AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **57** (2013), Pages 3–19

Group-magic labelings of graphs with deleted edges

W.C. Shiu

Department of Mathematics
Hong Kong Baptist University
224 Waterloo Road, Kowloon Tong
Hong Kong
wcshiu@hkbu.edu.hk

Richard M. Low

Department of Mathematics
San Jose State University
San Jose, CA 95192
U.S.A.
richard.low@sjsu.edu

Abstract

Let A be a non-trivial, finitely-generated abelian group and $A^* = A\backslash\{0\}$. A graph is A-magic if there exists an edge labeling using elements of A^* which induces a constant vertex labeling of the graph. In this paper, we analyze the group-magic property for complete n-partite graphs and composition graphs with deleted edges.

1 Introduction

Let $G = (V, E)$ be a connected simple graph. For any non-trivial, finitely generated abelian group A (written additively), let $A^* = A\backslash\{0\}$. A mapping $f : E \to A^*$ is called a *labeling* of G. Any such labeling induces a map $f^+ : V \to A$, defined by $f^+(v) = \sum_{uv\in E} f(uv)$. If there exists a labeling f whose induced map on V is a constant map, we say that f is an *A-magic labeling* of G and that G is an *A-magic* graph. The corresponding constant $\mathbf{x}$ is called an *A-magic value*. The *integer-magic spectrum* of a graph G is the set $\mathrm{IM}(G) = \{k \mid G \text{ is } \mathbb{Z}_k\text{-magic and } k \geq 1\}$. Note that the integer-magic spectrum of a graph is not to be confused with the set of achievable magic values. A *labeling matrix* for a labeling f of G is a matrix whose rows and columns are named by the vertices of G and the (u, v)-entry is $f(uv)$ if $uv \in E$, and is $*$ otherwise. In particular, if f is an A-magic labeling of G, then a labeling matrix of f is called an *A-magic labeling matrix* of G. Thus G is A-magic if and only if there exists a labeling $f : E \to A^*$ such that the row sums (and the column sums) of the labeling matrix for f are a constant value $\mathbf{x}$.

$\mathbb{Z}$-magic (or $\mathbb{Z}_1$-magic) graphs were considered by Stanley [26, 27], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [1–3] and others [7, 9, 15, 16, 23] have studied A-magic graphs and $\mathbb{Z}_k$-magic graphs were investigated in [4, 6, 8, 10–14, 17–20, 24, 25].

Within the mathematical literature, various definitions of magic graphs have been introduced. The original concept of an A-magic graph is due to J. Sedlacek [21, 22], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Previously, Kotzig and Rosa [5] had introduced yet another definition of a magic graph. Over the years, there has been great research interest in graph labeling problems. The interested reader is directed to Wallis' [28] monograph on magic graphs.

2 Group-magic labelings of $K_n - e$

First, let us make the following useful observations.

- Graph G is $\mathbb{Z}_2$-magic $\Longleftrightarrow$ every vertex of G is of the same parity.
- If A_1 is a subgroup of A and graph G is A_1-magic, then G is A-magic.

Since the Klein-4 group V_4 is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, we let $V_4 = \{0, a, b, c\}$, where $0 = (0, 0)$, $a = (1, 0)$, $b = (0, 1)$ and $c = (1, 1)$.

In this section, we focus on a complete graph with one edge missing $K_n - e$. Here, we set $V(K_n) = \{v_1, \ldots, v_n\}$. Without loss of generality, we let $e = v_1v_2$. Any labeling matrix of $K_n - e$ is defined according to this list of vertices. First, note that $K_n - e$ is not $\mathbb{Z}_2$-magic.

Theorem 2.1. *$K_4 - e$ is V_4-magic.*

Proof. The following matrix is a V_4-magic labeling matrix of $K_4 - e$.

$$\begin{pmatrix} * & * & a & a \\ * & * & b & b \\ a & b & * & c \\ a & b & c & * \end{pmatrix}$$

□

Theorem 2.2. $\mathrm{IM}(K_4 - e) = 2\mathbb{N} \setminus \{2\}$.

Proof. Let a $\mathbb{Z}_k$-magic labeling matrix of $K_4 - e$ be

$$\begin{pmatrix} * & * & x_1 & x_2 \\ * & * & x_3 & x_4 \\ x_1 & x_3 & * & x_5 \\ x_2 & x_4 & x_5 & * \end{pmatrix}.$$

Then, we have $x_2 + x_4 + x_5 = x_1 + x_2$ and $x_1 + x_3 + x_5 = x_3 + x_4$. This implies $2x_5 + x_1 + x_2 + x_3 + x_4 = x_1 + x_2 + x_3 + x_4$, and hence $2x_5 = 0$. Thus if k is odd, then $x_5 = 0$, which is not allowed. Thus, $k = 2n$ for some $n \geq 2$.

A $\mathbb{Z}_{2n}$-magic labeling matrix of $K_4 - e$ is

$$\begin{pmatrix} * & * & 1 & -1 \\ * & * & n-1 & n+1 \\ 1 & n-1 & * & n \\ -1 & n+1 & n & * \end{pmatrix},$$

where $n \geq 2$. □

Theorem 2.3. *$K_5 - e$ is A-magic, for all $A \neq \mathbb{Z}_2$.*

Proof. Suppose A contains a subgroup isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_n$ with $n \geq 3$. Then the following matrix is an A-magic labeling matrix of $K_5 - e$.

$$\begin{pmatrix} * & * & -1 & 2 & -1 \\ * & * & 2 & -1 & -1 \\ -1 & 2 & * & -2 & 1 \\ 2 & -1 & -2 & * & 1 \\ -1 & -1 & 1 & 1 & * \end{pmatrix}$$

Suppose A contains a subgroup isomorphic to V_4. Then, a V_4-magic labeling matrix of $K_5 - e$ is

$$\begin{pmatrix} * & * & a & b & c \\ * & * & a & b & c \\ a & a & * & c & c \\ b & b & c & * & c \\ c & c & c & c & * \end{pmatrix}.$$

Hence, $K_5 - e$ is A-magic. □

The following definition and Theorem 2.4 were introduced by Salehi [18] and will be used within this paper.

