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Matching extension in quadrangulations of the torus

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Abstract

A graph G is said to have the property $E(m,n)$ if, given any two disjoint matchings M and N where $|M| = m$ and $|N| = n$ respectively and $M \cap N = \emptyset$, there is a perfect matching F in G such that $M \subseteq F$ and $F \cap N = \emptyset$. This property has been previously studied for triangulations of the plane, projective plane, torus and Klein bottle. Here this study is extended to quadrangulations of the torus.

1 Introduction

A graph G with a perfect matching is said to be *m-extendable* if $|V(G)| \geq 2m+2$ and every matching of size m extends to (i.e., is a subset of) a perfect matching. (For a general reference on the subject of matching extension, see [20], and for three surveys on the subject, see also [16, 17, 18].)

More recently, extendability has been generalized as follows. We say that a graph G has property $E(m,n)$ (or more briefly, G is $E(m,n)$) if for every pair of matchings M and N where $|M| = m$ and $|N| = n$ respectively, and $M \cap N = \emptyset$, there is a perfect matching F of G such that $M \subseteq F$ and $F \cap N = \emptyset$. (Hence a graph is $E(0,0)$ if it contains a perfect matching and is $E(m,0)$ if and only if it is m-extendable.)

Considerable effort has been devoted to the study of the $E(m,n)$ properties for graphs embedded in surfaces. For a study of these properties as applied to general planar graphs see [1], and, more specifically, for planar triangulations and other face-regular planar graphs see [2].

The $E(m,n)$ properties for triangulations of the torus have been extensively studied in [3]. In the present paper we turn our attention to quadrangulations of the torus. We begin by noting that if an even toroidal quadrangulation is only 3-connected, it may not even have a perfect matching. (See Figure 2.1.)

Hence we immediately focus on those even toroidal graphs which are (at least) 4-connected. Dean [8] showed that no toroidal graph is 4-extendable. On the positive side, those values of m and n for which $E(m,n)$ may hold for 4-connected even toroidal graphs (not necessarily quadrangulations) have been extensively investigated in [3]. In the present paper we will show that 4-connected even quadrangulations of the torus must possess properties $E(1,0), E(1,1)$ and $E(0,k)$ for all k where $0 \leq k \leq 5$. Moreover, examples are provided to show that the properties $E(1,2)$ and $E(0,6)$ need not be satisfied.

Another of our main results deals with the property $E(2,0)$. Although not all 4-connected toroidal quadrangulations satisfy this property, we will characterize precisely which ones do.

We adopt the following notation. The neighborhood of vertex v will be denoted by $N(v)$ and $v \cup N(v)$ by $N[v]$. Also if $S, T \subseteq V(G)$, $S \cap T = \emptyset$, then we denote by $q(S,T)$ the number of edges with one endvertex in S and the other in T. Additional notation and terminology will be introduced as needed. For all other background material, we direct the reader to [7].

2 3-connected quadrangulations

Lemma 2.1: *If G quadrangulates the plane (or the projective plane), then*

$$\text{mindeg}\,(G) \leq 3$$

and hence G is at most 3-connected.

Proof: This is a simple application of Euler's formula. ∎

In Figures 2.1 and 2.2 we present examples of 3-connected even quadrangulations of the plane and the torus respectively that do not have perfect matchings. (In each of these examples, the boxed vertices constitute a Tutte set of size 30 the deletion of which leaves 32 isolated vertices.)

Hence we proceed to study 4-connected even quadrangulations of the torus. It is known that these graphs in fact do have perfect matchings. (Cf. [3, Theorem 4.3(a)].)

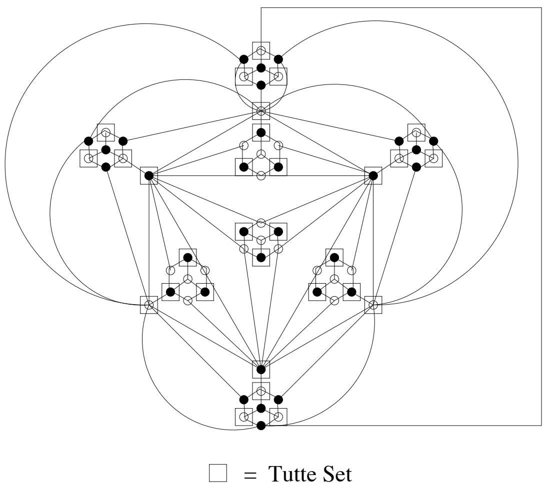


Figure 2.1: 3-connected even quadrangulations of the plane need not have perfect matchings.

