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Throttling zero forcing propagation speed on graphs

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Abstract

Zero forcing is a game played on a graph that starts with a coloring of the vertices as white and black and at each step any vertex colored black with a unique neighbor colored white "forces" the color of the white vertex to become black. In this note we look at what happens when we balance the size of the initial set of vertices colored black and the number of steps, called speed of propagaion, that it takes for all vertices to be colored black. We also give an example that shows it is possible in some graphs to slow down the speed of propagation in the graph by choosing larger initial sets. Finally, we give a tight relationship between the zero forcing number and the number of edges in the graph.

1 Introduction

Zero forcing is a game played on a graph where vertices have two possible colors, traditionally black and white. The goal is to make all the vertices have color black which is accomplished by initially coloring some subset of vertices black and the rest white and then applying a color change rule. Namely, if a vertex is colored black and has exactly one neighbor which is colored white (all other neighbors are colored black), then the neighbor colored white is changed to having color black.

A subset of vertices which when initially colored black can change all of the vertices to be colored black is known as a zero forcing set. The fewest number of vertices in a zero forcing set is known as the zero forcing number of the graph and denoted $Z(G)$. The parameter $Z(G)$ has connections to the minimum rank of a matrix associated with a graph (see [1]), and also has interpretations in physics (see [3]).

Hogben et al. [4] studied the time that it takes for a given zero forcing set to change the color of all the vertices to black where at each time step each valid application of the color change rule is used simultaneously. Given a zero forcing set S the amount of time required to change all the vertices, or the propagation time, is denoted $p(G; S)$. Hogben et al. focused on sets S which had $Z(G)$ elements, i.e., minimum zero forcing sets and determined various properties for some special classes of graphs.

In this note we will look at how we can affect the speed of propagation by looking at other possible sets. In Section 2 we will look at balancing the size of the zero forcing set with the propagation time and show that when the zero forcing number is bounded then we can get to within a constant factor of optimal for this situation. In Section 3 we give an example of the somewhat unexpected behavior of being able to slow down the propagation time by allowing for a larger set of initially colored black vertices. In Section 4 we give a tight relationship between the number of edges in a graph and the zero forcing number.

2 Balancing the initial set with propagation time

By adding additional vertices to a minimal zero forcing set we can speed up the propagation time. For instance, $p(G;V(G))=0$, but of course that requires coloring all of the vertices black to achieve such speed. Richard Brualdi [2] asked the following question about balancing the propagation time of the set with the initial size of the set.

Question. For a graph G, determine $\min_S \big(p(G;S)+|S|\big)$ where S ranges over all sets and $p(G;S)=\infty$ if S is not a zero forcing set.

This question has an interesting motivation. Suppose we want to spread some piece of information across a network, for example a patch of code on a computer network, or a viral marketing campaign in a social network. Then there are two natural costs, namely the initial distribution among some subset of the network (which corresponds to S) and the time that it takes for the information to propagate (which corresponds to $p(G;S)$). This question then looks at minimizing the total cost of sharing this information with the whole network.

Given a zero forcing set we can keep track of the forces as we propagate through the graph. We note that if a vertex ever forces once then it could not force a second time (i.e., that would imply there were two vertices incident which were colored white and so could not force the first time). So we can start with vertices in our initial zero forcing set S and keep track of the forces that are made, these are known as forcing chains and will correspond to induced paths inside of our graph G. Suppose G has n vertices, since each forcing chain can only grow by one vertex at each time step we immediately have that

$$|S|\cdot\big(p(G;S)+1\big)\geq n,$$

i.e., given an initial zero forcing set S we need enough time so that we can propagate and cover all of the vertices of the graph. Some simple optimization then gives that

$$\min_S \big(p(G;S)+|S|\big)\geq 2\sqrt{n}-1. \tag{1}$$

The lower bound (1) is tight for some graphs. Consider the path on n vertices and let s and t be chosen so that $s(t+1)\geq n$ and $s+t$ is minimal. Then we can embed the path on n vertices into an $s \times (t+1)$ box where we "snake" the path back and forth. An example is shown in Figure 1 for $n = 18$, $s = 4$ and $t = 4$.

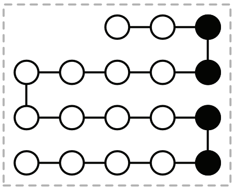


Figure 1: Snaking a path into a box, with a zero forcing set marked.

For the path we now select all the right-most vertices in each row in the box (as indicated in Figure 1). This gives us a zero forcing set of size s (the number of rows) and the propagation time is t (we force from right to left), giving us the minimal possible sum of the size of a zero forcing set plus the propagation time. If we let P_n denote the path on n vertices and we solve for the optimal s and t, we have established the following.

$$\min_S \big(p(P_n;S) + |S|\big) = \begin{cases} 2m-1 & \text{if } n = m^2; \\ 2m & \text{if } m^2 < n \le m^2+m; \\ 2m+1 & \text{if } m^2+m < n \le m^2+2m. \end{cases}$$

More generally, if the zero forcing number is bounded we can be within a constant multiple of the optimal lower bound given in (1). The basic idea is to split the forcing chains into small manageable chunks that allow for fast propagation similar to what was done for the path.

