

Construction of new larger (a, d) -edge antimagic vertex graphs by using adjacency matrices

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Abstract

Let $G = G(V, E)$ be a finite simple undirected graph with vertex set V and edge set E , where $|E|$ and $|V|$ are the number of edges and vertices on G . An (a, d) -edge antimagic vertex $((a, d)$ -EAV) labeling is a one-to-one mapping f from $V(G)$ onto $\{1, 2, \dots, |V|\}$ with the property that for every edge $xy \in E$, the edge-weight set is equal to $\{f(x) + f(y) : x, y \in V\} = \{a, a + d, a + 2d, \dots, a + (|E| - 1)d\}$, for some integers $a > 0$, $d \geq 0$. An (a, d) -edge antimagic total $((a, d)$ -EAT) labeling is a one-to-one mapping f from $V \cup E$ onto $\{1, 2, \dots, |V| + |E|\}$ with the property that for every edge $xy \in E$, the edge-weight set is equal to $\{f(x) + f(y) +$

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$f(xy) : x, y \in V, xy \in E\} = \{a, a + d, a + 2d, \dots, a + (|E| - 1)d\}$, where $a > 0, d \geq 0$ are two fixed integers. Such a labeling is called a super (a, d) -edge antimagic total $((a, d)$ -SEAT) labeling if $f(V) = \{1, 2, \dots, |V|\}$. A graph that has an (a, d) -EAV $((a, d)$ -EAT or (a, d) -SEAT) labeling is called an (a, d) -EAV $((a, d)$ -EAT or (a, d) -SEAT) graph. For an (a, d) -EAV (or (a, d) -SEAT) graph G , an adjacency matrix of G is a $|V| \times |V|$ matrix $A_G = [a_{ij}]$ such that the entry a_{ij} is 1 if there is an edge from vertex with index i to vertex with index j , and entry a_{ij} is 0 otherwise. This paper shows the construction of new larger (a, d) -EAV graph from an existing (a, d) -EAV graph using the adjacency matrix, for $d = 1, 2$. The results will be extended for (a, d) -SEAT graphs with $d = 0, 1, 2, 3$.

1 Introduction

In this paper, we consider finite simple undirected graphs. The set of vertices and edges of a graph G is denoted by V and E , respectively. Let $|V| = n$ and $|E| = m$.

Simanjuntak, Miller and Bertault [9] defined an (a, d) -edge-antimagic vertex $((a, d)$ -EAV) labeling for a graph $G(V, E)$ as an injective mapping f from V onto the set $\{1, 2, \dots, n\}$ with the property that the edge-weights $\{w(xy) : w(xy) = f(x) + f(y), xy \in E\}$, form an arithmetic sequence with the first term a and difference d , where $a > 0$ and $d \geq 0$ are two fixed integers.

Acharya and Hegde [1] (see also [6]) introduced the concept of a *strongly* (a, d) -indexable labeling which is equivalent to (a, d) -EAV labeling. The relationship between the sequential graphs and the graphs having an (a, d) -EAV labeling is shown in [3].

An (a, d) -edge antimagic total $((a, d)$ -EAT) labeling is a bijection f from $V \cup E$ onto $\{1, 2, \dots, n + m\}$ with the property that the sums of the label on the edges and the labels of their end points form an arithmetic sequence starting from a and having a common difference d . This labeling is a natural extension of the notion of *edge-magic labeling* which was originally introduced by Kotzig and Rosa in [7], where edge-magic labeling is called *magic valuation*. Relationships between (a, d) -EAT labeling and other labelings, namely, (a, d) -EAV labeling are presented in [2].

An (a, d) -EAT labeling is called *super* (a, d) -edge antimagic total $((a, d)$ -SEAT) labeling if $f(V) = \{1, 2, \dots, n\}$. This labeling is a natural extension of the notion of a *super edge-magic labeling* defined by Enomoto *et al.* in [5]. A graph that has an (a, d) -EAV $((a, d)$ -EAT or (a, d) -SEAT) labeling is called an (a, d) -EAV $((a, d)$ -EAT or (a, d) -SEAT) graph.

An adjacency matrix of G is a symmetric matrix $A_G = [a_{ij}]$ of order n such that the entry a_{ij} is 1 if there is an edge from the vertex with index i to the vertex with index j , and the entry a_{ij} is 0 otherwise.

There are many results on graph labeling, including on edge antimagic vertex labeling. Sugeng and Miller in [10] have explained the relationship between (a, d) -EAV

labeling and adjacency matrix and shown how to manipulate this matrix to construct new (a, d) -EAV graphs, for $d = 1$. In this paper, we give a construction of new larger (a, d) -EAV graph from an existing (a, d) -EAV graph by using adjacency matrix, for $d = 1$ and $d = 2$. The results will be extended for (a, d) -SEAT graphs.

2 Some Properties

2.1 Adjacency Matrix

Let $G = G(V, E)$ be a graph with an (a, d) -EAV labeling f . Label the vertices in G such that $f(v_i) = i$, for $i = 1, 2, \dots, n$. An $n \times n$ matrix $A_G = [a_{ij}]$, $i, j = 1, 2, \dots, n$, is called an *adjacency matrix* of G if

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let G be an (a, d) -EAV graph with adjacency matrix A_G . Since G is an undirected graph, A_G is a symmetric matrix. Beside that, A_G has another characteristic that shows that A_G is a matrix of an (a, d) -EAV graph. A skew diagonal S_r , $r = 3, 4, \dots, 2n - 1$, of A_G is $\{a_{ij} : i + j = r; i, j = 1, 2, \dots, n\}$ (see Figure 1).

