subsets of size 2 in F) are adjacent if they are disjoint[2]. The graph $\mathbb{M}_F := {\mathbb{E}_F}^{\mathrm{c}}$ is the complement graph of $\mathbb{E}_F$. If F is finite of size n, we replace it by the set $\{j \in \mathbb{N} \mid j < n\}$ of the same size, identifying $\mathbb{E}_n := \mathbb{E}_F$ and $\mathbb{M}_n := \mathbb{M}_F$.

2.2 Buttons and laces. For any set F of small size n, one obtains nice (and self-explaining) labels for the elements of $\binom{F}{2}$ and for subsets of $\binom{F}{2}$ by fixing a set of n points ("button holes") in a suitable arrangement, endowed with marks ("laces") that indicate the subsets in question. We have taken inspiration from [18, 4.3, p. 54] (cf. also [19] and [24]).

2.3 Definition. To any incidence structure $\mathbb{P} = (P, B, I)$ we associate the following graphs:

- The *incidence graph* $\mathbb{J}_\mathbb{P} := (P \coprod B, \{\{p, b\} \mid p \in P, b \in B, (p, b) \in I\}, \in)$ has the disjoint union[3] $P \coprod B$ as vertex set; a point and a block are adjacent precisely if they are incident.

- The *collinearity graph* $\mathbb{C}_\mathbb{P}$ of $\mathbb{P}$ is the graph with vertex set P where an edge is put between two vertices if, and only if, these points are joined by a block in $\mathbb{P}$.

- The *confluence graph* of $\mathbb{P}$ is the collinearity graph $\mathbb{C}_{\mathbb{P}^*}$ of the dual $\mathbb{P}^*$.

2.4 Examples. Let $F := \{0, 1, 2, 3, 4\}$ be the standard set of size 5.

1. The Desargues configuration $\mathbb{D} = \left(\binom{F}{2}, \binom{F}{3}, \subset\right)$ has $\mathbb{M}_5 = \mathbb{M}_F$ as its collinearity graph.

2. Various graphical representations of the incidence graph $\mathbb{J}_\mathbb{D}$ of $\mathbb{D}$ are shown in Fig. 3: in Fig. 3(a), one sees an automorphism of order 5 (and its powers) plus five involutions; these elements clearly form a dihedral group of order 10. The pictures in Figs. 3(b), 3(c) and 3(d) should be thought of as spatial representations of the graph. These pictures allow to "see" actions of groups isomorphic to $\mathrm{S}_2 \times \mathrm{S}_3$ and S_4, respectively. It is also possible to see certain polarities. See 5.3 below.

3. The graph $\mathbb{E}_5$ is the famous *Petersen graph*, see Fig. 4.

2.5 Definition. Let $\mathbb{G} = (V, E, \in)$ be a graph. A *clique* in $\mathbb{G}$ is a subset $C \subseteq V$ such that any two vertices in C are adjacent in $\mathbb{G}$; we call a clique C *maximal* if it is not contained as a proper subset in any other clique in $\mathbb{G}$.

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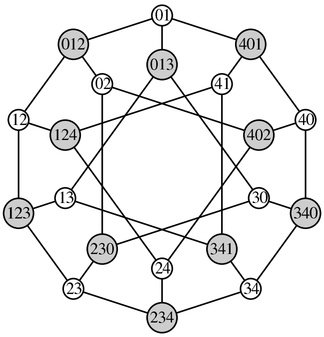
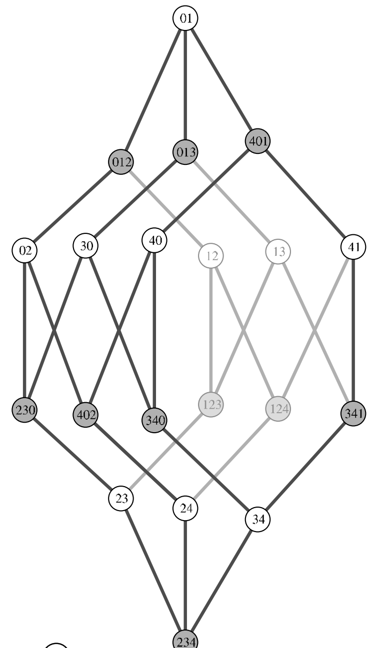
[2]We think of $\mathbb{M}$eeting sets and $\mathbb{E}$mpty intersections.

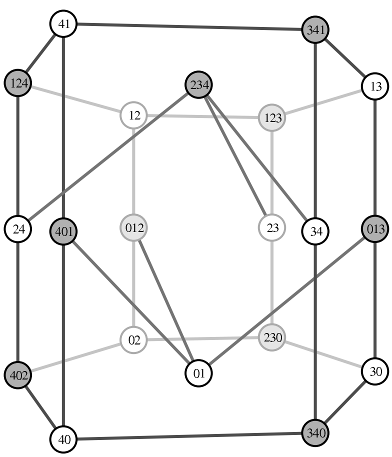
[3]If $P \cap B \neq \emptyset$ we have to use one of the standard tricks to replace them by a pair of disjoint sets.



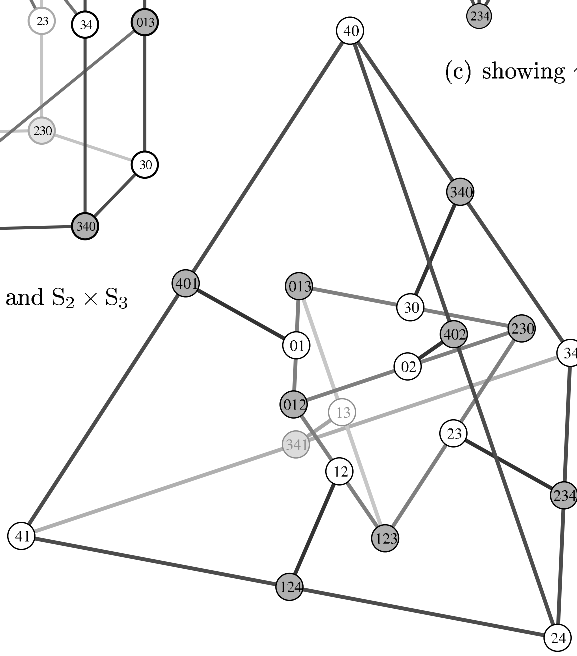
(a) showing a 5-cycle, and the polarity γ



(c) showing γ and $\gamma\delta$



(b) showing γ, $\gamma\tau$, and $S_2 \times S_3$



(d) showing S_4

Figure 3: Pictures of the incidence graph $\mathbb{J}_{\mathbb{D}}$; cf. 5.1, 5.2, and 5.3.

