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Covering cubic graphs with matchings of large size

S. Bonvicini

Dipartimento di Scienze e Metodi dell'Ingegneria
Università di Modena e Reggio Emilia
via Amendola 2, 42122 Reggio Emilia
Italy

G. Mazzuoccolo*

Laboratoire G-SCOP (Grenoble-INP, CNRS)
Grenoble
France

Abstract

Let m be a positive integer and let G be a cubic graph of order $2n$. We consider the problem of covering the edge-set of G with the minimum number of matchings of size m. This number is called the excessive $[m]$-index of G in the literature. The case $m = n$, that is, a covering with perfect matchings, is known to be strictly related to an outstanding conjecture of Berge and Fulkerson. In this paper we study in some detail the case $m = n - 1$. We show how this parameter can be large for cubic graphs with low connectivity and we furnish some evidence that each cyclically 4-connected cubic graph of order $2n$ has excessive $[n-1]$-index at most 4. Finally, we discuss the relation between excessive $[n-1]$-index and some other graph parameters such as oddness and circumference.

1 Introduction

Throughout this paper, a graph G always means a cubic simple connected finite graph (without loops and parallel edges). We refer to any introductory book for graph-theoretical notation and terminology not described in this paper (see for instance [2]).

The excessive $[m]$-index of a graph G, denoted by $\chi'_{[m]}(G)$, was first defined by Cariolaro and Fu in [6] as the minimum number of matchings of size m needed to cover the edge-set of G. In what follows, $[m]$-matching will stand for a matching of size m. The excessive $[m]$-index of particular classes of graphs is computed (for some

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values of m) in [6] and [7] and a general formula for small values of m is furnished in [5]. It is recently proved by the second author (see [12] and [13]) that the excessive $[m]$-index of a graph is strictly related to the well-known Berge-Fulkerson conjecture and its generalization given by Seymour in [16]. Mainly for this reason we will focus our attention on cubic graphs. We would like to stress that for small values of m the problem is already solved, whereas it remains open for large values of m. More precisely, by a result of Cariolaro, if $\frac{|E(G)|}{m} > \chi'(G)$ holds, then $\chi'_{[m]}(G) = \left\lceil \frac{|E(G)|}{m} \right\rceil$. Hence, by direct computation the following proposition holds:

Proposition 1. *Let G be a cubic graph of order $2n$. If $m < \frac{3n}{4}$, then $\chi'_{[m]}(G) = \left\lceil \frac{3n}{m} \right\rceil$.*

That naturally leads our attention to large values of m. The largest possible case, that is $m = n$, is completely open: it is conjectured (see Conjecture 2) that $\chi'_{[n]}(G) \leq 5$ for each 2-connected cubic graph G of order $2n$, but still unproved is the existence of a constant k such that $\chi'_{[n]}(G) \leq k$ for each 2-connected cubic graph G of order $2n$. To the best of our knowledge, the best upper bound for the excessive $[n]$-index of a 2-connected cubic graph in terms of its size is given in [14].

In the present paper we will focus our attention on the case $m = n - 1$. In particular, we will address the following question:

How many $[n-1]$-matchings do we need in order to cover the edge-set of a cubic graph of order $2n$?

Trivially, if a cubic graph G has an edge which is contained in no $[n-1]$-matching, then it is not possible to cover the edge-set of G with $[n-1]$-matchings; in this case we set $\chi'_{[n-1]}(G) = \infty$. For instance, the graph in Figure 1 is a graph of order 22 with $\chi'_{[10]}(G) = \infty$: one can easily check that the edge labelled e does not belong to a $[10]$-matching of G.