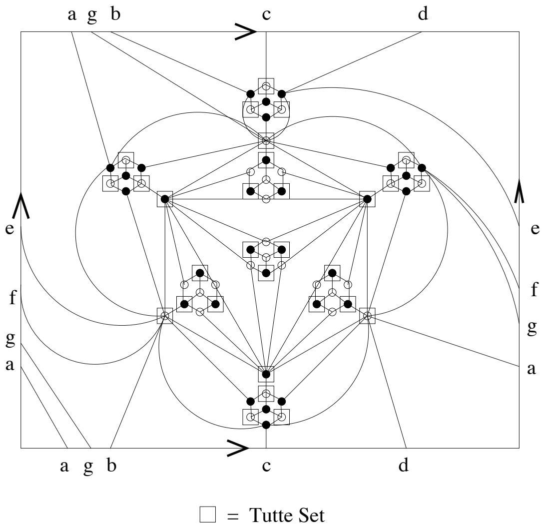


Figure 2.2: 3-connected even quadrangulations of the torus need not have perfect matchings.

3 4-connected quadrangulations

In order to present our results on 4-connected quadrangulations of the torus, we will require a more detailed analysis of the structure of these graphs. Although this was first accomplished by Altschuler [4], we will follow, for the most part, the presentation due to Nakamoto and Negami. (Cf. [14, 15, 13].)

The universal covering space of a torus is homeomorphic to the $x-y$ plane $\mathbb{R}^2$. Let $\tilde{G}$ be the union of the vertical and horizontal lines through the points of $\mathbb{R}$ which have integral coordinates; that is

$$\tilde{G} = \{(x,y) \in \mathbb{R}^2 | x \in \mathbb{Z}\} \cup \{(x,y) \in \mathbb{R}^2 | y \in \mathbb{Z}\}.$$

Let $\hat{G}$ denote the infinite 4-regular and 4-face-regular graph induced by $\tilde{G}$. We will denote by $\square(m,n,t)$ the set of all translations $T_{(\alpha,\beta)}$ of $\mathbb{R}^2$ onto $\mathbb{R}^2$ defined by

$$T_{(\alpha,\beta)}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \alpha\begin{pmatrix} 0 \\ m \end{pmatrix} + \beta\begin{pmatrix} n \\ -t \end{pmatrix}, \tag{3.1}$$

for positive integers m and n and non-negative integers t. Then $\square$ is a group under composition of translations and all members of $\square$ leave $\hat{G}$ invariant. The orbit space $\frac{\mathbb{R}^2}{\square(m,n,t)}$ of the group is homeomorphic to a torus and the projection $\frac{\hat{G}}{\square(m,n,t)}$ is a

4-regular and 4-face-regular graph on the torus which we denote by $Q(m,n,t)$. Informally, one can visualize $Q(m,n,t)$ as the graph on the torus obtained by forming a cylinder of length n and cross-sectional cycles of length m and then identifying the ends of the cylinder with a "twist" of t units.

As an example, see the graph $Q(4,5,1)$ displayed in the figure below.

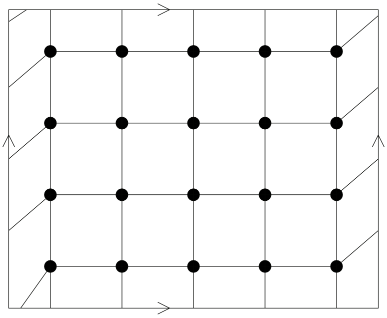


Figure 3.1: The quadrangulation $Q(4,5,1)$

We will always assume that $m \geq 3$ to forbid loops and parallel edges. For the same reason, we will also assume $0 \leq t \leq m-1$ and, when $n=1$, that $m \geq 5$ and $2 \leq t \leq m-2$, while if $n \geq 2$, we assume $1 \leq t \leq m-1$. Finally, since we are concerned with perfect matchings throughout this paper, we will also assume that mn is even.

Note that we will sometimes want to single out graphs of the type $Q(m,n,t)$ which contain triangles. We will denote this class by $3Q$ and note that $3Q = \{Q(3,n,t), Q(m,3,0),$
$Q(m,2,1), Q(m,2,m-1), Q(m,1,2), Q(m,1,m-2)\}$.

We will have need of the following results.