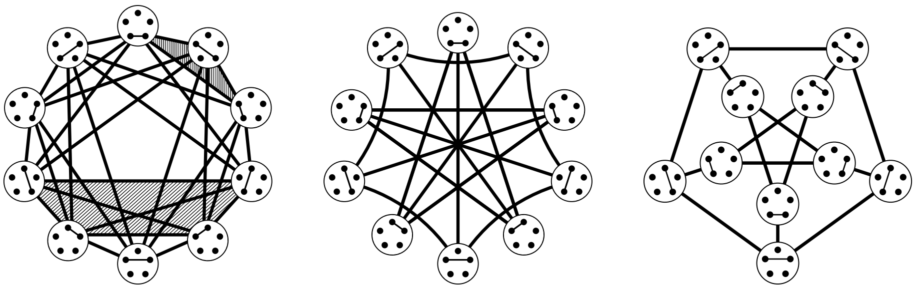


Figure 4: Collinearity graph $\mathbb{C}_{\mathbb{D}} = \mathbb{M}_5$ of the Desargues configuration (with two maximal cliques indicated, on the left, cf. 2.6), and its complement (in the middle and once more on the right): this is the Petersen graph $\mathbb{E}_5$.

2.6 Theorem. *Let F be a finite set with at least 4 elements.*

(a) *If $|F|$ is even then the set $\mathcal{P}_F$ of maximal cliques in $\mathbb{E}_F$ consists of all partitions of F into two-element subsets (of F).*

(b) *If $|F|$ is odd then $\mathcal{P}_F$ consists of all partitions of complements of singleton subsets in F into two-element subsets (of F).*

(c) *The maximal cliques in $\mathbb{M}_F$ fall into two disjoint sets, namely*[4] $\mathcal{T}_F := \big\{\{ab, bc, ca\} \;\big|\; abc \in \binom{F}{3}\big\}$ *and* $\mathcal{V}_F := \big\{\{ax \mid x \in F \smallsetminus \{a\}\} \;\big|\; a \in F\big\}$.

Proof. We note first that any clique in $\mathbb{E}_F$ must consist of a collection of pairwise disjoint elements of $\binom{F}{2}$. If $|F|$ is even then every such clique is contained in a partition by subsets of size two, and these partitions are the maximal cliques. If $|F|$ is odd, then there do not exist any partitions of F into subsets of size 2. The union over any clique in $\mathbb{E}_F$ will then miss at least one point of F, and the maximal cliques miss precisely one. This proves the first two assertions.

In order to prove the last one, we note first that the elements of $\mathcal{V}_F$ and those of $\mathcal{T}_F$ are in fact maximal cliques of $\mathbb{M}_F$. If ab and ac lie in a maximal clique C then $\{ab, ac\} \neq C$ because $|F| > 3$. For $yz \in C \smallsetminus \{ab, ac\}$ we then obtain $yz = bc$ or $a \in yz$. Thus $C \in \mathcal{T}_F$ or $C = \{ax \mid x \in F \smallsetminus \{a\}\} \in \mathcal{V}_F$, as claimed. □

3 Reconstruction, and full groups of automorphisms

We start with a general observation:

3.1 Lemma. *Let $\mathbb{G} = (V, E, \in)$ be a graph, and let $\mathcal{B}$ be a set of cliques such that for each edge $e \in E$ there exists at least one $B \in \mathcal{B}$ with $e \in B$. Then the collinearity graph $\mathbb{C}_{(V,\mathcal{B},\in)}$ coincides with $\mathbb{G}$, and* $\mathrm{Aut}\,(V, \mathcal{B}, \in) \leq \mathrm{Aut}\,(\mathbb{G})$.

[4]Here we think of $\mathcal{T}$riangles and $\mathcal{V}$ertices.

Proof. It follows immediately from our assumption that two vertices $x, y \in V$ are collinear in $(V, \mathcal{B}, \in)$ precisely if they form an edge in $\mathbb{G}$. Thus $\mathbb{C}_{(V,\mathcal{B},\in)} = \mathbb{G}$. It is true in general that every automorphism of an incidence structure induces an automorphism of the collinearity graph of that structure. As different automorphisms of $(V, \mathcal{B}, \in)$ act differently on V, we obtain our last assertion: $\mathrm{Aut}\,(V, \mathcal{B}, \in) \leq \mathrm{Aut}\,(\mathbb{C}_{(V,\mathcal{B},\in)})$. □

We return to our incidence geometries constructed from subsets of a set F. The group S_F of all permutations of F acts in an obvious way by automorphisms of the incidence geometries $\mathbb{M}_F$, $\mathbb{E}_F$, $\big(\binom{F}{2}, \mathcal{T}_F, \in\big)$, $\big(\binom{F}{2}, \mathcal{V}_F, \in\big)$, or $\big(\binom{F}{2}, \mathcal{P}_F, \in\big)$. The action in question is faithful if $|F| > 2$; we will then identify S_F with the induced group of automorphisms.

3.2 Lemma. *For any set F, the incidence structure* $\big(\binom{F}{2}, \mathcal{V}_F, \in\big)$ *is dual to* $\big(F, \binom{F}{2}, \in\big)$. *This is the complete graph on F, and* $\mathrm{Aut}\,\big(\binom{F}{2}, \mathcal{V}_F, \in\big) = \mathrm{Aut}\,\big(F, \binom{F}{2}, \in\big) = \mathrm{S}_F$.

3.3 Theorem. *Let F be any set, and let $\mathbb{X}$ be one of* $\mathbb{M}_F$, $\mathbb{E}_F$, $\big(\binom{F}{2}, \mathcal{T}_F, \in\big)$, $\big(\binom{F}{2}, \mathcal{V}_F, \in\big)$, *or* $\big(\binom{F}{2}, \mathcal{P}_F, \in\big)$. *If* $|F| \geq 5$ *then* $\mathrm{S}_F = \mathrm{Aut}\,(\mathbb{X})$.

Proof. We already know that S_F is contained in $\mathrm{Aut}\,(\mathbb{X})$. It is also true that $\mathrm{Aut}\,(\mathbb{M}_F) = \mathrm{Aut}\,(\mathbb{E}_F)$ because $\mathbb{E}_F$ is the complement of $\mathbb{M}_F$.