Definition. The *null set* of a graph G, denoted by $N(G)$, is the set of all numbers $k \in \mathbb{N}$, where G has a $\mathbb{Z}_k$-magic labeling with magic-value 0.

Theorem 2.4 ([18, Theorem 2.1]). *For $n \geq 4$, $N(K_n) = \mathbb{N} \setminus \{2\}$ if n is even and $N(K_n) = \mathbb{N}$ if n is odd.*

Lemma 2.5. *Let A be an abelian group, where $|A| \geq 3$. Then, there exist $x_1, x_2, x_3 \in A^*$ such that $x_1 + x_2 + x_3 = 0$.*

Proof. Suppose A contains an element x such that $2x \neq 0$. Then, choose $x_1 = x_2 = x$ and $x_3 = -2x$.

Suppose every element x in A satisfying $2x = 0$. Then, A contains a subgroup isomorphic to V_4. Then, choose $x_1 = a$, $x_2 = b$ and $x_3 = c$. □

Theorem 2.6. *$K_n - e$ is A-magic, for all $A \neq \mathbb{Z}_2$, where $n \geq 6$.*

Proof. Let A be an abelian group not equal to $\mathbb{Z}_2$. By Theorem 2.4, there exists an A-magic labeling matrix M of K_{n-2} such that the row sums are zero.

Let $\alpha = (x_1, x_2, \ldots, x_{n-2})$. Suppose n is odd. By Lemma 2.5 we can choose $x_1, x_2, x_3 \in A^*$ such that their sum is zero. Moreover, we choose $x_i = (-1)^i x$ for $i = 4, \ldots, n-2$, for some $x \in A^*$. Suppose n is even. We choose $x_i = (-1)^i x$ from some $x \in A^*$, for $i = 1, 2, \ldots, n-2$. For both cases, the sum of all components of α is zero.

Then the following matrix is an A-magic labeling matrix of $K_n - e$ with row sums equal to zero.

$$\left(\begin{array}{cc|c} * & * & \alpha \\ * & * & -\alpha \\ \hline \alpha^T & -\alpha^T & M \end{array}\right),$$

where M and α are defined above. □

Example. A $\mathbb{Z}_k$-magic labeling matrix of $K_8 - e$, where $k \neq 2$.

$$\left(\begin{array}{rr|rrrrrr} * & * & 1 & -1 & 1 & -1 & 1 & -1 \\ * & * & -1 & 1 & -1 & 1 & -1 & 1 \\ \hline 1 & -1 & * & 1 & -1 & 2 & -1 & -1 \\ -1 & 1 & 1 & * & -1 & -1 & 2 & -1 \\ 1 & -1 & -1 & -1 & * & 1 & -1 & 2 \\ -1 & 1 & 2 & -1 & 1 & * & -1 & -1 \\ 1 & -1 & -1 & 2 & -1 & -1 & * & 1 \\ -1 & 1 & -1 & -1 & 2 & -1 & 1 & * \end{array}\right)$$

From the results above, we obtain the following corollary.

Corollary 2.7. *$N(K_4 - e) = 2\mathbb{N} \setminus \{2\}$ and $N(K_n - e) = \mathbb{N} \setminus \{2\}$ for $n \geq 5$.*

Proof. This follows from Theorems 2.3 and 2.6. □

3 Group-magic labelings of $K_{r,s} - \{e\}$

In this section, we focus on the class of complete bipartite graphs with a deleted edge. Throughout this section, we shall use $X = \{u_1, \dots, u_r\}$ and $Y = \{v_1, \dots, v_s\}$ as the bipartition of $K_{r,s}$ with $r \geq s \geq 1$. Without loss of generality, we may let $e = u_1v_1$. Any labeling matrix of $K_{r,s} - e$ is defined according to this list of vertices. Since $K_{r,1} - e$ contains an isolated vertex, we do not consider these cases. So we may assume that $r \geq s \geq 2$. It is clear that $K_{r,s} - e$ is not $\mathbb{Z}_2$-magic.

Note that the labeling matrix of $K_{r,s} - e$ of any labeling is of the form $\begin{pmatrix} \bigstar_r & B \\ B^T & \bigstar_s \end{pmatrix}$, where B is an $r \times s$ matrix whose $(1,1)$-entry is $*$, $\bigstar_r$ and $\bigstar_s$ are square matrices of order r and s respectively with all entries being $*$. In order to find an A-magic labeling matrix of $K_{r,s} - e$ with magic value $\mathbf{x}$, it suffices to find such a matrix B (having non-zero entries) so that its row sums and column sums are $\mathbf{x}$.

The following theorems, which were proved by the authors in [23], will be used.

Theorem 3.1 ([23, Theorem 1]). *Let n_1 and n_2 be even. Then, K_{n_1,n_2} has an A-magic labeling with magic value 0 for all abelian groups A.*

Theorem 3.2 ([23, Theorem 3]). *Suppose n_1 is odd, with $n_1 \geq 3$ and $n_2 \geq 2$. For any abelian group A where $|A| \geq 3$, K_{n_1,n_2} has an A-magic labeling with magic value 0.*

Theorem 3.3. *Suppose $r \geq s \geq 5$. For any abelian group A where $|A| \geq 3$, $K_{r,s} - e$ has an A-magic labeling with magic value 0.*

Proof. Since $r \geq s \geq 5$, by Theorems 3.1 and 3.2, there is an A-magic labeling matrix M of $K_{r-2,s-2}$ such that its row sums and column sums are zero. We write M as the following partition matrix:

$$M = \begin{pmatrix} m_{11} & \beta \\ \gamma & M_1 \end{pmatrix},$$

where $m_{11} \in A$ and β, γ and M_1 are $1 \times (s-3)$, $(r-3) \times 1$ and $(r-3) \times (s-3)$ matrices, respectively.

