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A sharp lower bound on the number of hyperedges in a friendship 3-hypergraph

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Abstract

Let $(X, \mathcal{B})$ be a set system in which $\mathcal{B}$ is a set of 3-subsets of X. Then $(X, \mathcal{B})$ is a *friendship* 3-*hypergraph* if it satisfies the following property: for all distinct elements $u, v, w \in X$, there exists a unique fourth element $x \in X$ such that $\{u, v, x\}, \{u, w, x\}, \{v, w, x\} \in \mathcal{B}$. The element x is called the *completion* of u, v, w and we say u, v, w is *completed* by x. If a friendship 3-hypergraph contains an element $f \in X$ such that $\{f, u, v\} \in \mathcal{B}$ for all $u, v \in X \setminus \{f\}$, then the friendship 3-hypergraph is called a *universal friend* 3-*hypergraph* and the element f is called a *universal friend* of the hypergraph.

In this note, we show that if $(X, \mathcal{B})$ is a friendship 3-hypergraph with $|X| = n$, then $|\mathcal{B}| \geq \lceil 2(n-1)(n-2)/3 \rceil$. In addition, we show that this bound is met if and only if $(X, \mathcal{B})$ is a universal friend 3-hypergraph.

1 Introduction

Let $(X, \mathcal{B})$ be a set system in which $\mathcal{B}$ is a set of 3-subsets of X. Then $(X, \mathcal{B})$ is a *friendship* 3-*hypergraph* if it satisfies the following property: for all distinct elements $u, v, w \in X$, there exists a unique fourth element $x \in X$ such that $\{u, v, x\}, \{u, w, x\}, \{v, w, x\} \in \mathcal{B}$. The element x is called the *completion* of u, v, w and we say u, v, w is *completed* by x. If a friendship 3-hypergraph contains an element $f \in X$ such that $\{f, u, v\} \in B$ for all $u, v \in X \setminus \{f\}$, then the friendship 3-hypergraph is called a *universal friend* 3-*hypergraph* and the element f is called a *universal friend* of the hypergraph. The members of $\mathcal{B}$ are called *hyperedges* while we reserve the term

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137	124	235	346	457	156	267
013	012	023	034	045	015	026
037	024	035	046	057	056	067
017	014	025	036	047	016	027

Figure 1: The universal friend 3-hypergraph on eight points

012	456	014	016	234	236	024	135
013	457	015	017	235	237	026	137
023	467	045	067	245	267	046	157
123	567	145	167	345	367	246	357

Figure 2: The non-universal friend 3-hypergraph on eight points

triple for three elements that may or may not be a hyperedge. When writing out hyperedges in friendship 3-hypergraphs we omit brackets and commas.

Friendship hypergraphs were introduced by Sós [2]. She observed that if $n \equiv 2, 4 \pmod 6$, then there exists a universal friend 3-hypergraph. Such a hypergraph must be constructed from a Steiner Triple System in the following manner: start with a Steiner Triple System $(Y, \mathcal{A})$ on $n-1 \equiv 1, 3 \pmod 6$ elements. Then introduce a new element $f \notin Y$ to form the universal friend 3-hypergraph, $(X, \mathcal{B})$, where $X = Y \cup \{f\}$ and $\mathcal{B} = \mathcal{A} \cup \{\{f, x, y\} : x, y \in X, x \neq y\}$. Clearly f is the universal friend. Figure 1 gives a universal friend 3-hypergraph on eight points with universal friend 0.

In [1], Hartke and Vanderbussche showed that friendship 3-hypergraphs without a universal friend exist. Such hypergraphs are called *non-universal friend 3-hypergraphs*. They showed that there is exactly one such hypergraph when $n = 8$, at least three when $n = 16$ and at least one when $n = 32$. They also showed that any friendship 3-hypergraph must have at least $\frac{1}{2}n(n-2)$ hyperedges. In [3], Li, van Rees, Seo and Singhi showed that the known friendship 3-hypergraphs on less than 13 points are the only such hypergraphs. They also proved that that the number of hyperedges in a friendship 3-hypergraph on n points, is at least $n^2/2$. Figure 2 shows the non-universal friend 3-hypergraph on eight points.