Theorem 3.1 [5]: *The genus of a graph is the sum of the genera of its blocks.*

Corollary 3.2: *Let X be an edge-cut of a connected graph G where $\gamma(G)=1$. If $G-X$ has exactly two components G_1 and G_2, then at least one of G_1 and G_2 is planar.*

Proof: Delete all edges joining G_1 and G_2 until precisely one such edge $e=uv$ remains. Let the resulting graph be denoted by G'. Then clearly $\gamma(G') \leq 1$ and the blocks of G' are precisely the blocks $B_1^1, \ldots, B_i^1$ of G_1, the blocks $B_1^2, \ldots, B_j^2$ of G_2 and the edge e. So $\gamma(G') = \gamma(B_1^1) + \cdots + \gamma(B_i^1) + \gamma(B_1^2) + \cdots + \gamma(B_j^2) + \gamma(e)$ $= \gamma(G_1) + \gamma(G_2)$. Thus $\gamma(G_1) + \gamma(G_2) \leq 1$. Consequently, $\gamma(G_1) = 0$ or $\gamma(G_2) = 0$ and the result follows. ∎

Def.: A set of edges X in a graph G is said to be a *cyclic edge-cut* if $G - X$ contains at least two components each of which contains a cycle.

Def.: A graph G is said to be *cyclically k-edge-connected* if there exists no cyclic edge-cut X in G with $|X| < k$. The maximum value of k for which a graph G is cyclically k-edge-connected is called the *cyclic edge-connectivity* of G and is denoted by $c\lambda(G)$.

Lemma 3.3: *If G belongs to $3Q$ then (i) G is 4-connected and (ii) G has cyclic edge-connectivity 6.*

Proof: (i) Note that $Q(m, 3, 0) \cong Q(3, n, t)$ when $m = n$ and $t = 0$, while $Q(m, 1, 2)$ and $Q(m, 1, m - 2)$ are essentially the same, differing only in the direction of the twist. $Q(m, 2, 1)$ and $Q(m, 2, m - 1)$ are similarly related. Consequently, we need only consider three cases.

First suppose $G \cong Q(3, n, t)$, for some $n \geq 2$. Observe that deleting at most one vertex from each of the n concentric triangles that partition $V(G)$ leaves a connected graph. So any minimal vertex-cut must contain at least two vertices from the same triangle (in the n concentric triangles indicated). Clearly, deleting any such pair leaves a hamiltonian graph and thus G is 4-connected.

Suppose next that $G \cong Q(m, 2, 1)$ for some $m \geq 3$. Observe that deleting an independent set from G leaves a connected graph. Thus any minimum vertex-cut must contain a pair of adjacent vertices. Again it is clear that deleting any pair of adjacent vertices from G leaves a hamiltonian graph.

Finally, assume that $G \cong Q(m, 1, 2)$ for some $m \geq 5$. If we delete any set of vertices such that no two are consecutive on the m-cycle, C_m, the resulting graph is still connected. (Here we observe that G consists of an m-cycle C_m, together with all chords joining alternate vertices on C_m.) So a minimum vertex-cut must contain a pair of vertices which are consecutive on C_m. But then again it is clear that deleting any such pair leaves a 2-connected graph.

(ii) **Claim:** G contains no cyclic 4-edge-cuts.

Since G is 4-connected by part (i), if K is a cyclic 4-edge-cut in G, K is a matching in G. Moreover, $G - K$ has precisely two components which we denote by G_1 and G_2. Now one of G_1 and G_2 must contain no non-contractible cycles. If G_1 is such a component, then G_1 together with K and a 4-cycle on the endvertices of K in G_2 yields a quadrangulation of the plane with precisely four vertices of degree 3 and all other vertices of degree 4 which is impossible.

By parity G cannot contain cyclic 5-cuts.

Since all graphs in $3Q$ are 4-regular and contain triangles, we have obvious cyclic 6-edge-cuts, namely the coboundaries of triangles. The lemma now follows. ∎

Lemma 3.4: *$c\lambda(Q(m,n,t)) = 8$, except for the members in class $3Q$, in which case $c\lambda(Q(m,n,t)) = 6$.*

Proof: First, if G belongs to the class $3Q$ we are done by Lemma 3.3. So suppose $G = Q(m,n,t)$ has girth 4. Let S be a minimum cyclic edge-cut; i.e., $|S| = c\lambda(G)$. Then $G - S$ has exactly two components which we will denote by G_1 and G_2. By Corollary 3.2, at least one of G_1 and G_2 is planar. Without loss of generality, we may suppose that G_1 is planar and that its embedding on the sphere is that inherited from the embedding of G on the torus. Denote $|V(G_1)|$ by n_1 and $|E(G_1)|$ by m_1. If G_1 has exactly two faces that are not inherited from G (that is, G_1 is embedded as an annular graph) and the lengths of these face boundaries are denoted by ℓ_1 and ℓ_2, then by Euler's formula, $4n_1 - |S| = 2m_1$ and $n_1 - m_1 + 2 + (2m_1 - \ell_1 - \ell_2)/4 = 2$. Then, since G contains no triangles, $|S| = \ell_1 + \ell_2 \geq 8$.

On the other hand, if G_1 has exactly one face that is not inherited from G and the length of this face is ℓ_1, an argument similar to that in the preceding paragraph shows that $|S| = 4 + \ell_1 \geq 8$.