Since $r \geq s \geq 5$, by the argument of the proof of Theorem 2.6, we have a $1 \times (s-3)$ matrix α such that the sum of all its components is zero. By Lemma 2.5 there are $x_1, x_2, x_3 \in A^*$ such that $x_1 + x_2 + x_3 = 0$. There also exists a $1 \times (r-3)$ matrix δ, whose sum is 0. Hence the matrix

$$\begin{pmatrix} \bigstar_r & B \\ B^T & \bigstar_s \end{pmatrix}, \quad \text{where } B = \left(\begin{array}{ccc|c} * & x_1 & -x_1 & \alpha \\ x_2 & x_3 & x_1 & -\alpha \\ -x_2 & x_2 & m_{11} & \beta \\ \hline \delta^T & -\delta^T & \gamma & M_1 \end{array}\right)$$

is an A-magic labeling matrix of $K_{r,s} - e$. □

Example. A $\mathbb{Z}_3$-magic labeling matrix of $K_{6,5}-e$.

$$\begin{pmatrix} \bigstar_6 & B \\ B^T & \bigstar_5 \end{pmatrix}, \quad \text{where } B = \left(\begin{array}{rrr|rr} * & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 2 \\ \hline 1 & -1 & 1 & 1 & -2 \\ 1 & -1 & -1 & -1 & 2 \\ 1 & -1 & 1 & 1 & -2 \end{array}\right).$$

Corollary 3.4. *$N(K_{r,s}-e) = \mathbb{N}\setminus\{2\}$ for $r \ge s \ge 5$.*

Now we consider the graph $K_{r,2}-e$. Its labeling matrix is of the form:

$$\begin{pmatrix} \bigstar_r & M_{r,2} \\ M_{r,2}^T & \bigstar_2 \end{pmatrix}, \quad \text{where } M_{r,2} = \begin{pmatrix} * & m \\ x_1 & y_1 \\ \vdots & \vdots \\ x_{r-1} & y_{r-1} \end{pmatrix}$$

for some $m \in A^*$. Since $K_{2,2}-e \cong P_3$ (which is not A-magic, for any abelian group A), we assume that $r \ge 3$. If $M_{r,2}$ is to be used to construct an A-magic labeling matrix of $K_{r,2}-e$, then we must have

$$\left\{\begin{array}{l} x_1 = y_2 + \cdots + y_{r-1} + m \\ x_i = -y_i + m \quad (\text{for } 2 \le i \le r-1) \\ y_1 = -y_2 - \cdots - y_{r-1} \\ 0 = (r-2)m \end{array}\right\}. \tag{3.1}$$

The first equation follows from the fact that $x_1 + y_1 = m + \sum_{i=1}^{r-1} y_i$. The second equation follows from the facts that m is the magic value and $x_i + y_i = m$. The third equation follows from the fact that $m + \sum_{i=1}^{r-1} y_i = m$. To obtain the last equation, observe that $m = \sum_{i=1}^{r-1} x_i = \sum_{i=1}^{r-1}(-y_i + m) = (r-1)m + \sum_{i=1}^{r-1} -y_i$ which implies $m = (r-1)m + 0$ and hence, $(r-2)m = 0$. Clearly, $K_{r,2}-e$ is not $\mathbb{Z}$-magic. Now, we consider $A = \mathbb{Z}_k$. If g.c.d.$(r-2,k) = 1$, then $K_{r,2}-e$ is not $\mathbb{Z}_k$-magic. So we assume that g.c.d.$(r-2,k) = d > 1$. Let $k = dk_0$.

If $d = 2$, then r is even. We can choose $a \in \mathbb{Z}_k^* \setminus \{k_0\}$ for $k \ge 4$. Then, $2a \ne 0$ and thus

$$\begin{pmatrix} \bigstar_r & B \\ B^T & \bigstar_2 \end{pmatrix}, \quad \text{where } B = \left(\begin{array}{cc} * & k_0 \\ k_0+2a & -2a \\ k_0-a & a \\ k_0-a & a \\ \hline k_0-a & a \\ k_0+a & -a \\ \vdots & \vdots \\ k_0-a & a \\ k_0+a & -a \end{array}\right)$$

is a $\mathbb{Z}_k$-magic labeling matrix of $K_{r,2} - e$ [note that $(r-1)k_0 \equiv k_0 \pmod{k}$, since $(r-2)k_0 \equiv 0 \pmod{k}$], for even $r \geq 4$.

If $d \geq 3$, then $k = 3$ or $k \geq 5$. We can choose $a \in \mathbb{Z}_k^* \setminus \{k_0, -k_0, 2k_0\}$ for $k \geq 5$. Then

$$\begin{pmatrix} \bigstar_r & B \\ B^T & \bigstar_2 \end{pmatrix}, \quad \text{where } B = \left(\begin{array}{cc} * & k_0 \\ k_0 + a & -a \\ 2k_0 - a & -k_0 + a \\ \hline 2k_0 & -k_0 \\ \vdots & \vdots \\ 2k_0 & -k_0 \end{array}\right)$$

is a $\mathbb{Z}_k$-magic labeling matrix of $K_{r,2} - e$ with magic value k_0, for $k \geq 5$. Suppose $k = 3$. Since all unknowns including m are nonzero, from Eq. (3.1) we have $x_i = y_i$ for all i. Hence, the column sums of the matrix $M_{r,2}$ are not the same. So $K_{r,2} - e$ is not $\mathbb{Z}_k$-magic for g.c.d.$(r-2, 3) = 3$.