The members of $\mathcal{T}_F$ have size 3 while those of $\mathcal{V}_F$ have size $|F|-1$. For $|F| \neq 4$, this means that $\mathrm{Aut}\,(\mathbb{M}_F)$ leaves both $\mathcal{T}_F$ and $\mathcal{V}_F$ invariant. Now $\mathrm{Aut}\,(\mathbb{M}_F)$ is contained in the automorphism group of $\big(\binom{F}{2}, \mathcal{T}_F, \in\big)$, of $\big(\binom{F}{2}, \mathcal{V}_F, \in\big)$, and of $\big(\binom{F}{2}, \mathcal{P}_F, \in\big)$, respectively. Conversely, Lemma 3.1 yields $\mathrm{Aut}\,\big(\binom{F}{2}, \mathcal{T}_F, \in\big) \leq \mathrm{Aut}\,(\mathbb{M}_F)$ and $\mathrm{Aut}\,\big(\binom{F}{2}, \mathcal{P}_F, \in\big) \leq \mathrm{Aut}\,(\mathbb{E}_F) = \mathrm{Aut}\,(\mathbb{M}_F)$. Finally, we have $\mathrm{Aut}\,\big(\binom{F}{2}, \mathcal{V}_F, \in\big) = \mathrm{S}_F$ by Lemma 3.2. □

3.4 Corollary. *The full automorphism group of the Petersen graph* $\mathbb{E}_5$ *is* $\mathrm{Aut}\,(\mathbb{E}_5) = \mathrm{S}_5$. □

3.5 Remark. The graph $\mathbb{E}_F$ is also known as a *Kneser graph*, and denoted by $\mathrm{K}_{F,2}$. One knows in general that $\mathrm{Aut}\,(\mathbb{E}_F) = \mathrm{S}_F$ if $|F| \geq 5$, cf. [15] or [7, 7.8.2]. The automorphism groups $\mathrm{Aut}\,(\mathbb{E}_2)$ and $\mathrm{Aut}\,(\mathbb{E}_4)$ form exceptions to this general pattern, see Examples 3.7 below.

3.6 Theorem. *The full automorphism group of the Desargues configuration is* $\mathrm{Aut}\,(\mathbb{D}) = \mathrm{S}_5$.

Proof. The group $\mathrm{Aut}\,(\mathbb{D})$ contains S_5 and is embedded in the automorphism group of the collinearity graph $\mathbb{C}_{\mathbb{D}} = \mathbb{M}_5$, see Examples 2.4. Our claim follows from the fact that $\mathrm{Aut}\,(\mathbb{M}_5) = \mathrm{S}_5$, see Theorem 3.3. □

3.7 Examples. The small values $n \in \{2, 3, 4\}$ form exceptions to our general results, as follows:

1. The symmetric group S_2 acts trivially on $\binom{2}{2}$, and the trivial group $\mathrm{Aut}\,(\mathbb{E}_2)$ is a proper quotient of S_2.

2. The graph $\mathbb{M}_3$ *is* just one clique. There is only one type of maximal clique; the set $\mathcal{V}_3$ does not consist of *maximal* cliques.

3. The sets $\mathcal{T}_4$ and $\mathcal{V}_4$ are fused under the action of $\mathrm{Aut}(\mathbb{E}_4)$. The group $\mathrm{Aut}(\mathbb{E}_4)$ is indeed larger than S_4, it is a semi-direct product of S_3 with an elementary abelian group of order 2^3. Thus $|\mathrm{Aut}(\mathbb{E}_4)| = 2^4 \cdot 3 = 2|\mathrm{S}_4|$. Indeed, the graph $\mathbb{E}_4$ has 6 vertices and 3 edges; every vertex belongs to precisely one edge.

4 The doily

Generalised quadrangles play an important role in Jacques Tits' theory of buildings, but they are also a quite classical object of geometry because a large family of "classical" examples is provided by certain ruled quadrics. We will only consider the smallest specimen here (which represents the unique isomorphism type of generalised quadrangles of order 2, cf. [21]).

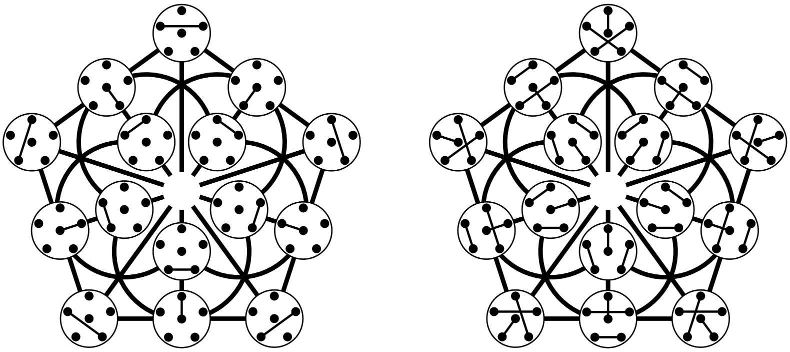


Figure 5: The doily $\mathbb{W}$, and its dual.

4.1 Example. Let $S := \{0, 1, 2, 3, 4, 5\}$, and recall that $\mathcal{P}_S$ denotes the set of all partitions of S into three subsets of size 2, cf. 2.6. The incidence structure $\mathbb{W} := \left(\binom{S}{2}, \mathcal{P}_S, \in\right)$ is a generalised quadrangle (i.e., its incidence graph $\mathbb{J}_{\mathbb{W}}$ has diameter 4 and girth 8, and every vertex has valency at least three). In fact, it represents the isomorphism type of the smallest generalised quadrangle, known as the *doily*, see Fig. 5.

The incidence structure $\mathbb{W}$ is shown in Fig. 5, its incidence graph in Fig. 7. The collinearity graph of the doily is $\mathbb{C}_{\mathbb{W}} = \mathbb{E}_6$, shown in Fig. 6 (it is isomorphic to the confluence graph because the doily admits dualities, see 5.4).

4.2 Theorem. *The automorphism group of the doily is* $\mathrm{Aut}(\mathbb{W}) = \mathrm{S}_6$.

Proof. This is a special case of 3.3. ☐

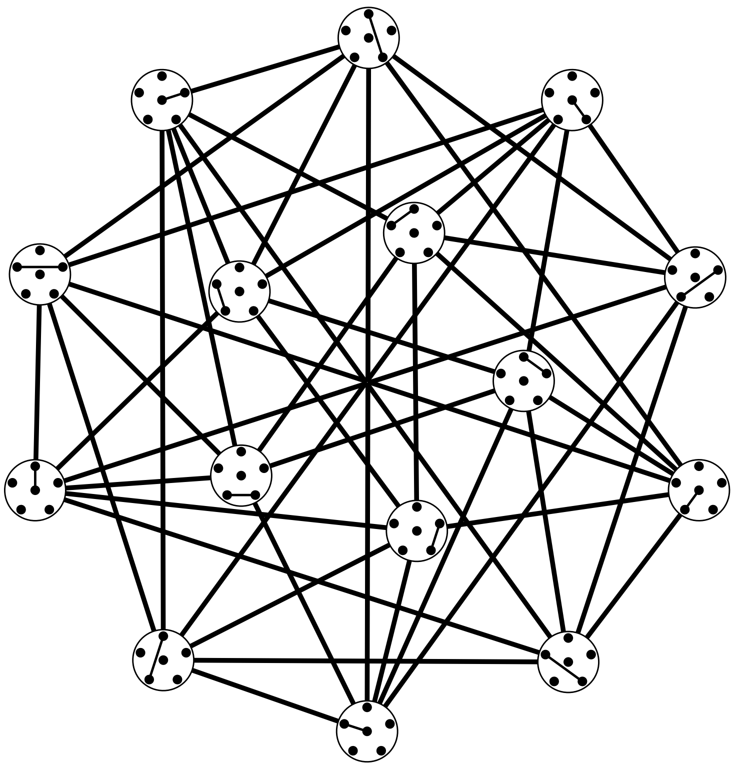


Figure 6: The collinearity graph $\mathbb{C}_{\mathbb{W}}$ of the doily.