Theorem 1. *If G is a graph on n vertices and $Z(G) \le k$ then*

$$\min_S\{|S| + p(G;S)\} \le (2k+1)\lceil\sqrt{n}\rceil + k = \Theta(\sqrt{n}).$$

Proof. We will produce an S which achieves the bound indicated in the theorem. Given that $Z(G) \le k$ there is some zero forcing set $S_0 = S'$ of size k. Associated with each element $v \in S'$ is a corresponding forcing chain C_v where the vertices are ordered along the chain by when they are forced. We now do the following procedure:

1. Given S_i we take $\lceil\sqrt{n}\rceil$ time steps where at each time step we make every possible force available *along the chains*. If all the vertices are colored black then we are done and we let $S = S_i$ and stop.

2. Given that not all the vertices were colored black let $u_1^{(i)}, \ldots, u_k^{(i)}$ be the last forces made in each chain and let $w_j^{(i)}$ be the subsequent vertex in the chain that follows after $u_j^{(i)}$ (if $u_j^{(i)}$ is not the last vertex in the chain). Now let $S_{i+1} = S_i \cup \big(\bigcup u_j^{(i)}\big) \cup \big(\bigcup w_j^{(i)}\big)$ and return to the previous step replacing S_i by S_{i+1}.

Note at the end we will only stop once we have a zero forcing set which has propagation time at most $\lceil\sqrt{n}\rceil$. We start out with k vertices in the zero forcing set and at each step we are adding at most $2k$ vertices to the zero forcing set. (We might be adding fewer because it is possible that in the given time steps that no forces happen along a particular chain so that we would not be adding new vertices, similarly we might already be at the end of a chain and not need the next element.) So to finish the proof it suffices to show that we will run through these steps at most $\lceil\sqrt{n}\rceil$ times.

The key observation to make is that at each time iteration i we can lop off the portions of the chain that come before the $u_j^{(i)}$ because we know that this portion can already be forced in $\lceil\sqrt{n}\rceil$ steps and because there cannot be an edge between a vertex in chain C_{j_1} lying *after* $u_{j_1}^{(i)}$ to a vertex in chain C_{j_2} lying *before* $u_{j_2}^{(i)}$. Otherwise we could not have forced at that vertex along the chain. In particular, we can use the vertices $\big(\bigcup u_j^{(i)}\big) \cup \big(\bigcup w_j^{(i)}\big)$ to finish forcing along the chains for the vertices that are left after the ith step, i.e., the new vertices by themselves can continue to make forces at each time step.

In particular, each iteration we will have at least $\lceil\sqrt{n}\rceil$ new vertices that were forced that were not forced in the previous iteration (if we did not force the whole graph). Therefore after ℓ iterations we will have that S_ℓ forces at least $\ell\lceil\sqrt{n}\rceil$ vertices black in $\lceil\sqrt{n}\rceil$ steps. But there are only n vertices so we must stop by the time we hit $\ell = \lceil\sqrt{n}\rceil$.

Finally, we observe that dropping the restriction that we only force along the chains can only speed up the propagation time so that S is our desired set. ☐

An example of how this evolves is shown in Figure 2 where the two forcing chains are the top and bottom paths. More generally, the same method of proof shows the following result which weighs the relative costs of the initial set and the cost of propagation.

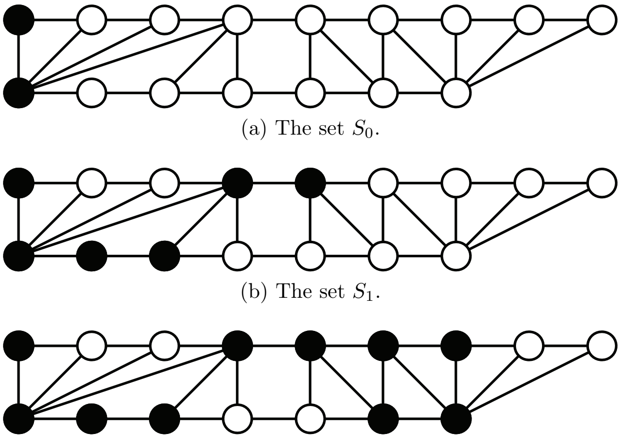
Theorem 2. *If G is a graph on n vertices, $Z(G) \le k$, and a and b are given weights, then*

$$\min_S\{a|S| + b\,p(G;S)\} \le (2ka+b)\lceil\sqrt{n}\rceil + ka = \Theta(\sqrt{n}).$$

3 Slowing down propagation time

Intuitively when we allow for a larger set to propagate, we would expect the propagation time to decrease. However it is possible to find a larger set which can have a longer propagation time.