Since the 8 edges exiting the boundary of a 4-face form a cyclic edge-cut, we have the result. ∎

We note that if deleting an edge cut from G leaves a plane component (one whose plane embedding is inherited from its embedding on the torus), then traversing the boundary of such a component clockwise we encounter precisely four more right-hand turns than left-hand turns. Each right-hand turn contributes two edges to the edge-cut. Passing through a vertex on the boundary without turning left or right corrsponds to one further edge in the edge-cut. Thus we have the following corollary.

Corollary 3.5: *If $G = Q(m,n,t)$ contains a cyclic edge-cut of size 8 whose deletion leaves a non-planar component, then the other (plane) component is a quadrangle.* ∎

Def.: If G' is a connected subgraph of G such that $q(V(G'), V(G) - V(G')) = k$, we call the subgraph G' a *k-shooter* (in G).

Corollary 3.6: *If $G = Q(m,n,t)$ and G contains a 4-shooter G', then either G' or $G - G'$ is a singleton.*

Proof: Suppose $G' \subseteq G$ is a 4-shooter. Then either G' is a singleton or else G' contains a cycle. Suppose G' contains a cycle. Then by Lemma 3.4, $G - G'$ must be acyclic. Hence $G - G'$ must be a singleton since all the edges exiting $G - G'$ must have their other endvertex in G' and there are only four such edges. ∎

Lemma 3.7: *A quadrangulation G of the torus is 4-regular if and only if it is 4-connected.*

Proof: If G is 4-connected, then it is an easy consequence of Euler's formula that G must be 4-regular.

To prove the converse, we proceed as follows. Suppose, to the contrary, that G contains a vertex-cut S with $|S| < 4$. Let G_1 and G_2 be two components of $G - S$. Since G is 4-regular, each of G_1 and G_2 contains at least two vertices. If G_1 does not contain a cycle, then $q(G_1, G - G_1) = 4|V(G_1)| - 2(|V(G_1)| - 1) = 2|V(G_1)| + 2 \geq 6$. On the other hand, suppose G_1 does contain a cycle. Then if $G[S \cup V(G_2)]$ also contains a cycle, then $q(G_1, G - G_1) \geq 8$ by Lemma 3.4. If $G[S \cup V(G_2)]$ does not contain a cycle, then $q(G_1, G - G_1) = q(G_1, S \cup V(G_2)) \geq 2|S| + 2|V(G_2)| + 2 \geq 6$. On the other hand, $q(S, G_1 \cup G_2) \leq 12$. Hence $G_1 = G_2 = K_2$ and G must contain triangles.

But then G belongs to the class $3Q$ and hence G is 4-connected by Lemma 3.3, part(i). ∎

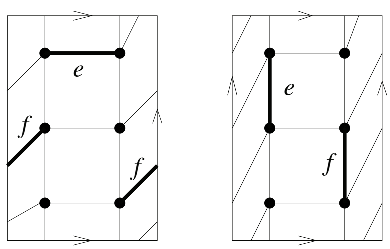
The following is a corollary of a theorem of Berge [6].

Theorem 3.8: *Every 4-connected 4-regular even graph is $E(1,0)$.*

Our next goal is to prove that a 4-regular quadrangulation of the torus is 2-extendable or else it belongs to a certain family of exceptions which we shall describe precisely.

First, we deal with the class $3Q$. We observe that:

(i) every graph of type $Q(3,n,t)$, with n even, is 2-extendable, except for $Q(3,2,1)$ or $Q(3,2,2)$;



Q(3,2,1) Q(3,2,2)

e and f do not extend

Figure 3.2: The graphs $Q(3,2,1)$ and $Q(3,2,2)$ are not 2-extendable.

(ii) every graph of type $Q(m,3,0)$ with $m \geq 4$ and m even is 2-extendable;

(iii) while if $m \geq 6$ is even, no graph of type $Q(m,1,2)$ or of type $Q(m,1,m-2)$ is 2-extendable.

We shall need the following well-known definition from matching theory.

Def.: A graph G is said to be *factor-critical* if $G - v$ contains a perfect matching for all $v \in V(G)$.

We are now prepared to state and prove our main theorem on 2-extendability of $Q(m, n, t)$.

Theorem 3.9: *Let $G = Q(m, n, t)$ with mn even. Then G is 2-extendable if and only if G does not belong to one of the following three classes:*

(i) $\{Q(3, 2, 1), Q(3, 2, 2), Q(m, 1, 2), Q(m, 1, m - 2)\}$*;*

(ii) $\{Q(m, 2, 2), Q(m, 2, m - 2) \mid m \text{ odd and } m \geq 5\}$*; or*

(iii) $\{Q(m, 1, m/2 + 1), Q(m, 1, m/2 - 1) \mid m/2 \text{ odd and } m \geq 6\}$.