4.3 Remark. There are two actions of $\mathrm{Aut}(\mathbb{W})$ on geometrically defined sets of size 6, namely the set of *ovoids* (i.e., maximal sets of pairwise non-collinear points in $\mathbb{W}$) and the set of spreads (defined dually, as maximal sets of pairwise non-confluent lines in $\mathbb{W}$).

One of those ovoids is $\big\{\{0,5\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\big\}$; the stabiliser of that ovoid is the stabiliser S_5 of the subset $\{0,1,2,3,4\} \subset S$, taken in S_6.

4.4 Remarks. Freudenthal [6] has used this description of the doily, extending it to a description of the generalised quadrangle of order $(2,4)$ (i.e., with 3 points per line and 5 lines per point). The extra points come as two copies S and $S' := \{0',1',2',3',4',5'\}$ of the set S of size six, new lines are obtained as sets $\{j,\{j,k\},k'\}$ for $\{j,k\} \in \binom{S}{2}$. That description shows that the action of S_6 extends to an action of S_6 on the larger generalised quadrangle. However, the determination of the full group of automorphisms is harder than the study of $\mathrm{Aut}(\mathbb{W})$.

See [16, 6.1.3] for a brief résumé of Freudenthal's paper, and [21] for another elementary approach to the generalised quadrangles with 3 points per line (including proofs of uniqueness).

It is quite amazing what can be done with a set of six elements, together with its subsets of size two and its partitions into sets of size two (viz., the points and lines of the doily). Among the structures accessible via constructions in this vein are the projective plane of order 4, the Hoffman-Singleton graph and the $5-(12,6,1)$ design, see [2, Ch. 8] (or the extended version in [3, Ch. 6]). The $5-(12,6,1)$ design can then be used to find an outer automorphism of the Mathieu group M_{12}; we do this for the small case S_6 in Section 5 below.

5 Dualities

Recall that a *duality* of an incidence structure $\mathbb{P} = (P, B, I)$ is an isomorphism from $\mathbb{P}$ onto its dual $\mathbb{P}^*$. Every duality induces an automorphism of the incidence graph $\mathbb{J}_{\mathbb{P}}$ interchanging P with B. The duality is called a *polarity* if this automorphism of $\mathbb{J}_{\mathbb{P}}$ is an involution.

Both $\mathbb{D}$ and $\mathbb{W}$ admit polarities. These induce (via conjugation) involutory automorphisms of $\mathrm{Aut}(\mathbb{D}) = \mathrm{S}_5$ and $\mathrm{Aut}(\mathbb{W}) = \mathrm{S}_6$, respectively. We study these in detail now.

5.1 Example. Let $F = \{0, 1, 2, 3, 4\}$. Interchanging X with its complement $F \setminus X$ gives a polarity γ of $\mathbb{D} = \left(\binom{F}{2}, \binom{F}{3}, \in\right)$. This polarity commutes with every element $\alpha \in \mathrm{Aut}(\mathbb{D}) = \mathrm{S}_5$ because $F \setminus X^\alpha = (F \setminus X)^\alpha$. Therefore, it induces the identity on $\mathrm{Aut}(\mathbb{D}) = \mathrm{S}_5$, and $\mathrm{Aut}(\mathbb{J}_{\mathbb{D}}) \cong \mathrm{S}_2 \times \mathrm{S}_5$.

Every other duality of $\mathbb{D}$ is obtained as $\gamma\alpha$ for some $\alpha \in \mathrm{Aut}(\mathbb{D})$, and induces an inner automorphism of S_5. We note that $\gamma\alpha$ is a polarity precisely if $\alpha^2 = \mathrm{id}$.

For $\beta, \sigma \in \mathrm{Aut}(\mathbb{D})$ we have $\beta^{-1}(\gamma\sigma)\beta = \gamma\beta^{-1}\sigma\beta$. This yields that dualities $\gamma\alpha_1$ and $\gamma\alpha_2$ are conjugates under S_5 precisely if α_1 and α_2 are conjugates in S_5. Using the representatives $\tau = (01)$ and $\delta = (01)(24)$ of involutions in S_5, we obtain:

5.2 Theorem. *The set of all polarities of $\mathbb{D}$ is the union of three conjugacy classes, represented by γ, $\gamma\tau$, and $\gamma\delta$.*

5.3 Remark. One sees γ as the reflection at the centre of the drawings of the incidence graph in Fig. 3(a), in Fig. 3(b), and also in Fig. 3(c) — note that the latter representations are meant to be three-dimensional. The reflection in a horizontal plane shows $\gamma\delta$ in Fig. 3(c), and a half-turn around the vertical axis describes $\gamma\tau$ in Fig. 3(b) (we also have to interchange the two inner vertices on the vertical axis).

An edge of the incidence graph is called an *absolute flag* with respect to a polarity if it is fixed by the polarity. One sees immediately that γ does not have any absolute flags at all. It is less obvious but still true that $\gamma\tau$ does not have absolute flags, cf. Fig. 3(b).

The set of absolute flags of $\gamma\delta$ is $\big\{\{02, 230\}, \{40, 340\}, \{12, 123\}, \{41, 341\}\big\}$; this can be seen conveniently with the help of Fig. 3(c) where $\gamma\delta$ occurs as the reflection in a horizontal plane.