From the discussion above, we have

Theorem 3.5. *Suppose $r \geq 3$. For $k \geq 4$, $K_{r,2} - e$ has a $\mathbb{Z}_k$-magic labeling if and only if g.c.d.$(r-2, k) > 1$. Moreover, $K_{r,2} - e$ is neither $\mathbb{Z}_3$-magic nor $\mathbb{Z}$-magic.*

Corollary 3.6. *For $r \geq 3$, $N(K_{r,2} - e) = \varnothing$.*

Example. A $\mathbb{Z}_5$-magic labeling matrix of $K_{7,2} - e$ with magic value 1.

$$\begin{pmatrix} \bigstar_7 & B \\ B^T & \bigstar_2 \end{pmatrix}, \quad \text{where } B = \begin{pmatrix} * & 1 \\ 4 & -3 \\ -1 & 2 \\ 2 & -1 \\ 2 & -1 \\ 2 & -1 \\ 2 & -1 \end{pmatrix}.$$

For the graph $K_{r,3} - e$, its labeling matrix is of the form

$$\begin{pmatrix} \bigstar_r & M_{r,3} \\ M_{r,3}^T & \bigstar_3 \end{pmatrix}, \quad \text{where } M_{r,3} = \begin{pmatrix} * & x & y \\ a_1 & b_1 & c_1 \\ \vdots & \vdots & \vdots \\ a_{r-1} & b_{r-1} & c_{r-1} \end{pmatrix}$$

for $x, y, a_i, b_i, c_i \in A^*$. If $M_{r,3}$ is to be used to construct an A-magic labeling matrix of $K_{r,3} - e$ (with magic value m), then we must have

$$\left\{\begin{array}{l} x = c_1 + \cdots + c_{r-1} \\ y = -c_1 - \cdots - c_{r-1} + m \\ a_1 = b_2 + \cdots + b_{r-1} + c_2 + \cdots + c_{r-1} \\ a_i = -b_i - c_i + m \quad (\text{for } 2 \leq i \leq r-1) \\ b_1 = -b_2 - \cdots - b_{r-1} - c_1 - \cdots - c_{r-1} + m \\ 0 = (r-3)m \end{array}\right\}. \tag{3.2}$$

If $r = 3$, we obtain

$$\left\{\begin{array}{l} x = c_1 + c_2 \\ y = -c_1 - c_2 + m \\ a_1 = b_2 + c_2 \\ a_2 = -b_2 - c_2 + m \\ b_1 = -b_2 - c_1 - c_2 + m \end{array}\right\}. \tag{3.3}$$

Then

$$\begin{pmatrix} \bigstar_3 & B \\ B^T & \bigstar_3 \end{pmatrix}, \quad \text{where } B = \begin{pmatrix} * & 2 & -2 \\ 2 & -3 & 1 \\ -2 & 1 & 1 \end{pmatrix}$$

is a $\mathbb{Z}_k$-magic labeling matrix of $K_{3,3} - e$ with magic value 0, where $k = 1, 4, 5, \ldots$.

For $r = 4$, we obtain

$$\left\{\begin{array}{l} x = c_1 + c_2 + c_3 \\ y = -c_1 - c_2 - c_3 + m \\ a_1 = b_2 + b_3 + c_2 + c_3 \\ a_2 = -b_2 - c_2 + m \\ a_3 = -b_3 - c_3 + m \\ b_1 = -b_2 - b_3 - c_1 - c_2 - c_3 + m \end{array}\right\}, \tag{3.4}$$

where m is the magic value. Then,

$$\begin{pmatrix} \bigstar_4 & B \\ B^T & \bigstar_3 \end{pmatrix}, \quad \text{where } B = \begin{pmatrix} * & 1 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}$$

is a $\mathbb{Z}_k$-magic labeling matrix of $K_{4,3} - e$ with magic value 0, where $k = 1, 4, 5, \ldots$.

Let $x_1, x_2, x_3 \in \mathbb{Z}_k^*$ such that $x_1 + x_2 + x_3 = 0$. Then,

$$\begin{pmatrix} \bigstar_r & B \\ B^T & \bigstar_3 \end{pmatrix}, \quad \text{where } B = \begin{pmatrix} * & 2 & -2 \\ 2 & -3 & 1 \\ -2 & 1 & 1 \\ x_1 & x_2 & x_3 \\ -x_1 & -x_2 & -x_3 \\ & \vdots & \\ x_1 & x_2 & x_3 \\ -x_1 & -x_2 & -x_3 \end{pmatrix}$$

for odd $r \geq 5$, and

$$\begin{pmatrix} \bigstar_r & B \\ B^T & \bigstar_3 \end{pmatrix}, \quad \text{where } B = \begin{pmatrix} * & 1 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \\ x_1 & x_2 & x_3 \\ -x_1 & -x_2 & -x_3 \\ & \vdots & \\ x_1 & x_2 & x_3 \\ -x_1 & -x_2 & -x_3 \end{pmatrix}$$

for even $r \geq 6$, are $\mathbb{Z}_k$-magic labeling matrices of $K_{r,3} - e$ with magic value 0, where $k = 1, 4, 5, \ldots$.