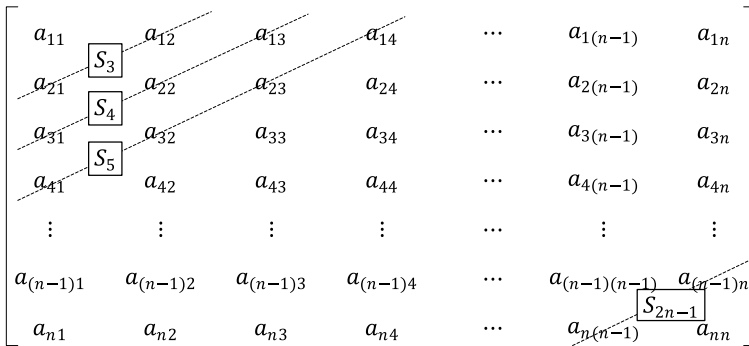


Figure 1: Skew diagonal S_r in a matrix A_G .

A skew diagonal S_r contains all entries of A_G that are related to edges with weight r . With respect to the symmetry of A_G , every skew diagonal of A_G has either zero or exactly two “1” elements. A skew diagonal that only contains the zero elements is called zero skew diagonal, while a skew diagonal that contains exactly two “1” elements is called non-zero skew diagonal.

Sugeng and Miller [10] explained that the set of edge-weights $\{f(x) + f(y) : x, y \in V\}$ in skew diagonal lines generate a sequence of integers of difference d . If $d = 1$ then the nonzero skew diagonal lines form a band of consecutive integers. If $d = 2$ then the non-zero skew diagonal lines form a band of difference 2 with a zero skew diagonal

line in between. We have similar skew diagonal line bands for $d = 3, 4, \dots$ and denote such a skew diagonal band as d -band.

2.2 Maximal (a, d) -EAV Graph

A maximal (a, d) -EAV graph of order n is a graph that has an (a, d) -EAV labeling and has the maximum possible number of edges. If G is a maximal $(a, 1)$ -EAV graph then $a = 3$. From the adjacency matrix of a maximal $(3, 1)$ -EAV graph, we can see that the first “1” elements will be in the position of $(1, 2)$ and $(2, 1)$.

Observation 1. [10] *The number of edges of a maximal (a, d) -EAV graph of order n is $\lceil \frac{n-1}{d} \rceil + \lceil \frac{n-2}{d} \rceil$.*

Consequently, a maximal (a, d) -EAV graph of order n cannot be connected for $d > 2$ since the maximum number of edges is less than the maximum number of edges for $d = 2$, i.e., $\lceil \frac{n-1}{2} \rceil + \lceil \frac{n-2}{2} \rceil = n - 1$.

We can construct adjacency matrices of maximal $(3, d)$ -EAV graphs for $d = 1, 2$ by putting “1” elements at the ends of each non zero skew diagonal. A *triangular book* $B_{n-2}(C_3)$ is the complete tripartite graph $K_{1,1,n-2}$. It is a graph consisting of $n - 2$ triangles all sharing a common edge. A double star obtained from two vertex disjoint copies of the star $S_{\frac{n}{2}}$ by connecting their centers we call the *twin star graph*, $Twin(n)$. Both $B_{n-2}(C_3)$ and $Twin(n)$ are maximal $(3, 1)$ -EAV graphs and maximal $(3, 2)$ -EAV graphs of order n , respectively. Figure 2(a) depicts the triangular book graph $B_6(C_3)$ of order 8 with $(3, 1)$ -EAV labeling and its adjacency matrix. Figure 2(b) shows the twin star graph $Twin(8)$ with $(3, 1)$ -EAV labeling and its adjacency matrix.

3 Constructing New Larger (a, d) -EAV Graph Using Adjacency Matrix

We can construct new (a^*, d) -EAV graphs from an existing (a, d) -EAV graph by using adjacency matrices manipulation. Here we only consider how adjacency matrix manipulation can be used to construct a new larger maximal $(3, d)$ -EAV graph. Given an (a, d) -EAV graph G , there are several ways to obtain a larger (a, d) -EAV graph, such as adding some vertices and edges, combining two (or more) given (a, d) -EAV graphs, and combining two (or more) given (a, d) -EAV graphs and adding some vertices and edges.