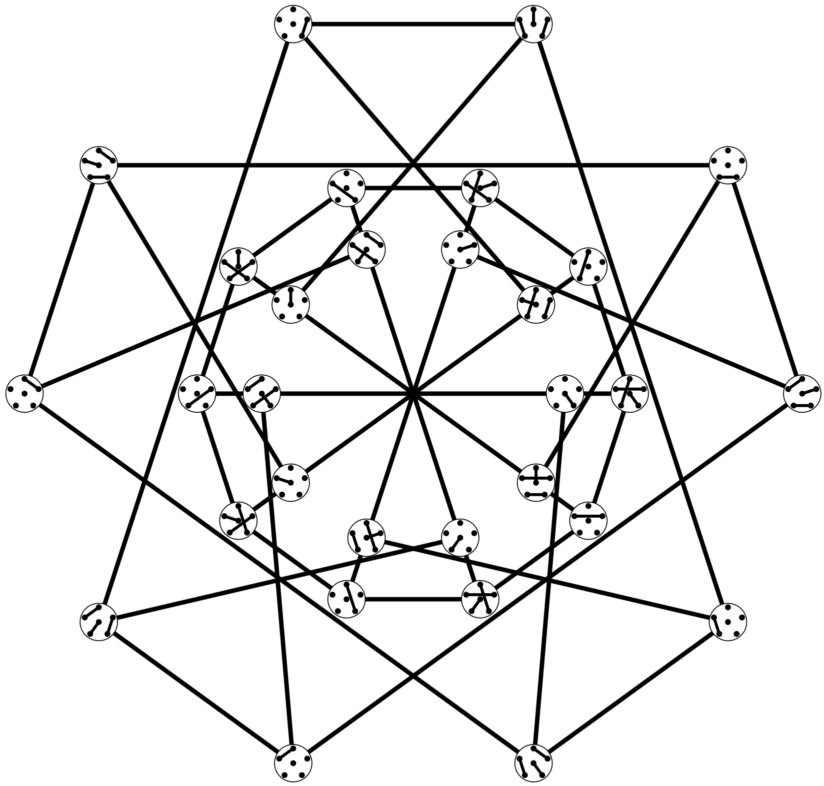
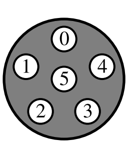


Figure 7: The incidence graph $\mathbb{J}_{\mathbb{W}}$ of the doily—and the complement of the confluence graph of the inversive plane Ip_3 of order 3.

5.4 Examples. The doily $\mathbb{W}$ admits polarities.

One of these can be seen in the representation of the incidence graph $\mathbb{J}_{\mathbb{W}}$ in Fig. 7, as the reflection ρ at a vertical axis. The conjugate $\pi = (35)\rho(35)$ of the polarity ρ is also visible in Fig. 5: we map each label $p \in \binom{S}{2}$ used in the left image to the partition p^π used as the label for the same position in the right image.

For the sake of easy reference, we number the six holes, as shown in the drawing on the right. Then the polarity π can be described explicitly, as follows (we write elements of $\binom{S}{2}$ as sets here, for the sake of clarity):



- For a point of the form $\{5, j\}$ with $j \in \{0, 1, 2, 3, 4\}$, the image under π is the partition $\big\{\{5, j\}, \{j+1, j-1\}, \{j+2, j-2\}\big\}$.
- For $\{j, j+2\}$ with $j \in \{0, 1, 2, 3, 4\}$; the image under π is the partition $\big\{\{5, j+1\}, \{j, j-2\}, \{j+2, j-1\}\big\}$.

- For $\{j, j+1\}$ with $j \in \{0,1,2,3,4\}$; the image under π is the partition $\big\{\{5, j-2\}, \{j, j-1\}, \{j+1, j+2\}\big\}$.

Of course, addition in $\{0,1,2,3,4\}$ is performed modulo 5.

We use different conjugates because the polarity ρ in Fig. 7 should invert[5] the "visible" automorphism $\varphi := (01234)$ of order 5, while the polarity π in Fig. 5 should centralise φ.

Conjugation by a duality of an incidence structure $\mathbb{P}$ induces an automorphism of $\mathrm{Aut}(\mathbb{P})$. In particular, the polarity π of $\mathbb{W}$ induces an involutory automorphism of $\mathrm{Aut}(\mathbb{W}) = \mathrm{S}_6$.

5.5 Theorem. *For any duality δ of $\mathbb{W}$, conjugation by δ induces an automorphism of S_6 that is not an inner automorphism of S_6. In particular, conjugation by the polarity π induces an involutory automorphism of S_6 that is not inner.*

Proof. For each $ab \in \binom{S}{2}$, the transposition $\tau = (ab) \in \mathrm{S}_6$ is an automorphism of $\mathbb{W}$ fixing precisely 7 points; namely, the point ab and each point collinear with ab. The lines fixed by τ are precisely those through the point ab. Conjugation by any duality δ (such as π) interchanges the set of fixed points of τ with the set of fixed lines of $\delta^{-1}\tau\delta$. Thus τ and $\delta^{-1}\tau\delta$ cannot be conjugates in the group S_6. □

The sets of points and lines of the doily $\mathbb{W}$ can be interpreted as the two conjugacy classes (represented by (01) and (01)(23)(45), respectively) of involutions in $\mathrm{S}_6 \smallsetminus \mathrm{A}_6$; two such involutions from different classes are incident if they commute. Every automorphism of S_6 either preserves each one of these classes (and induces an automorphism of $\mathbb{W}$) or interchanges them (and induces a duality of $\mathbb{W}$). In any case, that automorphism of S_6 induces an automorphism of the incidence graph $\mathbb{J}_{\mathbb{W}}$. Thus we obtain:

5.6 Theorem. *The group $\mathrm{Aut}(\mathbb{J}_{\mathbb{W}})$ of all automorphisms of the incidence graph of the doily is isomorphic to the group of all automorphisms of S_6. In particular, we have $|\mathrm{Aut}(\mathrm{S}_6)| = 2\,|\mathrm{S}_6|$.* □

5.7 Remark. Among all symmetric groups, the group S_6 is singled out by the existence of automorphisms that are not inner. For a nice proof, see [14]. An analogous result holds for the alternating groups, see [15], cf. [26, 2.4.1]. See [26, 2.4.2] for a purely group theoretic construction of an outer automorphism of S_6 (using the fact that S_5 has 6 Sylow 5-subgroups).

5.8 Classification of polarities. The doily is just the start $\mathbb{W} = \mathbb{W}(2)$ of the infinite family of (finite) *symplectic quadrangles* $\mathbb{W}(q)$ where q is a prime power. Note that $\mathbb{W}(q)$ admits dualities precisely if q is even, and that $\mathbb{W}(q)$ admits polarities precisely if q is even and not a square, cf. [20, 4.9]. One knows that there is just one conjugacy class of polarities, cf. [20, 5.4] (and use the fact that there is at most one Tits endomorphism in a *finite* field).

[5] The polarity ρ centralises $\langle(01245)\rangle$.

One could also derive directly from our present description that there is only one conjugacy class of polarities of the doily. However, this requires a detailed study of the action of π on (conjugacy classes in) S_6; cf. [24, 5.3].