Proof: If G contains a triangle, $G \in 3Q$ and we have already analyzed the 2-extendability of graphs in $3Q$. Henceforth, therefore, we will assume that G is triangle-free.

Suppose edges e and f in G are independent and no perfect matching of G contains both e and f. Thus $G' = G - V(e) - V(f)$ contains no perfect matching and hence by the Gallai-Edmonds theorem [9, 10, 11, 12] there is a set $S \subseteq V(G')$ with $G' - S$ consisting of at least $|S| + 2$ factor-critical (and hence odd) components.

Let $K = S \cup V(e) \cup V(f)$. Then by 4-regularity and 4-connectivity of G respectively, we have that

$$q(K, G - K) \leq 4|S| + 12 = 4|K| - 4$$

and

$$q(G - K, K) \geq 4(|S| + 2) = 4(|K| - 2).$$

Hence by parity, we have three cases to consider:

Case 1: $q(K, G - K) = 4(|K| - 2) = 4|K| - 8$;

Case 2: $q(K, G - K) = 4|K| - 6$; or

Case 3: $q(K, G - K) = 4|K| - 4$.

Case 1: Suppose $q(K, G - K) = 4|K| - 8$. (Note that this count ensures that the number of odd components in $G - K$ is exactly $|S| + 2$.) Consequently, each of the exactly $|K| - 2$ factor-critical components in $G - K$ contributes exactly four edges to $q(K, G - K)$ and is thus a singleton by Corollary 3.6. Also $G[K]$ has precisely four edges including e and f.

Now if $\varepsilon \in E(G[K])$, since ε lies in the boundaries of two 4-faces we have one of the two configurations shown in Figure 3.3, where all of a, b, c, x, y and z are distinct. Thus we cannot have both c and x in $G - K$, nor can we have both a and z in $G - K$. So each edge $\varepsilon \in E(G[K])$ is incident with at least two other edges in $E(G[K])$. Moreover, these two edges cannot belong to the boundary of the same face; in other words, these two edges lie on "opposite sides" of edge ε.

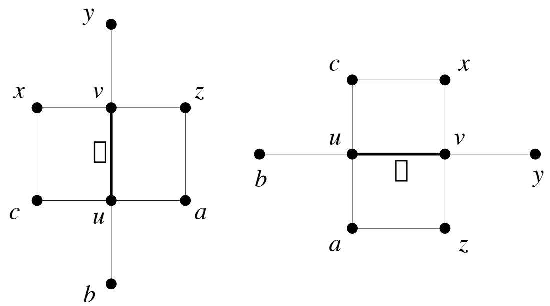


Figure 3.3: The edge ε must lie in two quadrangular faces.

We note from the above that, in particular, $G[K]$ does not contain a 4-cycle. Also, since e and f are independent, no vertex in K is incident with all four edges of $G[K]$.

Suppose $v \in K$ has three edges of $E(G[K])$ incident upon it. Exactly one edge incident with v is in $\{e, f\}$. Without loss of generality, we may assume that it is $e = uv$ and the other two edges incident with v will be denoted by xv and yv. Since f is the only remaining edge in $G[K]$ and f has two edges from $E(G[K])$ incident with its endvertices, $f = xy$. This contradicts our assumption that G is triangle-free.

So no vertex has degree greater than 2 in $G[K]$. Since each edge is a boundary edge of two 4-faces, each endvertex of e and f has degree 2 in $G[K]$. But $|E(G[K])| = 4$, so it follows that $G[V(e) \cup V(f)]$ is a 4-cycle. Moreover, this 4-cycle does not bound a face. Thus this 4-cycle must have the appearance of one of the four subgraphs represented in Figure 3.4, where the bold edges are identified in the sense of the arrow. We now wish to determine the possible translations that could emulate the paths shown in Figure 3.4. Note that since these paths each represent a horizontal or vertical translation of at most two units, in equation (3.1), both α and β are at most 2.

We begin with Case (1).

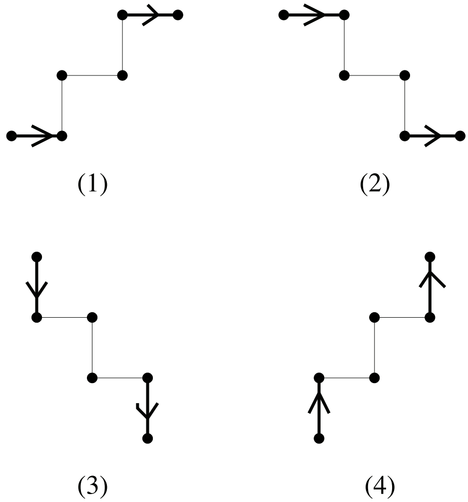


Figure 3.4: The four cases giving rise to non-contractible 4-cycles.