We illustrate this with an example. In the graph shown in Figure 3a we give a graph along with a zero forcing set with 7 vertices which has a propagation time of 5 (in fact this is a minimum zero forcing set, and every minimum zero forcing set has a propagation time of 5 for this graph). By comparison in Figure 3b we give the same graph with a different zero forcing set with 13 vertices which has a propagation time of 13.

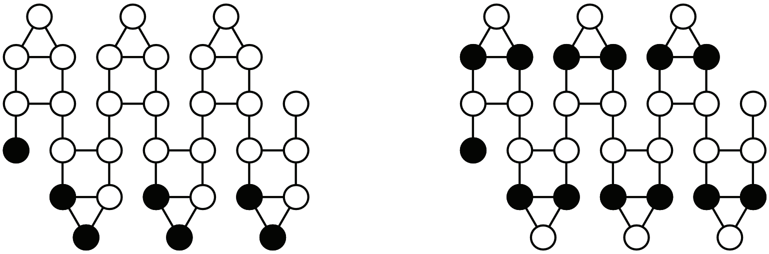


(a) The set S_0.

(b) The set S_1.

(c) The set $S_2 = S$.

Figure 2: An example of the construction from Theorem 1.



(a)

(b)

Figure 3: An example of two zero forcing sets

In Figure 3a the forcing chains run in parallel while in Figure 3b the forcing chains run in serial. This construction can easily be extended to construct graphs where the propagation time for any minimum zero forcing set is 5 but the maximum propagation time over all zero forcing sets can be made arbitrarily large.

4 A tight relationship between edges and zero forcing

One of the options we considered looking at while preparing this note was trying to bound the number of high degree vertices. In particular, we were able to derive a tight relationship between the zero forcing number of a graph and the number of edges in the graph. While our proof ended up using a completely different approach we still include the result because it has independent interest. First we will establish a property of graphs with $Z(G) = 2$.

Lemma 1. *If $Z(G) = 2$ and G is connected, then G is outerplanar.*

Proof. Since $Z(G) = 2$ place one forcing chain on the x-axis with vertices going from left to right in the order they are forced. Similarly place the other forcing chain on the line $y = 1$ with vertices again going from left to right in the order they are forced. This places all vertices in the plane and the edges in the forcing chains, any remaining edges can then be drawn with straight lines between these two paths. If any two lines crossed then these could not be a pair of forcing chains starting with the left-most vertices. Therefore we have found an outerplanar embedding of G. □

A well known result of outerplanar graphs is that they have at most $2n-3$ edges. We now prove our main result.

Theorem 3. *For a graph G with n vertices, if $Z(G) \le k$ then $|E(G)| \le kn - \binom{k+1}{2}$. Further this bound is tight.*

Proof. If $Z(G) = 1$ then the graph must be a path with $|E(G)| = n-1$. If $Z(G) = 2$ then by the lemma the graph is an outerplanar graph and $|E(G)| \le 2n - 3$. Now suppose that $Z(G) \ge 3$. Let x_i be the number of vertices in the ith forcing the chain. The graph induced by two forcing chains, say the ith and jth chain with x_i and x_j vertices respectively, has zero forcing number at most 2, and so has at most $2(x_i+x_j)-3$ edges. In particular the number of edges between the two forcing chains is at most

$$(2(x_i + x_j) - 3) - (x_i - 1) - (x_j - 1) = x_i + x_j - 1.$$

Therefore we have

$$|E(G)| \le \sum_{i \ne j}(x_i + x_j - 1) + \sum_i (x_i - 1) = k\sum_i x_i - \binom{k}{2} - k = kn - \binom{k+1}{2}.$$

To see that this is tight consider the graph $K_{k-1} \vee P_{n-k+1}$ (the join of the clique on $k-1$ vertices with a path on $n-k+1$ vertices). This graph has $Z(G) = k$ (the vertices of the clique and one end of the path) and the graph has $kn - \binom{k+1}{2}$ edges. □

If m denotes the number of edges then the above can be restated $k^2 + (1 - 2n)k + 2m \le 0$. We can use this to get a lower bound for k by solving for a root of this quadratic equation corresponding to this inequality. If we also use that $\sqrt{1-x} \le 1 - \frac{1}{2}x$ then we get the following.

Corollary 1. *For a graph G with n vertices and m edges, $Z(G) > \frac{m}{n}$, in particular $Z(G)$ is greater than half of the average degree of the graph.*

Another lower bound for the zero forcing number is the minimum degree (i.e., we have to have at least the minimum degree number of vertices colored black to get our first force). So the above bound works best for graphs with highly irregular degree sequences.

Acknowledgment

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