6 Inversive planes

6.1 Definition. An incidence structure $\mathbb{I} = (P, C, I)$ is called a finite *inversive plane of order* n if for each point $p \in P$ the *affine derivation* $\mathbb{I}_p := (P \smallsetminus \{p\}, C_p, I_p)$ at p is an affine plane of order n; here I_p denotes the intersection of I with $(P \smallsetminus \{p\}) \times C_p$.

We note that this is not the standard definition of inversive plane, but equivalent to the usual one (see [4, 6.1, p. 253]) — and very well suited for our discussion here. The blocks of an inversive plane are called *circles* because of the following fact (which is, in fact, one of the axioms in the standard definition):

6.2 Lemma. *Through any set of three points in an inversive plane there is precisely one circle.* □

6.3 Example. We construct an inversive plane Ip_3 of order 3, as follows. As point set, we take the set P of all partitions of $S := \{0, 1, 2, 3, 4, 5\}$ into three-element subsets. The set C of circles is the union of $\binom{S}{2}$ and the set of all partitions of S into elements of size two. We resume our convention that ab denotes $\{a, b\}$ and xyz denotes $\{x, y, z\}$.

If $xyz \in \binom{S}{3}$ then $[xyz]$ denotes the partition $\{xyz, S \smallsetminus xyz\}$. A point $[xyz]$ and a circle ab are incident if $xyz \cap ab$ has an even number of elements (i.e., if ab is completely contained in one of the members of the partition). The point $[xyz]$ is incident with the partition $\{ab, cd, ef\}$ if each one of the intersections $xyz \cap ab$, $xyz \cap ab$, and $xyz \cap ef$ has precisely one element.

Clearly S_6 acts by automorphisms on this incidence structure, and that action is transitive on the set of points.

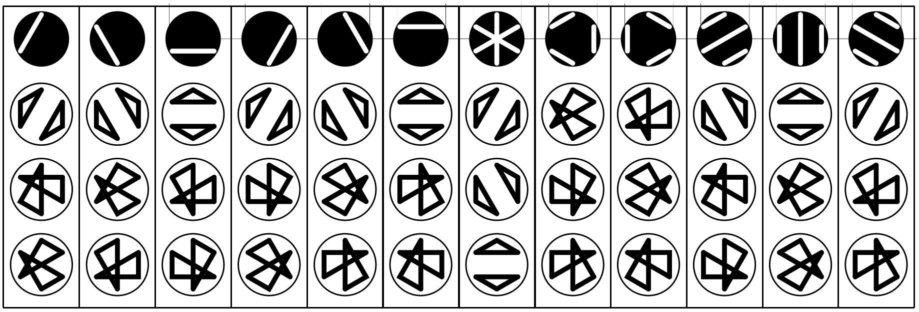


Table 1: The lines of the affine derivation of Ip_3 at 024.

The blocks of Ip_3 through 024 are those in

$$\left\{ 02, 13, 24, 35, 40, 51, \begin{array}{lll} \{03, 14, 25\}, & \{01, 23, 45\}, & \{12, 34, 50\}, \\ \{01, 25, 34\}, & \{12, 30, 45\}, & \{23, 41, 50\} \end{array} \right\}.$$

In Table 1, we list the point rows of these lines in columns (skipping the point 024 which belongs to each of these blocks); the black button on top indicates the line; the white buttons below denote the points. In Fig. 8 we have labelled the points of the affine plane of order 3 with these point labels; one checks without difficulty that the lines of Table 1 are those in the image of the affine plane. Thus we have verified that the affine derivation of Ip_3 at 024 is an affine plane of order 3. Transitivity of the automorphism group on the point set now gives that Ip_3 is indeed an inversive plane of order 3.

The incidence graph and the collinearity or confluence graph of Ip_3 are quite complicated to draw. However, the *complement* of the confluence graph is rather simple: it is the same as the incidence graph of the doily, see Fig. 7.

6.4 Lemma. *The automorphism group of the inversive plane* Ip_3 *contains* S_6 *and is contained in* $\mathrm{Aut}\,(\mathbb{J}_\mathbb{W}) = \mathrm{Aut}\,(\mathrm{S}_6)$.

Proof. We have already observed the inclusion $\mathrm{S}_6 \le \mathrm{Aut}\,(\mathrm{Ip}_3)$. The automorphism group of Ip_3 acts on the confluence graph $\mathbb{C}_{\mathrm{Ip}_3^*}$ and then also on its complement graph. The latter is isomorphic to the incidence graph of the doily, and we obtain $\mathrm{Aut}\,(\mathrm{Ip}_3) \le \mathrm{Aut}\,(\mathbb{J}_\mathbb{W}) = \mathrm{Aut}\,(\mathrm{S}_6)$, cf. 5.6. □

6.5 Remark. There is only one isomorphism class of inversive planes, respectively, of order 2 or 3. In fact, each inversive plane of order 2 is isomorphic to $\left(F, \binom{F}{3}, \in\right)$, for any set F of order 5. The case of order 3 can be solved by direct and elementary arguments ([27], see 7.2 below); it can be deduced with more help from geometry and group theory (as in [13, (2.1)–(2.5)]), and it also follows from a general — and much deeper — result, cf. [25].

The inversive planes of orders 2 and 3 are members of an infinite family of finite inversive planes, constructed as follows: Take the field $\mathbb{K} := \mathbb{F}_q$ of order q, and let $\mathbb{L} = \mathbb{F}_{q^2}$ denote the quadratic extension field. The point set of the projective line over $\mathbb{K}$ is embedded in the point set $\mathrm{PG}(1,\mathbb{L}) = \{\mathbb{L}\,v \mid v \in \mathbb{L}^2 \smallsetminus \{0\}\}$ of the projective line over $\mathbb{L}$ as

$$\mathrm{PG}(1,\mathbb{K}) := \{\mathbb{L}\,v \mid v \in \mathbb{K}^2 \smallsetminus \{0\}\}\,.$$

Let $C_\mathbb{K}$ denote the orbit of this subset under the natural action of $\mathrm{P\Gamma L}(2,\mathbb{L})$. Then $\mathrm{Ip}_\mathbb{K} = (\mathrm{PG}(1,\mathbb{L}), C_\mathbb{K}, \in)$ is an inversive plane of order $q = |\mathbb{K}|$. The full group of automorphisms of $\mathrm{Ip}_\mathbb{K}$ is just the group $\mathrm{P\Gamma L}(2,\mathbb{L})$, cf. [4, 6.4.1]. In 6.6 below, we will only need the rather obvious part of this assertion; namely, that $\mathrm{P\Gamma L}(2,\mathbb{L})$ is contained in $\mathrm{Aut}\,(\mathrm{Ip}_\mathbb{K})$.

Of course, we have $\mathrm{Ip}_3 \cong \mathrm{Ip}_{\mathbb{F}_3}$, and $\mathrm{Aut}\,(\mathrm{Ip}_3) \cong \mathrm{P\Gamma L}(2,9)$ follows. This group has order $2^5 \cdot 3^2 \cdot 5 = 6! \cdot 2$.