We will improve the lower bound for hyperedges in a friendship 3-hypergraph to $|\mathcal{B}| \geq \lceil 2(n-1)(n-2)/3 \rceil$. We will also show that the universal friend 3-hypergraph meets this bound and no other friendship 3-hypergraphs meet this bound. Therefore this bound is met if and only if $n \equiv 2, 4 \pmod 6$.

2 A new lower bound

In this section, we will show that if $(X, \mathcal{B})$ is a friendship hypergraph on n points, then $|\mathcal{B}| \geq \lceil 2(n-1)(n-2)/3 \rceil$. The proof will follow from Proposition 2 of [1]. Before we proceed, we state two useful observations from [1].

1. Every pair of points from X occurs together in at least one member of $\mathcal{B}$, and

2. every hyperedge must be contained in a unique K_4^3, the complete 3-hypergraph on four points.

From the second observation, it is simple to deduce that $\mathcal{B}$ can be partitioned into copies of K_4^3. In Figures 1 and 2, each column consists of the hyperedges that are in the same K_4^3.

Theorem 2.1 *If $(X, \mathcal{B})$ is a friendship 3-hypergraph, then $|\mathcal{B}| \geq \frac{2}{3}(n-1)(n-2)$.*

Proof Let $a \in X$. As in [1], we partition the hyperedges into three sets:

1. Let E_2 be the graph consisting of the edges formed by taking all hyperedges containing element a, and removing element a.

2. E_3^A consists of the hyperedges that are contained in a K_4^3 with the element a.

3. E_3^B consists of the remaining hyperedges.

Note that hyperedges containing a and the hyperedges in E_3^B can both be partitioned into copies of K_4^3. To get a lower bound on $|\mathcal{B}|$, we will compute a lower bound for $|E_2| + |E_3^A| + |E_3^B|$.

As in [1], we first prove that each $xyw \in E_3^A$ is counted three times. If $xyw \in E_3^A$, then the vertices x, y, w, a form a K_4^3, and each vertex of the K_4^3 is the completion for the other three. So xyw is counted for all three pairs xy, yw and xw. Since $xyw \in E_3^A$ only if xy, yw, $xw \in E_2$, then $|E_3^A| = \frac{1}{3}|E_2|$.

The key to getting a better lower bound is to carefully consider E_3^B. There are $\binom{n-1}{2} - |E_2|$ pairs xy that do not occur in E_2. Let xy be such a pair and let a be as above. Consider a, x, y and the element u that completes them. By definition, there must be hyperedges axu, ayu and uxy. Now hyperedge uxy must be in E_3^B. When this occurs, we say that the triple a, x, y is *finished* by the K_4^3 that contains the hyperedge uxy. Note that only one K_4^3 can finish a triple a, x, y. We claim that at most three triples of the form a, x, y, where $xy \notin E_2$ can be finished by any given K_4^3 of E_3^B. To see this, let a, x, y be a triple finished by a K_4^3 of E_3^B. Let this K_4^3 consist of the hyperedges uxy, wxy, uwx, uwy. Then the completion of a, x, y must be either u or w. Without loss of generality, suppose it is u. Then there are hyperedges axu, ayu and xyu in $(X, \mathcal{B})$, where xu and yu are in E_2 and hyperedge xyu must be in E_3^B. That leaves only pairs uw, wx and wy in the K_4^3 that have not been assigned to E_2 or its complement. If three more triples of the form a, x, y where $xy \notin E_2$ are finished by this K_4^3, then these triples must be a, w, x; a, w, y and a, u, w. This implies that $wx, wy, uw \notin E_2$. But this is impossible, since a, w, x must be completed by element u or y. If it completed by u, then $uw \in E_2$, which is a contradiction. Similarly, if it is completed by y, then $wy \in E_2$, which is also a contradiction. Therefore, each K_4^3 can finish at most three triples of the form a, x, y where $xy \notin E_2$. This implies that there must be at least $(\binom{n-1}{2} - |E_2|)/3$ K_4^3s in E_3^B, or equivalently $|E_3^B| \geq \frac{4}{3}(\binom{n-1}{2} - |E_2|)$. Then we have $|E_2| + |E_3^A| + |E_3^B| \geq |E_2| + \frac{1}{3}|E_2| + \frac{4}{3}(\binom{n-1}{2} - |E_2|) = \frac{2(n-1)(n-2)}{3}$. □