For (1) we have that

$$\binom{1}{0} + \binom{0}{1} + \binom{1}{0} + \binom{0}{1} = \alpha\binom{0}{m} + \beta\binom{n}{-t}.$$

Then $\beta n = 2$ and $\alpha m - \beta t = 2$.

Suppose $n = 1$ and $\beta = 2$. Then $\alpha m = 2t + 2$. If $\alpha = 1$, then $t = m/2 - 1$ and we are considering $Q(m, 1, m/2 - 1)$.

But this graph is not 2-extendable. For example, consider $G = Q(10, 1, 4)$.

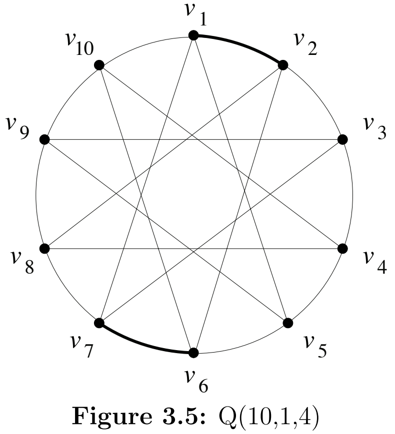


Figure 3.5: Q(10,1,4)

It is easy to see that edges v_1v_2 and v_6v_7 do not extend to a perfect matching, for the vertices v_4 and v_9 are a 2-cut in $G - \{v_1, v_2, v_6, v_7\}$ leaving four odd components, namely, the singletons v_3, v_5, v_8 and v_{10}. Hence by Tutte's theorem there is no perfect matching in $G - \{v_1, v_2, v_6, v_7\}$. A completely similar argument shows that every $Q(m, 1, m/2 - 1)$, where $m/2$ is odd, fails to be 2-extendable.

If $\alpha = 2$, then $t = m - 1$ and we are considering $Q(m, 1, m - 1)$. But this graph has multiple edges which we do not allow.

Finally, if $n = 2$, then $\beta = 1$ and $\alpha m = t + 2$.

If $\alpha = 1$ we obtain $t = m - 2$ and we are considering $Q(m, 2, m - 2)$.

When m is odd, these graphs are listed in the exceptional cases of the theorem. Otherwise it is a straightforward exercise to determine that $Q(m, 2, m - 2)$ is 2-extendable when $m \geq 4$ is even. If $\alpha = 2$ we get that $t = 2m - 2$ which contradicts our bounds on possible values of t.

Consider now (2). We have

$$\binom{1}{0} + \binom{0}{-1} + \binom{1}{0} + \binom{0}{-1} = \alpha\binom{0}{m} + \beta\binom{n}{-t}.$$

Suppose $n = 1$. Then $\beta = 2$ and $\alpha m - 2t = -2$ or $\alpha m = 2t - 2$. If $\alpha = 2$, we obtain $m = t - 1$ or $t = m + 1$ which violates our restrictions on t. If $\alpha = 0$, then $t = 1$ and since $n = 1$ we get multiple edges in our graph which is not allowed. If $\alpha < 0$ then $t < 0$ which also is not allowed.

So we must have $\alpha = 1$ and we obtain $t = m/2 + 1$; that is, we are considering $Q(m, 1, m/2 + 1)$. Moreover, m is even. Now if $m/2$ is odd, $Q(m, 1, m/2 + 1)$ is not 2-extendable, while if $m/2$ is even, then $Q(m, 1, m/2 + 1)$ is 2-extendable.

If $n = 2$, then $\beta = 1$ and $\alpha m - t = -2$, or $\alpha m = t - 2$.

If $\alpha = 0$, we have $Q(m, 2, 2)$ which, it can be routinely verified, is not 2-extendable when m is odd and is 2-extendable when m is even.

If $\alpha \geq 1$, we obtain $t \geq m + 2$ which violates our restriction on t.

Configuration (3) is equivalent to (2) and configuration (4) is equivalent to (1), so this completes Case 1.

Case 2: Suppose $q(K, G - K) = 4|K| - 6$.

Thus $|E(G[K])| = 3$ and $G - K$ has precisely one factor-critical component, call it C_1, with $q(C_1, G - C_1) = 6$ and all other odd components singletons.

The six edges exiting component C_1 form a cyclic edge-cut since $|V(C_1)| \geq 3$. But then by Lemma 3.4, G has no cyclic 6-edge-cuts, so Case 2 cannot occur.

Case 3: Suppose finally that $q(K, G - K) = 4|K| - 4$.