Now, we consider $A = \mathbb{Z}_3$. Since $(r-3)m = 0$, $rm = 0$ (in $\mathbb{Z}_3$). Suppose r is not a multiple of 3. Then, $m = 0$. Since $a_i \neq 0$ in Eq. (3.2), from $a_i = -b_i - c_i$ for $2 \leq i \leq r-1$, we have $b_i = c_i$. Hence Eq. (3.2) becomes

$$\left\{ \begin{array}{l} x = c_1 + (c_2 + \cdots + c_{r-1}) \\ y = -c_1 - (c_2 + \cdots + c_{r-1}) \\ a_1 = 2(c_2 + \cdots + c_{r-1}) \\ a_i = -b_i - c_i \quad (\text{for } 2 \leq i \leq r-1) \\ b_1 = -c_1 - 2(c_2 + \cdots + c_{r-1}) = -c_1 + (c_2 + \cdots + c_{r-1}) \end{array} \right\}. \tag{3.5}$$

It follows that $x - b_1 = 2c_1$ or equivalently $x + c_1 = b_1$. This implies $x = c_1$ as $b_1 \neq 0$. Then $c_2 + \cdots + c_{r-1} = 0$ and hence $a_1 = 0$, which is not allowed. So, $K_{r,3} - e$ is not $\mathbb{Z}_3$-magic when r is not a multiple of 3.

Suppose r is a multiple of 3. By the same proof above, $m \neq 0$. We consider $r = 3$ first. Then

$$\begin{pmatrix} \bigstar_3 & B \\ B^T & \bigstar_3 \end{pmatrix}, \quad \text{where } B = \begin{pmatrix} * & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

is a $\mathbb{Z}_3$-magic labeling matrix of $K_{3,3} - e$ with magic value 2. Now we consider $r = 3\ell$. We add $\ell - 1$ copies of the matrix C to the bottom of B defined above, where

$$C = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}.$$

So, $K_{r,3} - e$ is $\mathbb{Z}_3$-magic when r is a multiple of 3.

From the discussion above, we obtain the following theorem.

Theorem 3.7. *Suppose $r \geq 3$. Then $K_{r,3} - e$ has a $\mathbb{Z}_k$-magic labeling with magic value 0 for $k = 1, 4, 5, \ldots$ Moreover if r is a multiple of 3, then $K_{r,3} - e$ is $\mathbb{Z}_3$-magic.*

Corollary 3.8. *$N(K_{r,3}-e)=\mathbb{N}\setminus\{2,3\}$, for $r\geq 3$.*

Now, we consider $K_{r,4}-e$ for $r\geq 4$. It is clear that

$$\begin{pmatrix} \bigstar_4 & B \\ B^T & \bigstar_4 \end{pmatrix}, \quad \text{where } B=\begin{pmatrix} * & 1 & -2 & 1 \\ 1 & 2 & -2 & -1 \\ 1 & -2 & 2 & -1 \\ -2 & -1 & 2 & 1 \end{pmatrix}$$

is an A-magic labeling matrix of $K_{4,4}-e$ with magic value 0, for $A=\mathbb{Z}$ or $A=\mathbb{Z}_k$ where $k\geq 3$. Now, suppose $r\geq 4$ and even. Let $A=\mathbb{Z}$ or $\mathbb{Z}_k$ for $k\geq 3$. Then

$$\begin{pmatrix} \bigstar_r & B \\ B^T & \bigstar_4 \end{pmatrix}, \quad \text{where } B=\left(\begin{array}{rrrr} * & 1 & -2 & 1 \\ 1 & 2 & -2 & -1 \\ 1 & -2 & 2 & -1 \\ -2 & -1 & 2 & 1 \\ \hline 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{array}\right)$$

is an A-magic labeling matrix of $K_{r,4}-e$ with magic value 0, for even $r\geq 4$. For odd $r\geq 5$, we obtain the following A-magic labeling matrix of $K_{r,4}-e$ where $A=\mathbb{Z}$ or $\mathbb{Z}_k$, $k\geq 4$.

$$\begin{pmatrix} \bigstar_r & B \\ B^T & \bigstar_4 \end{pmatrix}, \quad \text{where } B=\left(\begin{array}{rrrr} * & 2 & -1 & -1 \\ 1 & -3 & 3 & -1 \\ -1 & 1 & -2 & 2 \\ \hline 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{array}\right)$$

Finally,

$$\begin{pmatrix} \bigstar_r & B \\ B^T & \bigstar_4 \end{pmatrix}, \quad \text{where } B=\left(\begin{array}{cccc} * & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ \hline 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \end{array}\right)$$

is a $\mathbb{Z}_3$-magic labeling matrix of $K_{r,4} - e$ with magic value 0, for odd $r \geq 5$.

From the above discussion, we have the following theorems.

Theorem 3.9. *Let $r \geq 4$. Then $K_{r,4} - e$ has a $\mathbb{Z}_k$-magic labeling with magic value 0 for $k \geq 3$. Furthermore, $K_{r,4} - e$ is $\mathbb{Z}$-magic, and hence,* $\mathrm{IM}(K_{r,4} - e) = \mathbb{N} \setminus \{2\}$ *for $r \geq 4$.*

Theorem 3.10. $N(K_{r,4} - e) = \mathbb{N} \setminus \{2\}$ *for $r \geq 4$.*

From Eq. (3.1), if we choose $A = V_4$, then r must be even. Without loss of generality, we may assume $m = a$. Then, $x_i = y_i + a$ for $1 \leq i \leq r - 1$. This implies that $y_i \neq a$ for $1 \leq i \leq r-1$. So, $y_i \in \{b, c\}$. But $\sum_{i=1}^{r-1} y_i = 0$ and r is even, which is impossible. Hence $K_{r,2} - e$ is not V_4-magic for $r \geq 2$.