We point out that if the above bound is met, and a hyperedge xyz is in E_3^B and xyz is the edge used by the K_4^3 containing it to finish the triple a, x, y where $xy \notin E_2$, then yz and xz are in E_2 and therefore, the hyperedge xyz can be used by the K_4^3 to finish exactly one pair (xy in this case) not in E_2. In addition, each K_4^3 in E_3^B must finish exactly three triples of the form a, x, y, where $xy \notin E_2$, if the above bound is to be met.

Lemma 2.2 *Let $(X, \mathcal{B})$ be a friendship 3-hypergraph with $|\mathcal{B}| = \frac{2}{3}(n-1)(n-2)$ hyperedges. Let a be an element of X, and let E_2 be defined as in Theorem 2.1. If $xy \notin E_2$, then the pair xy occurs in exactly two hyperedges in the friendship 3-hypergraph.*

Proof In the proof of the previous theorem, if the lower bound is attained then every K_4^3 in E_3^B must finish three triples of the form a, x, y where a is used to define E_2 and $xy \notin E_2$. In order to get a contradiction let some pair $12 \notin E_2$ be in more than 2 hyperedges; i.e., 12 is in at least two K_4^3s. We will now use the observation immediately after the proof of Theorem 2.1. Let 1234 and 1256 be two K_4^3s in E_3^B containing pair 12. Let one of the hyperedges, say 123 be used by 1234 to finish the triple $a, 1, 2$. Now consider the hyperedges $125, 126$ in the other K_4^3. If 125 is used by 1256 to finish some triple a, x, y, where $xy \notin E_2$, then this implies $xy = 15$ or $xy = 25$. In either case, this would imply that $12 \in E_2$, which is a contradiction. The same holds for 126. Therefore, a pair like $xy \notin E_2$ can not occur in more than one K_4^3 in E_3^B and so can and must occur in exactly two hyperedges in the friendship 3-hypergraph. □

Suppose we have a friendship 3-hypergraph with $|\mathcal{B}| = \frac{2}{3}(n-1)(n-2)$ hyperedges and a pair xy that does not occur with some element $z \in X \setminus \{x, y\}$. Then choosing the element z to construct E_2 shows that the pair $xy \notin E_2$, and therefore by Lemma 2.2, occurs in exactly two hyperedges of the friendship hypergraph. So we have the following observation: Any pair of elements xy in this situation occurs in either $n-2$ hyperedges or two hyperedges of the friendship hypergraph.

We now show that if $\frac{4}{3}\binom{n-1}{2}$ is an integer and $(X, \mathcal{B})$ is friendship hypergraph on $|X| = n$ elements having $\frac{4}{3}\binom{n-1}{2}$ hyperedges, then $(X, \mathcal{B})$ must be the universal friend hypergraph.