Thus $|E(G[K])| = 2$ and $G - K$ either has two 6-shooters and all other factor-critical components are singletons or $G - K$ has exactly one 8-shooter and all other factor-critical components are singletons. Since each 6-shooter contains a cycle, we cannot have two 6-shooters without giving rise to a cyclic edge-cut of size 6. Therefore, we must have one 8-shooter C_1 and all other components of $G - K$ are singletons.

Let $e = u_1v_1$ and $f = u_2v_2$. Then for $i = 1$ and $i = 2$ we have $N_G(u_i) = \{a_i, b_i, c_i, v_i\}$ and $N_G(v_i) = \{x_i, y_i, z_i, u_i\}$ and $N_G(u_i) \cap N_G(v_i) = \emptyset$. (See Figure 3.6.)

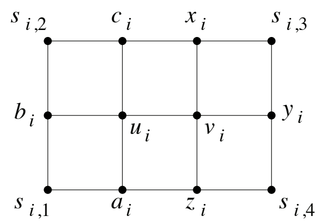


Figure 3.6: The structure forced about each of the non-extending pair of edges.

With the labelling indicated in Figure 3.6, c_i, x_i, a_i and z_i must all be in component C_1. So the eight edges from $V(C_1)$ to K are those in $T = \{u_1a_1, u_1c_1, v_1x_1, v_1z_1, u_2a_2, u_2c_2, v_2x_2, v_2z_2\}$. Thus, in particular, vertices b_1 and y_1 are distinct singleton components of $G - K$. Consequently, $\{s_{1,1}, s_{1,2}, s_{1,3}, s_{1,4}\} \in K$. Moreover, $s_{1,1}a_1, s_{1,2}c_1, s_{1,3}x_1$ and $s_{1,4}z_1$ are all edges in T. Since $|T| = 8$, we must have $\{s_{1,1}, s_{1,2}, s_{1,3}, s_{1,4}\} = \{u_2, v_2\}$; that is, $u = s_{1,2} = s_{1,4}$ and $v = s_{1,1} = s_{1,3}$. But then uvb_1u is a triangle, a contradiction.

This completes the proof in Case 3 and the theorem follows. ∎

Theorem 3.10: *Let G be a 4-connected even quadrangulation of the torus. Then G is $E(1,1)$.*

Proof: Suppose, to the contrary, that G is not $E(1,1)$ and that edges $e, f \in E(G)$ are such that no perfect matching of G contains e, but avoids f. Thus $G' = G - V(e) - f$ contains no perfect matching. By Tutte's theorem, we have $S \subseteq V(G')$ such that $G' - S$ has at least $|S| + 2$ odd components. Indeed, since we know that G is 1-extendable by Theorem 3.8, we have exactly $|S| + 2$ odd components. Moreover, in G, the two endvertices of edge f lie in different odd components of $G' - S$.

Now we count edges in G between vertices in $K = S \cup V(e)$ and vertices in $G' - S$ in two different ways. Viewed from the vertices in K, there can be at most $4|S| + 6$ edges with their other endvertex in $G' - S$ by 4-regularity. On the other hand, viewed from the odd components of $G' - S$, we have at least $4(|S| + 2) - 2 = 4|S| + 6$ edges having their other endvertex in K by 4-connectivity of G. Thus we must have precisely $4|S| + 6$ such edges; that is, e is the only edge in $G[K]$ and each odd component of $G' - S$ is a singleton by Corollary 3.6.

Recalling that G is a quadrangulation, the edge f must lie in the boundaries of two different 4-cycles. Let $f = uv$. Then these two 4-face boundaries are clockwise of the form uvk_1k_2u and vuk_3k_4v respectively, where $\{k_1, k_2, k_3, k_4\} \subseteq K$. But then k_1k_2 and k_3k_4 are distinct edges in $E(G[K])$, a contradiction. (Note we cannot have $k_1 = k_3$ and $k_2 = k_4$ as this gives an induced K_4 which is impossible.) The result follows. ∎

We would point out that in view of Theorem 3.9 and the fact that property $E(2, 0)$ implies property $E(1, 1)$, one could also prove Theorem 3.10 by showing directly that the exceptional graphs in Theorem 3.9 are each $E(1, 1)$. However, this alternate proof is not as clean as the one we present above.

We would also point out that it is *not* true that all 4-connected even toroidal graphs are $E(1, 1)$. In fact, such graphs need not even be $E(1, 0)$. For an example, the reader is referred to [3].

Theorem 3.11: *If G is a 4-connected even quadrangulation of the torus, then G is $E(0, k)$, for all k, $0 \le k \le 5$.*

Proof: We already know that G is $E(0, 0)$, and in fact, $E(0, 1)$ by Theorem 3.8 and the fact that $E(1, 0)$ implies $E(0, 1)$ in general. Let k be the smallest value for which G is not $E(0, k)$. So $k \ge 2$.