It is clear that

$$\begin{pmatrix} * & b & b \\ b & a & c \\ b & c & a \end{pmatrix}, \quad \begin{pmatrix} * & b & b \\ b & a & c \\ c & a & b \\ a & b & c \end{pmatrix}, \quad \begin{pmatrix} * & b & c & a \\ b & a & a & b \\ b & c & b & c \end{pmatrix}, \text{ and } \begin{pmatrix} * & b & c & a \\ b & a & a & b \\ c & a & c & a \\ a & b & a & b \end{pmatrix}$$

can be used (as the B and B^T partitions) to construct V_4-magic labeling matrices of $K_{3,3}-e$, $K_{4,3}-e$, $K_{3,4}-e$ and $K_{4,4}-e$ with magic value 0, respectively. By adjoining copies of

$$\begin{pmatrix} b & a & c \\ b & a & c \end{pmatrix}$$

to the bottom of the first two matrices and adjoining copies of

$$\begin{pmatrix} b & a & a & b \\ b & a & a & b \end{pmatrix}$$

to the bottom of the last two matrices, we obtain the following theorem.

Theorem 3.11. *There exist V_4-magic labelings* (*having magic value* 0) *for $K_{r,3} - e$ and $K_{r,4} - e$ where $r \geq 3$.*

4 Group-magic labelings of complete N-partite graphs with deleted edges

Definition. Let G be an n-partite graph with (vertex) partite sets $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$. An edge e of G joining a vertex in $\mathcal{P}_i$ to a vertex in $\mathcal{P}_j$ is said to *join partite pair* $\langle \mathcal{P}_i, \mathcal{P}_j \rangle$.

Theorem 4.1. *Let $min\{n_1, n_2, \ldots, n_l\} \geq 3$ and $G = K_{n_1,n_2,\ldots,n_l} - \{e_1, e_2, \ldots, e_s\}$ be a connected graph. If each of the e_j (in $K_{n_1,n_2,\ldots,n_l}$) joined different partite pairs, then G has a V_4-magic labeling with magic value* 0.

Proof. This follows from examining the cases involving all possible partite pairs of G and applying Theorems 3.1, 3.2, 3.3 and 3.11. $\square$

Theorem 4.2. *Let* $min\{n_1, n_2, \ldots, n_l\} = 3$ *and* $G = K_{n_1,n_2,\ldots,n_l} - \{e_1, e_2, \ldots, e_s\}$ *be a connected graph. If each of the* e_j *(in* $K_{n_1,n_2,\ldots,n_l}$*) joined different partite pairs, then* G *has a* $\mathbb{Z}_k$*-magic labeling with magic value* 0*, for* $k \in \mathbb{N}\backslash\{2, 3\}$.

Proof. This follows from examining the cases involving all possible partite pairs of G and applying Theorems 3.1, 3.2, 3.3, 3.7 and 3.9. $\square$

The following two corollaries are direct consequences of Theorem 4.2.

Corollary 4.3. *Let* $min\{n_1, n_2, \ldots, n_l\} = 3$ *and* $G = K_{n_1,n_2,\ldots,n_l} - \{e_1, e_2, \ldots, e_s\}$ *be a connected graph. If each of the* e_j *(in* $K_{n_1,n_2,\ldots,n_l}$*) joined different partite pairs (not involving partite sets of cardinality* 3*), then* G *has a* $\mathbb{Z}_k$*-magic labeling with magic value* 0*, for* $k \in \mathbb{N}\backslash\{2\}$.

Corollary 4.4. *Let* $min\{n_1, n_2, \ldots, n_l\} \geq 4$ *and* $G = K_{n_1,n_2,\ldots,n_l} - \{e_1, e_2, \ldots, e_s\}$ *be a connected graph. If each of the* e_j *(in* $K_{n_1,n_2,\ldots,n_l}$*) joined different partite pairs, then* G *has a* $\mathbb{Z}_k$*-magic labeling with magic value* 0*, for* $k \in \mathbb{N}\backslash\{2\}$.

5 Group-magic labelings of composition graphs and composition graphs with a deleted edge

Definition. Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, the *lexicographic product* (or *composition*) of G with H, denoted by $G[H]$ (or $G \circ H$), is the graph with vertex set $V_G \times V_H$ where (u_1, v_1) is adjacent with (u_2, v_2) if and only if $u_1u_2 \in E_G$, or $u_1 = u_2$ and $v_1v_2 \in E_H$.

Let N_n be the null graph of order n. Let J_n be the $n \times n$ matrix whose entries are 1. It is easy to see that if $M = (m_{u,v})$ is the adjacency matrix of $G = (V, E)$, where $u, v \in V$, then (under lexicographic order) the adjacency matrix of $G[N_n]$ is $M \otimes J_n$.

Example. Let G be the left graph of Figure 1.

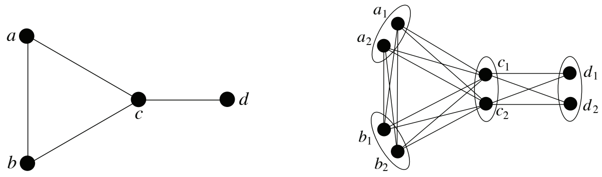

Figure 1: The graph G and the graph $G[N_2]$.

The adjacency matrix of G is

$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

and the adjacency matrix of $G[N_2]$ (the right graph of Figure 1) is

$$\begin{array}{c} \\ a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \\ d_1 \\ d_2 \end{array} \begin{array}{c} \begin{matrix} a_1 & a_2 & b_1 & b_2 & c_1 & c_2 & d_1 & d_2 \end{matrix} \\ \left(\begin{array}{cc|cc|cc|cc} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \end{array} = M \otimes J_2.$$

Let A be a non-trivial abelian group. One obtains a labeling matrix of $G[N_n]$ in the following way: Each sub-matrix block $\mathcal{B}_{uv}$ of 1s in the upper triangular part of $M \otimes J_n$ is replaced by an $n \times n$ matrix L_{uv} with entries in A^*. Since the labeling matrix of a graph is symmetric, the corresponding sub-matrix block $\mathcal{B}_{vu}$ of 1s is replaced by L_{uv}^T. Each sub-matrix block of 0s in $M \otimes J_n$ is replaced by an $n \times n$ matrix $\bigstar_n$.