3.1 Constructing New Larger Maximal $(3, 1)$ -EAV Graph Using Adjacency Matrix

We will construct a new larger maximal $(3, 1)$ -EAV graph by adding some vertices and some edges to an existing maximal $(3, 1)$ -EAV graph using adjacency matrix. It

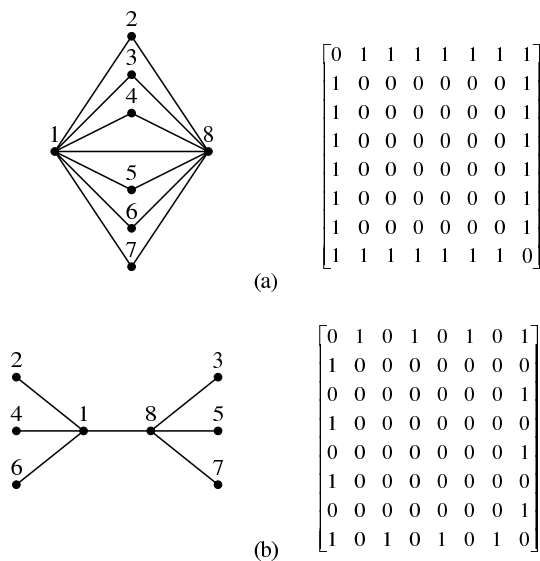


Figure 2: Graphs $B_6(C_3)$ and $Twin(8)$ with $(3, 1)$ -EAV labelings and corresponding adjacency matrices.

can be done by adding some columns and rows in the adjacency matrix and make it deal with the properties of maximal $(3,1)$ -EAV graph. Let us note that the *transpose* A' of a matrix A is the matrix obtained from A by writing its rows as columns.

Theorem 1. *Let G be a maximal $(3, 1)$ -EAV graph of order n , $n \geq 2$, with adjacency matrix A_G . Let $t = [t_{i1}]$ be $n \times 1$ matrix with*

$$t_{i1} = \begin{cases} 1, & \text{for } i = 1, 2 \\ 0, & \text{for } i = 3, 4, \dots, n. \end{cases}$$

Then the matrix

$$M = \begin{bmatrix} 0 & t' \\ t & A_G \end{bmatrix}$$

is the adjacency matrix of maximal $(3, 1)$ -EAV graph of order $n + 1$.

Proof. Matrix M contains A_G as its diagonal block matrix starting in position $(2, 2)$. Therefore, each vertex v_i in G with label i is now labeled with $i + 1$ and it makes the S_r , $r = 5, 6, \dots, 2n + 1$, of M non-zero skew diagonals. The matrices t' and t in M fill the S_3 and S_4 and make them non-zero skew diagonals. Now M is an $(n + 1) \times (n + 1)$ symmetric matrix with nonzero skew diagonal lines induce a band of consecutive integers started with S_3 until S_{2n+1} . ■

Since the matrix M in Theorem 1 is an adjacency matrix of a $(3, 1)$ -EAV graph, it

can be considered as A_G . Thus repeating the construction from Theorem 1 leads to the following corollary.

Corollary 1. *Let G be a maximal $(3, 1)$ -EAV graph of order n , $n \geq 2$, with adjacency matrix A_G . Let $t_k = [t_{i1}]$, $k = 1, 2, \dots$, be a $(n + k - 1) \times 1$ matrix with*

$$t_{i1} = \begin{cases} 1, & \text{for } i = 1, 2 \\ 0, & \text{for } i = 3, 4, \dots, n + k - 1 \end{cases}$$

and let M_1 be a $(n + 1) \times (n + 1)$ matrix with

$$M_1 = \begin{bmatrix} 0 & t'_1 \\ t_1 & A_G \end{bmatrix}.$$

Then the matrix

$$M_k = \begin{bmatrix} 0 & t'_k \\ t_k & M_{k-1} \end{bmatrix}, \quad k = 2, 3, \dots$$

is the adjacency matrix of a maximal $(3, 1)$ -EAV graph of order $n + k$.

Theorem 1 and Corollary 1 show a construction of new larger maximal $(3, 1)$ -EAV graphs by adding several columns and rows on the left and top side of the adjacency matrix of an existing maximal $(3, 1)$ -EAV graph. We also can add several columns and rows on the right and bottom sides of an adjacency matrix.

Theorem 2. *Let G be a maximal $(3, 1)$ -EAV graph of order n , $n \geq 2$, with adjacency matrix A_G . Let $t = [t_{i1}]$ and $t^* = [t^*_{i1}]$ be $n \times 1$ matrices with*

$$t_{i1} = \begin{cases} 1, & \text{for } i = 1, 2 \\ 0, & \text{for } i = 3, 4, \dots, n \end{cases}$$

$$t^*_{i1} = \begin{cases} 0, & \text{for } i = 1, 2, \dots, n - 2 \\ 1, & \text{for } i = n - 1, n. \end{cases}$$

Then the matrix

$$M = \begin{bmatrix} 0 & t' & 0 \\ t & A_G & t^* \\ 0 & (t^*)' & 0 \end{bmatrix}$$

is the adjacency matrix of a maximal $(3, 1)$ EAV graph of order $n + 2$.

Proof. Matrix M contains A_G as its diagonal block matrix starting in position $(2, 2)$. Therefore, each vertex v_i in G with label i is labeled with $i + 1$ and it makes the S_r , $r = 5, 6, \dots, 2n + 1$, of M non-zero skew diagonals. The matrices t' and t in M fill the S_3 and S_4 and make them non-zero skew diagonals and the matrices $(t^*)'$ and t^* in M fill the S_{2n+2} and S_{2n+3} and make them non-zero skew diagonals. Now M is an $(n + 2) \times (n + 2)$ symmetric matrix with nonzero skew diagonal lines induce a band of consecutive integers starting with S_3 until S_{2n+3} . ■

Theorem 2 can be done repeatedly which leads to the following corollary.