Suppose that $F = \{f_1, \ldots, f_k\} \subseteq E(G)$ is a set of k independent edges such that $G' = G - F$ contains no perfect matching. Thus again by Tutte's theorem there is a set $S \subseteq V(G)$ such that $G' - S$ has at least $|S| + 2$ odd components.

Counting the edges from S to $G' - S$, we have at most $4|S|$ by 4-regularity, while counting edges from the odd components of $G' - S$ to S, we have at least $4(|S| + 2) - 2k$ by 4-connectivity. Thus, if $k \le 5$, there are two possibilities:

(i) there are $4|S| - 2$ edges from S to $G' - S$ or

(ii) there are $4|S|$ edges from S to $G' - S$.

In Case (i) we note that there is exactly one edge in $G[S]$, each odd component in $G' - S$ is a singleton by Corollary 3.6, $k = 5$ and in G each edge in F joins two odd components of $G' - S$. Consider $f_1 = uv \in F$. Since G is a quadrangulation, f_1 lies in two 4-face boundaries; call them uvs_1s_2u and vus_3s_4v, where $\{s_1, s_2, s_3, s_4\} \subseteq S$ where s_1s_2 and s_3s_4 are distinct edges in $G[S]$, a contradiction.

In Case (ii) we have no edges in $G[S]$. Thus either $k = 4$, in which case all odd components in $G' - S$ are singletons and each f_i joins two odd components of $G' - S$ and we can proceed as in Case (i) to obtain a contradiction by considering 4-face boundaries containing f_1, say, or $k = 5$ and $G' - S$ has precisely one odd component with six exiting edges and all other odd components have exactly four exiting edges (and hence again by Corollary 3.6, are singletons). Moreover, in G each edge $f_i \in F$ joins two odd components of $G' - S$.

Since $G[S]$ has no edges, as in (i), no f_i joins two singleton odd components in $G' - S$. Hence each f_i has one end in the (unique) odd component of $G' - S$, call it C_1, which has exactly six exiting edges in G. This accounts for five of the edges exiting component C_1.

Again, now, let us consider the two 4-face boundaries containing edge $f_1 = uv$. These may be denoted by uvs_1xu and $vuys_2v$, where $u \in V(C_1)$ and s_1 and s_2 are in S. But then edge s_1x and edge ys_2 are two additional edges exiting component C_1 (since $\{x, y\} \cap S \neq \emptyset$ contradicts the observation that $G[S]$ has no edges). This contradicts the fact that C_1 has only six exiting edges. ∎

Remark: In connection with Theorem 3.11 we hasten to point out that $E(0, 5)$ does not imply $E(0, 4)$ does not imply $E(0, 3)$ for arbitrary graphs. (Cf. [19].)

We now present examples to show that we need not always have $E(0, 6)$ for 4-connected quadrangulations of the torus. Our examples contain $3 \times 2p = 6p$ vertices, where $p \geq 3$. The smallest of these examples is shown in Figure 3.7. (Note that the same idea gives a $k \times 2p$ grid graph which is not $E(0, 2k)$, if k is odd.)

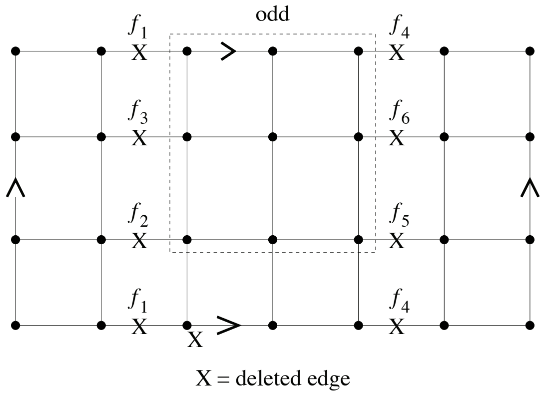


Figure 3.7: 4-connected even quadrangulations of the torus need not be $E(0, 6)$.

Finally, in Figure 3.8 we present an example to show that for quadrangulations of the torus we need not have $E(1, 2)$ (and therefore, not $E(3, 0)$ nor $E(2, 1)$ either). In particular, there is no perfect matching containing edge e, but missing both edges f_1 and f_2.

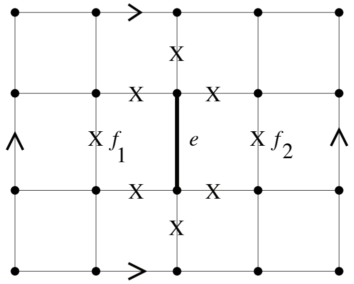


Figure 3.8: 4-connected even quadrangulations of the torus need not be $E(1,2)$.

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