Using the example above, suppose that f is a labeling of $G[N_2]$. Then, we have

$$\begin{pmatrix} \bigstar_2 & L_{ab} & L_{ac} & \bigstar_2 \\ L_{ab}^T & \bigstar_2 & L_{bc} & \bigstar_2 \\ L_{ac}^T & L_{bc}^T & \bigstar_2 & L_{cd} \\ \bigstar_2 & \bigstar_2 & L_{cd}^T & \bigstar_2 \end{pmatrix},$$

where

$$L_{ab} = \begin{pmatrix} f(a_1b_1) & f(a_1b_2) \\ f(a_2b_1) & f(a_2b_2) \end{pmatrix}, \; L_{ac} = \begin{pmatrix} f(a_1c_1) & f(a_1c_2) \\ f(a_2c_1) & f(a_2c_2) \end{pmatrix},$$
$$L_{bc} = \begin{pmatrix} f(b_1c_1) & f(b_1c_2) \\ f(b_2c_1) & f(b_2c_2) \end{pmatrix} \text{ and } L_{cd} = \begin{pmatrix} f(c_1d_1) & f(c_1d_2) \\ f(c_2d_1) & f(c_2d_2) \end{pmatrix}.$$

By examining the parity of the vertex degrees of G, it is easy to determine if 2 is an element of $N(G)$. Thus, we assume that $|A| \geq 3$. By Theorems 3.1 and 3.2, there exists an A-magic labeling matrix

$$\begin{pmatrix} \bigstar_n & L \\ L^T & \bigstar_n \end{pmatrix}$$

of $K_{r,r}$ with A-magic value 0, for $r \geq 2$. By Theorems 3.3, 3.7, 3.9 and 3.11, there exists an A-magic labeling matrix

$$\begin{pmatrix} \bigstar_n & B \\ B^T & \bigstar_n \end{pmatrix}$$

of $K_{r,r} - e$ with A-magic value 0, for $r \geq 4$. Here, if $r = 3$, then we assume that $|A| \geq 4$. The existence of these A-magic labeling matrices will be used to construct A-magic labeling matrices for $G[N_n]$ and $G[N_n] - e$.

Theorem 5.1. *Let G be a simple connected graph and let A be an abelian group with $|A| \geq 3$. For $n \geq 2$, $G[N_n]$ has an A-magic labeling with magic value 0. Moreover, for $n \geq 4$, $G[N_n] - e$ has an A-magic labeling with magic value 0; and $G[N_3] - e$ has an A-magic labeling with magic value 0, if $|A| \geq 4$.*

Proof. Replace each sub-matrix block of 1s in the upper triangular part of the adjacency matrix of $G[N_n]$ with L, each sub-matrix block of 1s in the lower triangular part with L^T, and each sub-matrix block of 0s with $\bigstar_n$. This yields an A-magic labeling matrix for $G[N_n]$, with magic-value 0.

Now, consider $G[N_n] - e$. By slightly modifying the A-magic labeling matrix (that we just obtained) for $G[N_n]$, we construct an A-magic labeling matrix for $G[N_n] - e$. Replace the L (and L^T) with B (and B^T), respectively, where B is determined by the particular deleted edge e. This yields an A-magic labeling matrix of $G[N_n] - e$, with magic value 0. □

Example. Let us again consider the graph G in Figure 1. Let A be the Klein-4 group $V_4 = \{0, a, b, c\}$ and $n = 3$. Then,

$$L = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \text{ and } B = \begin{pmatrix} * & b & b \\ b & a & c \\ b & c & a \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} \bigstar_3 & L & L & \bigstar_3 \\ L^T & \bigstar_3 & L & \bigstar_3 \\ L^T & L^T & \bigstar_3 & L \\ \bigstar_3 & \bigstar_3 & L^T & \bigstar_3 \end{pmatrix} \text{ and } \begin{pmatrix} \bigstar_3 & B & L & \bigstar_3 \\ B^T & \bigstar_3 & L & \bigstar_3 \\ L^T & L^T & \bigstar_3 & L \\ \bigstar_3 & \bigstar_3 & L^T & \bigstar_3 \end{pmatrix}$$

are V_4-magic labeling matrices of $G[N_3]$ and $G[N_3] - a_1b_1$, respectively.

Moreover,

$$\begin{pmatrix} \bigstar_3 & L & B & \bigstar_3 \\ L^T & \bigstar_3 & L & \bigstar_3 \\ B^T & L^T & \bigstar_3 & B \\ \bigstar_3 & \bigstar_3 & B^T & \bigstar_3 \end{pmatrix} \text{ and } \begin{pmatrix} \bigstar_3 & B & B & \bigstar_3 \\ B^T & \bigstar_3 & B & \bigstar_3 \\ B^T & B^T & \bigstar_3 & B \\ \bigstar_3 & \bigstar_3 & B^T & \bigstar_3 \end{pmatrix}$$

are V_4-magic labeling matrices of $G[N_3] - \{a_1c_1, c_1d_1\}$, and $G[N_3] - \{a_1b_1, a_1c_1, b_1c_1, c_1d_1\}$, respectively.

As the reader may have noted (as illustrated in the example above), an A-magic labeling matrix for certain $G[N_n]-\{e_1, e_2, \ldots, e_k\}$ can be obtained by carefully modifying the A-magic labeling matrix of $G[N_n]$.

Corollary 5.2. *Suppose $m \geq 3$. For $n \geq 2$, $N(C_m[N_n]) = \mathbb{N}$. Moreover, $\mathbb{N}\setminus\{2,3\} \subseteq N(C_m[N_3] - e)$. For $n \geq 4$, $N(C_m[N_n] - e) = \mathbb{N} \setminus \{2\}$.*

Proof. This follows immediately from Theorem 5.1. ☐

Since $K_{\underbrace{n,\ldots,n}_{m}} \cong K_m[N_n]$ is an $n(m-1)$-regular graph, we have the following result.