6.6 Theorem. *The automorphism group of the inversive plane* Ip_3 *coincides with the automorphism group of the incidence graph of the doily; and thus with* $\mathrm{Aut}\,(\mathrm{S}_6)$.

Proof. We have remarked before that $\mathrm{P\Gamma L}(2,9)$ acts on $\mathrm{Ip}_{\mathbb{F}_3}$. The isomorphism $\mathrm{Ip}_{\mathbb{F}_3} \cong \mathrm{Ip}_3$ from 6.5 thus yields

$$|\mathrm{Aut}\,(\mathrm{Ip}_3)\,| = |\mathrm{Aut}\,(\mathrm{Ip}_{\mathbb{F}_3})\,| \geq |\mathrm{P\Gamma L}(2,9)| = 2^5 \cdot 3^2 \cdot 5 = 6! \cdot 2 = 2\,|\mathrm{S}_6|\,.$$

On the other hand, we know $\mathrm{Aut}\,(\mathrm{Ip}_3) \leq \mathrm{Aut}\,(\mathbb{J}_{\mathbb{W}}) = \mathrm{Aut}\,(\mathrm{S}_6)$ from 6.4. Thus we find $|\mathrm{Aut}\,(\mathrm{Ip}_3)\,| \leq |\mathrm{Aut}\,(\mathbb{J}_{\mathbb{W}})\,| = 2\,|\mathrm{Aut}\,(\mathbb{W})\,| = 2\,|\mathrm{S}_6| = |\mathrm{Aut}\,(\mathrm{S}_6)\,|$, and the assertion follows. □

As a corollary, we obtain one of the famous exceptional isomorphisms between finite simple groups:

6.7 Theorem. *The groups* $\mathrm{P\Gamma L}(2,9)$ *and* $\mathrm{Aut}\,(\mathrm{S}_6)$ *are isomorphic, and so are their commutator subgroups* $\mathrm{PSL}(2,9)$ *and* A_6*.*

Note that $\mathrm{PGL}(2,9)$ and S_6 are not isomorphic; there are three different groups of order 6! between A_6 and $\mathrm{Aut}\,(\mathrm{S}_6)$.

7 Uniqueness of small inversive planes

It is easy to see that affine planes and inversive planes of order 2 are unique up to isomorphism—we just obtain $\left(A, \binom{A}{2}, \in\right)$ and $\left(F, \binom{F}{3}, \in\right)$, with $|A| = 4$ and $|F| = 5$, respectively. The automorphism groups are $\mathrm{Aut}\left(A, \binom{A}{2}, \in\right) = \mathrm{S}_A \cong \mathrm{S}_4$ and $\mathrm{Aut}\left(F, \binom{F}{3}, \in\right) = \mathrm{S}_F \cong \mathrm{S}_5$, respectively, cf. 3.2. Figure 8 gives graphical representations of the affine planes of order 2 and 3.

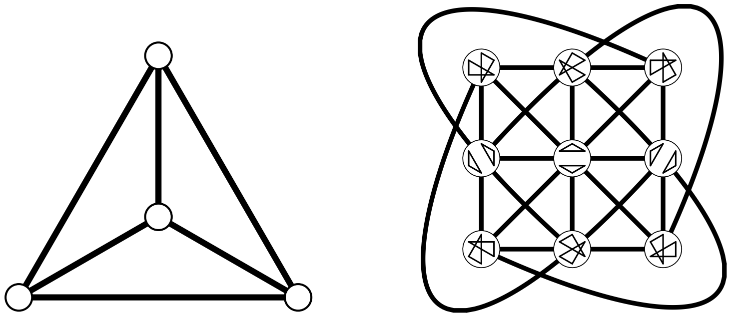


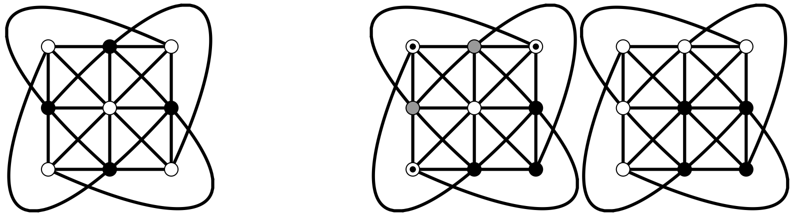
Figure 8: The affine planes of order 2 and 3.

7.1 Proposition. *Any two affine planes of order* 3 *are isomorphic. The automorphism group of any affine plane of order* 3 *acts transitively on the set of non-degenerate quadrangles (i.e., sets of four points such that no three of them are collinear).*

Proof. In any non-degenerate quadrangle Q in an affine plane of order 3, the six lines joining the four points form three sets such that two of them consist of parallel lines, the third contains two lines intersecting in a point outside Q.

We choose a quadrangle and draw the two classes of parallels to the two non-intersecting pairs of joining lines. Adding the missing lines (avoiding to add lines that contain two points that are joined already) one finds that these missing lines are just those marked in the drawing in Fig. 8. Thus there is just one isomorphism type of affine planes of order 3. As our drawing of the plane may start with any non-degenerate quadrangle, we obtain the transitivity assertion, as well. □

Now if we want to construct an inversive plane $\mathbb{I}$ of order 3, we may start with *the* affine plane of order 3 and take it as the affine derivation $\mathbb{I}_p$. Then it remains to find the circles not passing through p; each one of these is a set of four points in the affine plane such that no three of them are collinear. From 7.1 we infer that we may choose one circle S arbitrarily among the non-degenerate quadrangles. We have fixed such a choice in Fig. 9(a); it consists of the black points.



(a) starting with circle S (b) finding the circle Q_0

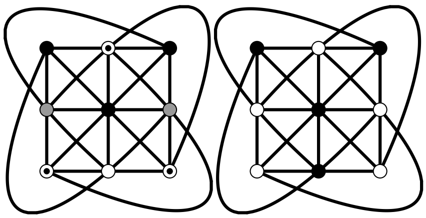
Figure 9: Finding circles for an inversive plane of order 3.

In $\mathbb{I}$, there are $1 + 9 = 10$ points, we have 12 circles through any given point (this is the number of lines in an affine plane of order 3), and every circle contains $1+3 = 4$ points. This yields that there are $10 \cdot 12$ flags, and thus $120/4 = 30$ circles. We know $12+1 = 13$ of them already, and proceed to determine the remaining ones. We use the observation 6.2 that each circle is determined by any three of its points.