Corollary 2. Let G be a maximal $(3, 1)$ -EAV graph of order n , $n \geq 2$, with adjacency matrix A_G . Let $t_k = [t_{i1}]$ and $t_k^* = [t_{i1}^*]$, $k = 1, 2, \dots$, be $(n + 2k - 2) \times 1$ matrices with

$$t_{i1} = \begin{cases} 1, & \text{for } i = 1, 2 \\ 0, & \text{for } i = 3, 4, \dots, n + 2k - 2 \end{cases}$$

$$t_{i1}^* = \begin{cases} 0, & \text{for } i = 1, 2, \dots, n + 2k - 4 \\ 1, & \text{for } i = n + 2k - 3, n + 2k - 2 \end{cases}$$

and let M_1 be a $(n + 2) \times (n + 2)$ matrix with

$$M_1 = \begin{bmatrix} 0 & t'_1 & 0 \\ t_1 & A_G & t_1^* \\ 0 & (t_1^*)' & 0 \end{bmatrix}.$$

Then the matrix

$$M_k = \begin{bmatrix} 0 & t'_k & 0 \\ t_k & M_{k-1} & t_k^* \\ 0 & (t_k^*)' & 0 \end{bmatrix}, \quad k = 2, 3, \dots$$

is the adjacency matrix of a maximal $(3, 1)$ -EAV graph of order $n + 2k$.

Let us start with the triangular book graph $B_2(C_3)$. According to Corollary 2, the matrix M_k , k even, produces a new maximal $(3, 1)$ -EAV graph of order $4 + 2k$. This graph is a *triangular ladder* \mathbb{L}_{2+k} which can be obtained from Cartesian product of two paths P_{2+k} and P_2 with $V(P_{2+k} \times P_2) = \{u_i, v_i : 1 \leq i \leq 2 + k\}$ and $E(P_{2+k} \times P_2) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq 1 + k\} \cup \{u_i v_i : 1 \leq i \leq 2 + k\}$ by completing the edges $u_{2i-1} v_{2i}$, for $1 \leq i \leq \frac{k}{2} + 1$, and $v_{2i} u_{2i+1}$, for $1 \leq i \leq \frac{k}{2}$, (see Figure 3).

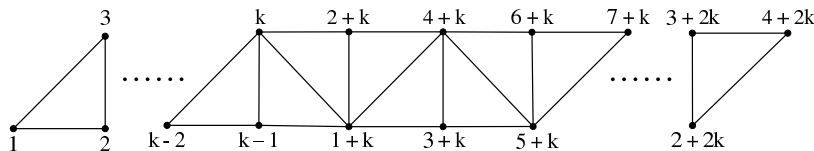


Figure 3: Constructing new larger $(3, 1)$ -EAV graphs by using Theorem 2.

Corollary 3. Every triangular ladder \mathbb{L}_{2+k} , $k \geq 2$ even, is a maximal $(3, 1)$ -EAV graph.

Graph $G(H, \mathbb{L}_{2+k})$ is called a *triangular ladder towered graph* if it is obtained from a graph H and the disjoint union of two copies of the triangular ladder \mathbb{L}_{2+k} in such a way that only two different edges in G are mutual with the edges $u_{2+k} v_{2+k}$ in each copy of \mathbb{L}_{2+k} .

Let us start with $B_6(C_3)$, see Figure 4(a). A triangular ladder towered graph $G(B_6(C_3), \mathbb{L}_4)$, see Figure 4(b), is a maximal $(3, 1)$ -EAV graph. The form of the

triangular ladder towered graph $G(B_{n-2}(C_3), \mathbb{L}_{2+k})$, $n \geq 4$ and $k \geq 2$ even, is shown in Figure 4(c). For any maximal $(3, 1)$ -EAV graph H , the general form of the triangular ladder towered graph $G(H, \mathbb{L}_{2+k})$ is shown in Figure 5.

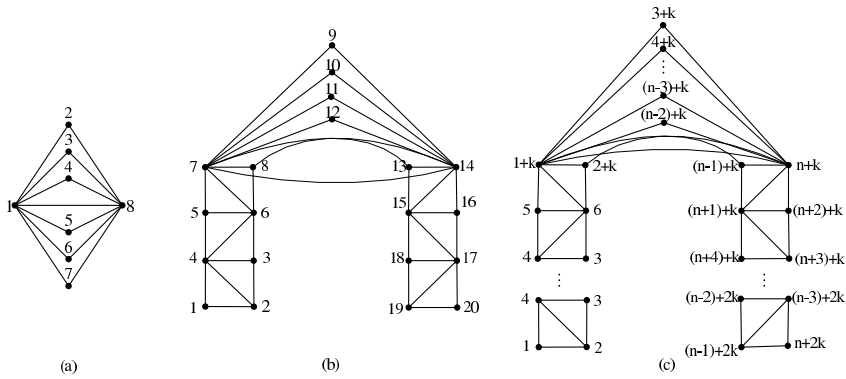


Figure 4: Triangular ladder towered graphs.