Theorem 2.3 *If $\frac{4}{3}\binom{n-1}{2}$ is an integer and $(X, \mathcal{B})$ is friendship hypergraph on $|X| = n$ elements having $\frac{4}{3}\binom{n-1}{2}$ hyperedges, then $(X, \mathcal{B})$ must be the universal friend hypergraph. Consequently, $n \equiv 2, 4 \pmod 6$.*

Proof By the remark after Lemma 2.2, we know that a pair of points occurs exactly 2 or $n-2$ times in the friendship 3-hypergraph. Let α denote the number of pairs that appear in exactly 2 hyperedges, and let β denote the number of pairs that appear in exactly $n-2$ hyperedges. Then, by counting pairs and total occurrence of pairs in the friendship 3-hypergraph, we have the following set of linear equations:

$$\alpha + \beta = \binom{n}{2}$$

and

$$2\alpha + (n-2)\beta = \frac{2}{3}(n-1)(n-2) \times 3.$$

Solving for α and β, we get the unique solution of $\alpha = \binom{n-1}{2}$ and $\beta = n-1$. Therefore, the hypergraph must contain $\binom{n-1}{2}$ pairs that occur in exactly two hyperedges and $n-1$ pairs that occur in exactly $n-2$ hyperedges of the hypergraph.

By the pigeonhole principle, there are two pairs, each occurring $n-2$ times in the hypergraph, that have a common element. Without loss of generality, suppose the pairs 12 and 13, each occur in $n-2$ hyperedges of the hypergraph. Then the hyperedges $123, 124, \ldots, 12n$ and $134, 135, \ldots, 13n$ are in the hypergraph. Note that if the pair 14 occurs more than 2 times then it must occur $n-2$ times. But then every other pair containing 1 must also occur more than 2 times and so occurs $n-2$ times. Then we see that all pairs containing element 1 occur $n-2$ times, giving the universal friend hypergraph where element 1 is the universal friend. So suppose the pair 14 occur exactly two times. We will show that this leads to a contradiction.

We proceed to show that the pair 23 cannot occur $n-2$ times in the hypergraph given that pairs 12 and 13 occur $n-2$ times. To see this, suppose that 23 does occur $n-2$ times in the hypergraph. Then the hyperedges $123, 234, 235, \ldots, 23n$ must be in the hypergraph. Since the hyperedges $123, 124, 134, 234$ are in the hypergraph, 1234 is a K_4^3 in the hypergraph. In addition, the hyperedges $123, 125, 135, 235$ are also in the hypergraph implying 1235 is a K_4^3. This contradicts observation 2 on Page 2.

Let ij be a pair of elements where $i, j \notin \{1, 2, 3\}$. Suppose that the pair ij occurs in $n-2$ hyperedges in the hypergraph. Note that the hyperedges $12i$, $12j$, $1ij$ and $2ij$ are in the hypergraph. In addition, the hyperedges $13i, 13j, 1ij$ and $3ij$ are in the hypergraph. These two facts imply that $12ij$ and $13ij$ are K_4^3s in the hypergraph which contradicts the fact that the K_4^3s in a friendship hypergraph must be disjoint. Therefore the ij pairs must occur in exactly two hyperedges of the hypergraph.

Notice we have not handled pairs of the form $3i$ where $4 \leq i \leq n$. Suppose some pair $3i$ (for $4 \leq i \leq n$) occurs in $n-2$ hyperedges of the hypergraph. Then $23i$ must be one of these hyperedges. As the pair 23 occurs in exactly two hyperedges (see the paragraph before the previous paragraph) and it occurs in 123 and $23i$, then no other pair $3j$ where $j \neq i, j \geq 4$ occur in a hyperedge with element 2. This means that these pairs $3j$ must occur in exactly two hyperedges of the hypergraph. So, we have shown that at most one pair of the form $3i$ (for $4 \leq i \leq n$) may occur in $n-2$ hyperedges. By symmetry, at most one pair of the form $2i$ (for $4 \leq n$) may occur in $n-2$ hyperedges of the hypergraph. That means there are at most 4 pairs that occur $n-2$ times in the hypergraph. But that contradicts the fact that there must be exactly $n-1$ such pairs. $\square$

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