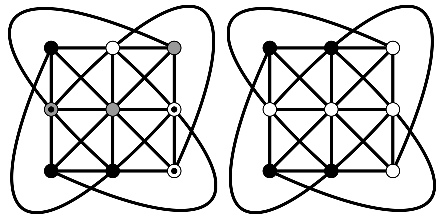
Consider the circle Q_0 through the three points on the lower right corner of our drawing, marked black in the first image in 9(b). The circle S contains two of these points. Therefore, no other point of S (marked grey in the image) lies in Q_0. The points marked by dots are on lines containing two black points; thus none of these points belongs to Q_0. Only one point remains to complete Q_0. Anti-clockwise rotation of the triplet of points yields the circles Q_1, Q_2, and Q_3.

Now we start with the three black points in the image of 10(a). The grey points belong to Q_1 and Q_2, respectively. The dots mark points on lines through two of the black points. We obtain the circle Y_0, and rotation gives Y_1, Y_2, and Y_3.

We proceed in 10(b) and 11(a) in analogous ways: grey points belong to already established circles sharing two points with the new circle in question (we use Q_3, Y_0 in 10(b) and Q_3, S in 11(a)), and dots mark points on joining lines.

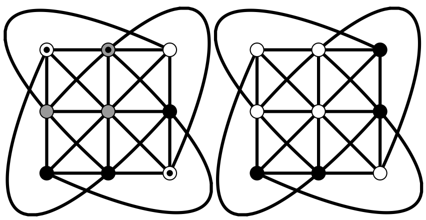


(a) finding the circle Y_0

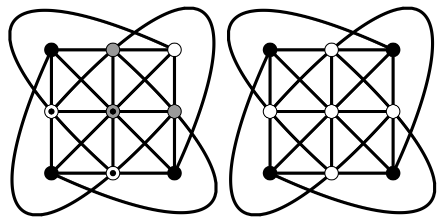


(b) finding the circle R_0

Figure 10: Finding more circles for an inversive plane of order 3.



(a) finding the circle J_0



(b) finding the circle L

Figure 11: Finding the last circles for an inversive plane of order 3.

We have thus found the 17 circles

$$\begin{array}{llll} S, & Q_0, Q_1, Q_2, Q_3, & Y_0, Y_1, Y_2, Y_3, \\ & R_0, R_1, R_2, R_3, & J_0, J_1, J_2, J_3 \end{array}$$

in addition to the 12 lines (i.e., circles through p). It remains to find one last circle L; this is done in 11(b) using Y_1 and Y_2 (or R_0 and R_1). After the choice of S (which was unique up to an automorphism of the affine plane) the set of the remaining circles is determined uniquely. This means:

7.2 Theorem. *There is precisely one isomorphism type of inversive planes of order* 3. □

8 Further exceptional isomorphisms

We briefly mention some more isomorphisms between classical groups that are related to the topics discussed in this note.

It is well known (cf. [11, I, 8.14]) that there is only one isomorphism class of simple groups of order 60, containing A_5, $\mathrm{PSL}(2,4)$, $\mathrm{PSL}(2,5)$ and the commutator group $\Omega^-(4,2)$ of the orthogonal group with respect to a quadratic form of Witt index 1 on a 4-dimensional vector space over the field with two elements.

The semi-direct product $\mathrm{Aut}(\mathbb{F}_4) \ltimes \mathrm{SL}(2,4) =: \Sigma\mathrm{L}(2,4) \cong \mathrm{P}\Sigma\mathrm{L}(2,4)$ is contained in $\mathrm{Sp}(4,2) \cong \mathrm{PSp}(4,2) = \mathrm{P}\Gamma\mathrm{Sp}(4,2)$, the automorphism group of the symplectic quadrangle over the field with 2 elements. This yields a natural action of $\mathrm{P\Sigma L}(2,4)$ on the doily; a spread is left invariant, and we find $\mathrm{P\Sigma L}(2,4) \cong \mathrm{S}_5$ (cf. 4.3). See [23] for details on the embedding and the action, for general ground fields.

The dual of the symplectic quadrangle is (in the finite case) the orthogonal quadrangle defined by a quadratic form of Witt index 2 on a vector space of dimension 5. If the ground field admits a suitable quadratic extension (in particular, if the ground field is finite) then that vector space contains a hyperplane such that the restriction of the quadratic form has Witt index 1. This hyperplane then defines an ovoid in the orthogonal quadrangle, invariant under the corresponding orthogonal group. For the case of the finite field of order q, we obtain the group $\mathrm{O}^-(4,q)$, and for $q = 2$ we find the group $\mathrm{O}^-(4,2)$ in the stabiliser of an ovoid in the orthogonal quadrangle of order 2 (that quadrangle is isomorphic to the doily, by uniqueness). The stabiliser in this smallest case is isomorphic to S_5 (for instance, any member of $\mathcal{V}_6$ is such an ovoid in $\mathbb{W}$, cf. [24, Sect. 4]), the orders coincide, and we obtain $\mathrm{O}^-(4,2) \cong \mathrm{S}_5$. The restriction of that isomorphism to the commutator groups is the isomorphism $\Omega^-(4,2) \cong \mathrm{A}_5$ mentioned above.

The isomorphism from S_6 onto $\mathrm{Aut}(\mathbb{W})$ (cf. 4.2) yields group homomorphisms from S_6 into $\mathrm{O}(5,2)$ (using the isomorphism from $\mathbb{W}$ onto the orthogonal quadrangle over $\mathbb{F}_2$) and from S_6 into $\mathrm{Sp}(4,2)$ (using the isomorphism from $\mathbb{W}$ onto the symplectic quadrangle over $\mathbb{F}_2$), respectively.

The linear representations of S_5 on $\mathbb{F}_2^4$ and S_6 on $\mathbb{F}_2^5$ afforded by the embeddings into orthogonal groups can also be interpreted as the actions on the sets of all subsets of even size in a set of size 5 or 6, respectively. Here symmetric difference serves as addition; the quadratic form maps such a subset X to the residue of $|X|/2$ modulo 2. The embedding of S_6 into $\mathrm{Sp}(4,2)$ comes from the action on the quotient modulo the radical of the polar form associated with that quadratic form.

Finally, we remark that the isomorphism from A_6 onto $\mathrm{PSL}(2,9)$ also gives an isomorphism onto $\Omega^-(4,3)$ because there is a generic isomorphism $\mathrm{PSL}(2,q^2) \cong \Omega^-(4,q)$ ([5, 198, p. 194], see [1, Thm. 5.21]).

Acknowledgements

The authors gratefully acknowledge the hospitality and generous support at the University of Canterbury (Christchurch, New Zealand) where the present paper was finished during a stay as a Visiting Erskine Fellow.

We thank the anonymous referee for suggesting the picture in Fig. 2, and several other very helpful remarks.

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(Received 4 Oct 2012; revised 14 Mar 2013)