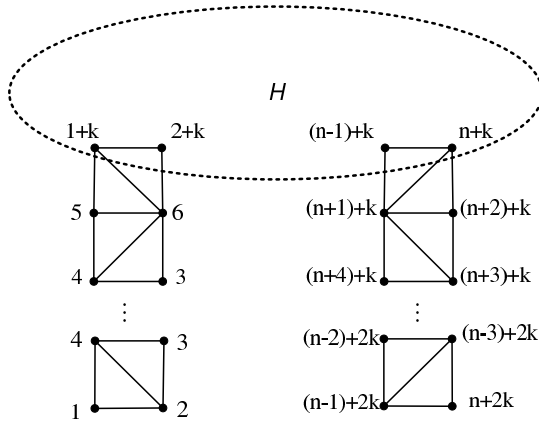


Figure 5: General form of triangular ladder towered graph $G(H, \mathbb{L}_{2+k})$.

Corollary 4. *Let H be any maximal $(3, 1)$ -EAV graph. Then the triangular ladder towered graph $G(H, \mathbb{L}_{2+k})$, $k \geq 2$ even, is also a maximal $(3, 1)$ -EAV graph.*

3.2 Constructing New Larger Maximal $(3,2)$ -EAV Graph Using Adjacency Matrix

Similarly to the construction of a new larger $(3, 1)$ -EAV graph, a new larger $(3, 2)$ -EAV graph will be constructed by adding some vertices and some edges to an $(a, 2)$ -

EAV graph by using adjacency matrix. It can be done by adding some columns and rows in the adjacency matrix and make it deal with the properties of (3,2)-EAV graph. Some of the results presented in this subsection are already discussed in [8].

Theorem 3. *Let G be a maximal (3, 2)-EAV graph of order n , $n \geq 1$, with adjacency matrix A_G . Let $s = [s_{i1}]$ and $s^* = [s_{i1}^*]$ be $n \times 1$ matrices with*

$$s_{i1} = \begin{cases} 1, & \text{for } i = 1 \\ 0, & \text{for } i = 2, 3, \dots, n \end{cases}$$

$$s_{i1}^* = \begin{cases} 0, & \text{for } i = 1, 2, \dots, n - 1 \\ 1, & \text{for } i = n. \end{cases}$$

Then the matrix

$$M = \begin{bmatrix} 0 & s' & 0 \\ s & A_G & s^* \\ 0 & (s^*)' & 0 \end{bmatrix}$$

is the adjacency matrix of a maximal (3, 2)-EAV graph of order $n + 2$.

Proof. Matrix M contains A_G as its diagonal block matrix starting in position (2, 2). Therefore, each vertex v_i in G with label i is labeled with $i + 1$ and it makes the S_r , $r = 5, 7, \dots, 2n + 1$, of M non-zero skew diagonals. The matrices s' and s in M fill the S_3 and make it non-zero skew diagonal and the matrices $(s^*)'$ and s^* in M fill the S_{2n+3} and make it non-zero skew diagonal. Now M is an $(n + 2) \times (n + 2)$ symmetric matrix with nonzero skew diagonal lines induce a band of the arithmetic sequence of difference 2 starting with S_3 until S_{2n+3} . ■

Since the matrix M in Theorem 3 is an adjacency matrix of a (3, 2)-EAV graph, it can be considered as A_G . Thus repeating the construction from Theorem 3 leads to the following corollary.

Corollary 5. *Let G be a maximal (3, 2)-EAV graph of order n , $n \geq 1$, with adjacency matrix A_G . Let $s_k = [s_{i1}]$ and $s_k^* = [s_{i1}^*]$, $k = 1, 2, \dots$, be $(n + 2k - 2) \times 1$ matrices with*

$$s_{i1} = \begin{cases} 1, & \text{for } i = 1 \\ 0, & \text{for } i = 2, 3, \dots, n + 2k - 2 \end{cases}$$

$$s_{i1}^* = \begin{cases} 0, & \text{for } i = 1, 2, \dots, n + 2k - 3 \\ 1, & \text{for } i = n + 2k - 2 \end{cases}$$

and let M_1 be a $(n + 2) \times (n + 2)$ matrix with

$$M_1 = \begin{bmatrix} 0 & s'_1 & 0 \\ s_1 & A_G & s_1^* \\ 0 & (s_1^*)' & 0 \end{bmatrix}.$$

Then the matrix

$$M_k = \begin{bmatrix} 0 & s'_k & 0 \\ s_k & M_{k-1} & s^*_k \\ 0 & (s^*_k)' & 0 \end{bmatrix}, \quad k = 2, 3, \dots$$

is the adjacency matrix of a maximal $(3, 2)$ -EAV graph of order $n + 2k$.

Graph $G(H, P_k)$ is called *path towered graph* if it is obtained from a graph H of order n and the disjoint union of two copies of the path P_k in such a way that an end vertex of each path P_k is adjoined to a vertex of the graph H . Thus $G(H, P_k)$ is graph of order $n + 2k - 2$.

Let us start with the twin star graph $Twin(8)$, see Figure 6(a). Then, forming the matrix M_1 by using Corollary 5 produces the new $(3, 2)$ -EAV graph $G(Twin(8), P_2)$ of order 10, see Figure 6(b). Forming the matrix M_k produces the new maximal $(3, 2)$ -EAV graph $G(Twin(8), P_{k+1})$ of order $8 + 2k$, see Figure 6(c).