Corollary 5.3. *For $n \geq 2$, $N(K_{\underbrace{n,\ldots,n}_{m}}) = \mathbb{N}$, if $n(m-1)$ is even, and $N(K_{\underbrace{n,\ldots,n}_{m}}) = \mathbb{N} \setminus \{2\}$, if $n(m-1)$ is odd. Moreover, $N(K_{\underbrace{n,\ldots,n}_{m}} - e) = \mathbb{N} \setminus \{2\}$ if $n \geq 4$ and $N(K_{\underbrace{n,\ldots,n}_{m}} - e) \supseteq \mathbb{N} \setminus \{2,3\}$ if $n = 3$.*

Corollary 5.4. *If n is even, then $N(P_m[N_n]) = \mathbb{N}$. If n is odd and m is even, then $N(P_m[N_n]) = \mathbb{N} \setminus \{2\}$. Moreover, $N(P_m[N_n] - e) = \mathbb{N} \setminus \{2\}$ if $n \geq 4$ and $N(P_m[N_3] - e) \supseteq \mathbb{N} \setminus \{2,3\}$.*

Proof. This follows immediately from Theorem 5.1. ☐

Acknowledgments

The authors wish to thank the referees for their valuable comments and suggestions.

References

[1] M. Doob, On the construction of magic graphs, *Proc. Fifth S.E. Conf. Combin., Graph Theory and Computing* (1974), 361–374.

[2] M. Doob, Generalizations of magic graphs, *J. Combin. Theory Ser. B* **17** (1974), 205–217.

[3] M. Doob, Characterizations of regular magic graphs, *J. Combin. Theory Ser. B* **25** (1978), 94–104.

[4] M.C. Kong, S.-M. Lee and H. Sun, On magic strength of graphs, *Ars Combin.* **45** (1997), 193–200.

[5] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451–461.

[6] S.-M. Lee, Yong-Song Ho and R.M. Low, On the integer-magic spectra of maximal planar and maximal outerplanar graphs, *Congr. Numer.* **168** (2004), 83–90.

[7] S.-M. Lee, A. Lee, Hugo Sun and Ixin Wen, On group-magic graphs, *J. Combin. Math. Combin. Comput.* **38** (2001), 197–207.

[8] S.-M. Lee and F. Saba, On the integer-magic spectra of two-vertex sum of paths, *Congr. Numer.* **170** (2004), 3–15.

[9] S.-M. Lee, F. Saba, E. Salehi and H. Sun, On the V_4-group magic graphs, *Congr. Numer.* **156** (2002), 59–67.

[10] S.-M. Lee, F. Saba and G. C. Sun, Magic strength of the k-th power of paths, *Congr. Numer.* **92** (1993), 177–184.

[11] S.-M. Lee and E. Salehi, Integer-magic spectra of amalgamations of stars and cycles, *Ars Combin.* **67** (2003), 199–212.

[12] S.-M. Lee, E. Salehi and H. Sun, Integer-magic spectra of trees with diameters at most four, *J. Combin. Math. Combin. Comput.* **50** (2004), 3–15.

[13] S.-M. Lee, L. Valdes and Yong-Song Ho, On group-magic spectra of trees, double trees and abbreviated double trees, *J. Combin. Math. Combin. Comput.* **46** (2003), 85–95.

[14] R.M. Low and S.-M. Lee, On the integer-magic spectra of tessellation graphs, *Australas. J. Combin.* **34** (2006), 195–210.

[15] R.M. Low and S.-M. Lee, On the products of group-magic graphs, *Australas. J. Combin.* **34** (2006), 41–48.

[16] R.M. Low and S.-M. Lee, On group-magic eulerian graphs, i *J. Combin. Math. Combin. Comput.* **50** (2004), 141–148.

[17] R.M. Low and L. Sue, Some new results on the integer-magic spectra of tessellation graphs, *Australas. J. Combin.* **38** (2007), 255–266.

[18] E. Salehi, Zero-sum magic graphs and their null sets, *Ars Combin.* **82** (2007), 41–53.

[19] E. Salehi, On zero-sum magic graphs and their null sets, *Bull. Inst. Math., Academia Sinica* **3** (2008), 255–264.

[20] E. Salehi and P. Bennett, On integer-magic spectra of caterpillars, *J. Combin. Math. Combin. Comput.* **61** (2007), 65–71.

[21] J. Sedlácek, On magic graphs, *Math. Slov.* **26** (1976), 329–335.

[22] J. Sedlácek, Some properties of magic graphs, in Graphs, Hypergraph, and Bloc Syst. 1976, *Proc. Symp. Comb. Anal., Zielona Gora* (1976), 247–253.

[23] W.C. Shiu and R.M. Low, Group-magicness of complete N-partite graphs, *J. Combin. Math. Combin. Comput.* **58** (2006), 129–134.

[24] W.C. Shiu and R.M. Low, Integer-magic spectra of sun graphs, *J. Comb. Optim.* **14** (2007), 309–321.

[25] W.C. Shiu and R.M. Low, $\mathbb{Z}_k$-magic labelings of fans and wheels with magic-value zero, *Australas. J. Combin.* **45** (2009), 309–316.

[26] R.P. Stanley, Linear homogeneous diophantine equations and magic labelings of graphs, *Duke Math. J.* **40** (1973), 607–632.

[27] R.P. Stanley, Magic labeling of graphs, symmetric magic squares, systems of parameters and Cohen-Macaulay rings, *Duke Math. J.* **40** (1976), 511–531.

[28] W.D. Wallis, *Magic Graphs*, Birkhauser Boston, (2001).

(Received 9 June 2011; revised 23 Apr 2013, 16 May 2013)