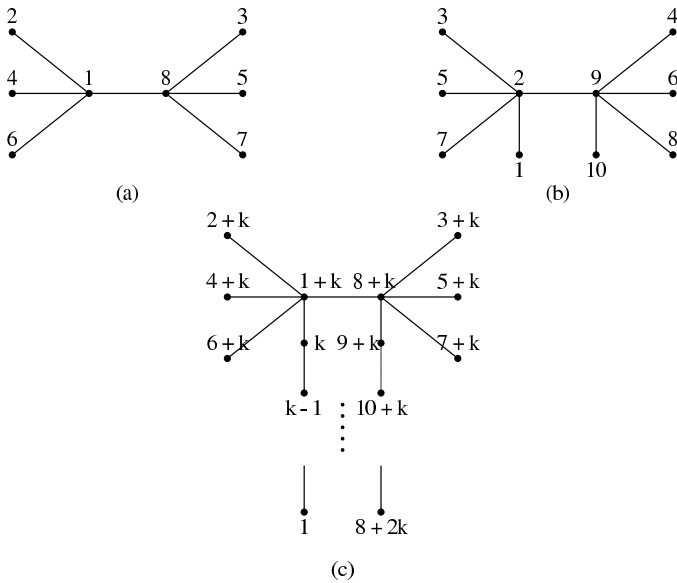


Figure 6: Constructing new larger $(3, 2)$ -EAV graph by using Theorem 3.

As an immediate consequence of Theorem 3 and Corollary 5 we can state the following corollary.

Corollary 6. *Let H be any maximal $(3, 2)$ -EAV graph. Then the path towered graph $G(H, P_k)$, $k \geq 2$, is also a maximal $(3, 2)$ -EAV graph.*

3.3 Other Constructions

Sugeng and Miller [10] have proved the following theorem.

Theorem 4. [10] *Let $G_i, i = 1, 2, \dots, p$, be an $(a, 1)$ -EAV graph of order n_i . Then there are $(a, 1)$ -EAV graphs of order w , where $\sum_{i=1}^p n_i - 2(p - 1) \leq w \leq \sum_{i=1}^p n_i$, and each contains G_i as induced subgraph.*

The proof of Theorem 4 uses a construction of a new adjacency matrix where its main diagonal contains adjacency matrices of graphs $G_i, i = 1, 2, \dots, p$, to obtain a new adjacency matrix of a maximal $(a, 1)$ -EAV graph.

Let $B_{n_i-2}(C_3)$ be the triangular book of order n_i with adjacency matrix $A_i, i = 1, 2, \dots, p$. Then combining the graphs using manipulation of adjacency matrix as the main diagonal block matrices produces a new class of maximal $(3, 1)$ -EAV graphs with order $\sum_{i=1}^p n_i - 2(p - 1)$, see Figure 7, or with order $\sum_{i=1}^p n_i - (p - 1)$, see Figure 8. In the first case we obtain a *ladder of triangular books* $LB(n_1 - 2, n_2 - 2, \dots, n_p - 2)$ and in the second case we obtain a *chain of triangular books* $CB(n_1 - 2, n_2 - 2, \dots, n_p - 2)$.

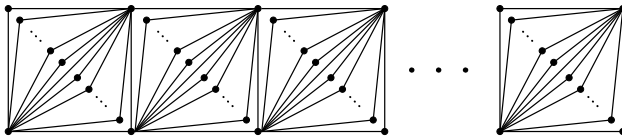


Figure 7: Ladder of triangular books.

Using the same construction as in Theorem 4 when the main diagonal of an adjacency matrix contains adjacency matrices of $(b, 2)$ -EAV graphs we are able to obtain a $(b, 2)$ -EAV graph. Thus we have

Theorem 5. *Let $G_i, i = 1, 2, \dots, p$, be $(b, 2)$ -EAV graphs of order n_i , respectively. Then there are $(b, 2)$ -EAV graphs of order w , where $\sum_{i=1}^p n_i - 2(p - 1) \leq w \leq \sum_{i=1}^p n_i$, and each contains G_i as induced subgraph.*

Another way to construct a new larger graph that has the same labeling as a given graph was introduced by Cavalier [4]. Using a similar idea, we have the following theorem.

Theorem 6. *Let G be a maximal $(3, 2)$ -EAV graph of order n with adjacency matrix A_G . Let $s = [s_{i1}]$ and $s^* = [s_{i1}^*]$ be $n \times 1$ matrices with*

$$s_{i1} = \begin{cases} 1, & \text{for } i = 1 \\ 0, & \text{for } i = 2, 3, \dots, n \end{cases}$$

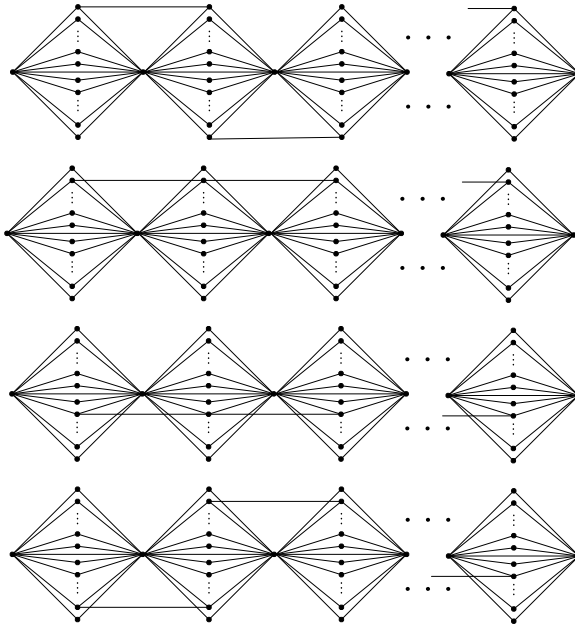


Figure 8: Chain of triangular books.

$$s_{i1}^* = \begin{cases} 0, & \text{for } i = 1, 2, \dots, n - 1 \\ 1, & \text{for } i = n, \end{cases}$$

and let $\mathbf{0}$ be the $n \times 1$ matrix of all zeros and \mathbb{O} be the $n \times n$ matrix of all zeros. Then a $(2pn + 2) \times (2pn + 2)$ matrix M constructed from $2p$ copies of A_G 's

$$M = \begin{bmatrix} 0 & s' & \mathbf{0}' & s' & \dots & \mathbf{0}' & 1 \\ s & A_G & \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & \mathbf{0} \\ \mathbf{0} & \mathbb{O} & A_G & \mathbb{O} & \dots & \mathbb{O} & s^* \\ s & \mathbb{O} & \mathbb{O} & A_G & \dots & \mathbb{O} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \mathbf{0} \\ \mathbf{0} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \dots & A_G & s^* \\ 1 & \mathbf{0}' & (s^*)' & \mathbf{0}' & \dots & (s^*)' & 0 \end{bmatrix}$$

is the adjacency matrix of a maximal $(3, 2)$ -EAV graph of order $2pn + 2$.

Proof. According to Theorem 5 if we add the element 1 as the last element in the first row and as the first element in the last row, and also the matrices s and s^* then the resulting matrix M with A_G as its main diagonal block matrices forms a new adjacency matrix for $(3, 2)$ -EAV graph. ■

Theorem 6 can be done repeatedly and it leads to the following corollary.

Corollary 7. *Let G be a maximal $(3, 2)$ -EAV graph of order n with adjacency matrix $A_G = M_0$ and let $q_k, k = 0, 1, 2, \dots$ be the order of the matrix M_k . Let $s_k = [s_{i1}]$ and $s_k^* = [s_{i1}^*]$ be $q_k \times 1$ matrices with*

$$s_{i1} = \begin{cases} 1, & \text{for } i = 1 \\ 0, & \text{for } i = 2, 3, \dots, q_k \end{cases}$$

$$s_{i1}^* = \begin{cases} 0, & \text{for } i = 1, 2, \dots, q_k - 1 \\ 1, & \text{for } i = q_k, \end{cases}$$

and $\mathbf{0}$ be the $q_k \times 1$ matrix of all zeros and \mathbb{O} be the $q_k \times q_k$ matrix of all zeros and let M_1 be a matrix of order $q_1 = 2pn + 2$ constructed from $2p$ copies of M_0 ,

$$M_1 = \begin{bmatrix} 0 & s' & \mathbf{0}' & s' & \dots & \mathbf{0}' & 1 \\ s & M_0 & \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & \mathbf{0} \\ \mathbf{0} & \mathbb{O} & M_0 & \mathbb{O} & \dots & \mathbb{O} & s^* \\ s & \mathbb{O} & \mathbb{O} & M_0 & \dots & \mathbb{O} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \mathbf{0} \\ \mathbf{0} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \dots & M_0 & s^* \\ 1 & \mathbf{0}' & (s^*)' & \mathbf{0}' & \dots & (s^*)' & 0 \end{bmatrix}.$$

Then the matrix M_k constructed from $2p$ copies of M_{k-1}

$$M_k = \begin{bmatrix} 0 & s' & \mathbf{0}' & s' & \dots & \mathbf{0}' & 1 \\ s & M_{k-1} & \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & \mathbf{0} \\ \mathbf{0} & \mathbb{O} & M_{k-1} & \mathbb{O} & \dots & \mathbb{O} & s^* \\ s & \mathbb{O} & \mathbb{O} & M_{k-1} & \dots & \mathbb{O} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \mathbf{0} \\ \mathbf{0} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \dots & M_{k-1} & s^* \\ 1 & \mathbf{0}' & (s^*)' & \mathbf{0}' & \dots & (s^*)' & 0 \end{bmatrix}$$

is the adjacency matrix of a maximal $(3, 2)$ -EAV graph of order $q_k = 2pq_{k-1} + 2$.

A graph containing only one vertex is a trivial $(3, 2)$ -EAV graph. We can combine a finite even number of copies of that trivial graph and construct a new $(3, 2)$ -EAV graph with adjacency matrix M_1 by using Corollary 7. For example, we combine 6 graphs of one vertex and produce a new $(3, 2)$ -EAV graph of order 8, see Figure 9(a). Then we construct the adjacency matrix M_2 by combining 6 M_1 and produce the new larger $(3, 2)$ -EAV graph of order 50, see Figure 9(b). Constructing M_3 by combining 6 M_2 will produce a new larger $(3, 2)$ -EAV graph, see Figure 10.

4 Results for (a, d) -Super Edge Antimagic Total Graphs

In [2] there is proved the following theorem.

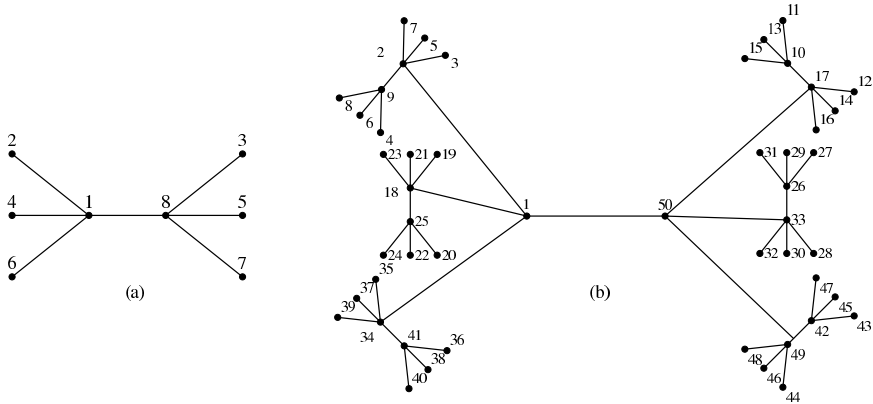


Figure 9: Constructing new larger $(3, 2)$ -EAV graph by using Corollary 7.

Theorem 7. [2] *If G of order n and size m has an (a, d) -EAV labeling, then*

- (i) G has an $(a + n + 1, d + 1)$ -SEAT labeling, and
- (ii) G has an $(a + n + m, d - 1)$ -SEAT labeling.

According to Theorem 7 from the previous results it follows that

Corollary 8. *Triangular book graph $B_{n-2}(C_3)$ and triangular ladder \mathbb{L}_n , both of order n and size $2n - 3$, admit an $(n + 4, 2)$ -SEAT labeling and a $(3n, 0)$ -SEAT labeling.*

Corollary 9. *The triangular ladder towered graph $G(B_{n-2}(C_3), \mathbb{L}_{2+k})$ of order $n+2k$ and size $2n + 8k + 5$, $k \geq 2$ even, admits an $(n + 2k + 4, 2)$ -SEAT labeling and a $(3n + 10k + 8, 0)$ -SEAT labeling.*

Corollary 10. *The ladder of triangular books $LB(n_1 - 2, n_2 - 2, \dots, n_p - 2)$ of order $\sum_{i=1}^p n_i - 2(p - 1)$ admits a $(\sum_{i=1}^p n_i - 2(p - 1) + 4, 2)$ -SEAT labeling and a $(3 \sum_{i=1}^p n_i - 2(p - 1), 0)$ -SEMT labeling.*

Corollary 11. *The chain of triangular books $CB(n_1 - 2, n_2 - 2, \dots, n_p - 2)$ of order $\sum_{i=1}^p n_i - (p - 1)$ admits a $(\sum_{i=1}^p n_i - (p - 1) + 4, 2)$ -SEAT labeling and a $(3 \sum_{i=1}^p n_i - (p - 1), 0)$ -SEMT labeling.*

Corollary 12. *The twin star graph $Twin(n)$ of order n and size $n - 1$ admits a $(2n + 2, 1)$ -SEAT labeling and an $(n + 4, 3)$ -SEAT labeling.*

Corollary 13. *The path towered graph $G(H, P_k)$, $k \geq 2$, of order $n + 2k - 2$ and size $n + 2k - 3$ admits a $(2n + 4k - 2, 1)$ -SEAT labeling and an $(n + 2k + 2, 3)$ -SEAT labeling.*

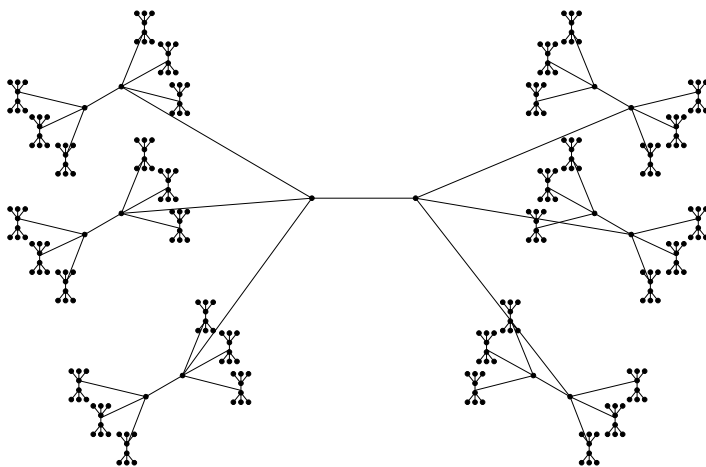


Figure 10: Graph given by adjacency matrix M_3 .

5 Conclusion

In this paper we showed how to construct a new larger (a, d) -EAV graph from a given graph with an (a, d) -EAV labeling, $d = 1, 2$, by using adjacency matrices. We also extended the results to the SEAT labelings.

Acknowledgements

The research for this article was partly supported by Indonesian Higher Education Department Competence Based Grant No. 3686/H2.R12/HKP.05.00/2012 and Slovak VEGA Grant 1/0130/12.

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(Received 12 Dec 2012)