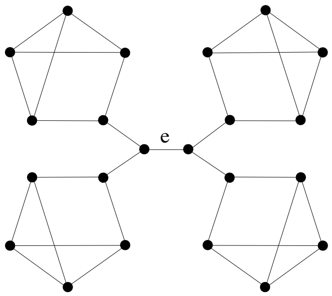

Figure 1: A graph with no $[n-1]$-covering

In Section 2, we show that, even under the assumption that the excessive $[n-1]$-index of G is finite, there exists a family of cubic graphs with arbitrarily large excessive $[n-1]$-index. In Section 3, we prove some general lemmas and propositions that we will use in the proofs of the last section. In Section 4, we consider 2-connected cubic graphs. First of all, by assuming the Berge-Fulkerson conjecture to be true, we obtain that $\chi'_{[n-1]}(G)$ is at most five for each 2-connected cubic graph G of order

$2n$. In particular, we provide examples of 2-connected and 3-connected cubic graphs with excessive $[n-1]$-index equal to five. After that, we prove that $\chi'_{[n-1]}(G) \leq 4$ holds for cubic graphs of oddness at most 4, for cubic graphs with circumference at least $2n-2$ and for the class of 3^*-connected cubic graphs introduced by Albert, Aldred, Holton and Sheehan in [1].

2 1-connected cubic graphs

In this section, we consider 1-connected cubic graphs such that every edge of the graph is contained in at least one $[n-1]$-matching (where $2n$ is the order of the graph).

In the next proposition, we show that there exist graphs with finite excessive $[n-1]$-index as large as we want.

Proposition 2. *For every positive integer m, there exists a positive integer n and a cubic graph G of order $2n$ such that $\chi'_{[n-1]}(G) \geq m$.*

Proof. Let $C_m = (v_1, \ldots, v_m)$ be a cycle of length $m \geq 3$. For $i = 1, \ldots, m$, denote by G_i a connected graph sharing no vertex with C_m, having $2m_i$ vertices of degree 3 and exactly one vertex of degree 2, say u_i. Furthermore, G_i is such that each edge is contained in at least one $[m_i]$-matching of G_i. Let G be the graph having $V(G) = V(C_m) \cup V(G_1) \cup \cdots \cup V(G_m)$ as vertex-set and $E(G) = E(C_m) \cup E(G_1) \cup \cdots \cup E(G_m) \cup \{\{u_i, v_i\} | 1 \leq i \leq m\}$ as edge-set. The edges $\{u_i, v_i\}$'s are bridges and will be called *spokes* (in Figure 2, we depicted an example with 30 vertices). We set $2n = |V(G)| = 2m + \sum_{i=1}^{m} 2m_i$. First observe that every edge of G is contained in an $[n-1]$-matching of G. We now show that no $[n-1]$-matching of G can contain two edges of C_m.

Assume there exists an $[n-1]$-matching of G, say M, containing $h \geq 2$ edges of C_m. Then M can contain at most $m-2h$ spokes. Since M can contain at most m_i edges of G_i, we have $|M| \leq \sum_{i=1}^{m} m_i + (m-2h) + h = n-h \leq n-2$, a contradiction, since M has size $n-1$. Hence, each $[n-1]$-matching of G contains at most one edge of C_m. Since $|E(C_m)| = m$, we need at least m $[n-1]$-matchings to cover the edges of G. □

3 General properties

Lemma 1. *Let G be a cubic graph of order $2n$. Let $\mathcal{M} = \{M_1, \ldots, M_t\}$ be a covering of G such that $\sum_{i=1}^{t} |M_i| = (n-1)t$. Then, there exists a $[n-1]$-covering of G of size t.*

Proof. Suppose $\mathcal{M}$ is not a $[n-1]$-covering of G. Then there exist two matchings M_i and M_j, where $i \neq j$, such that $|M_i| < n-1 < |M_j|$. Consider the subgraph $M_i \cup M_j$, $i \neq j$, of G: a connected component of $M_i \cup M_j$ is either a path (possibly a single edge belonging both to M_i and M_j) or a cycle of even length. In the latter case the connected component of $M_i \cup M_j$ has the same number of edges of M_i and

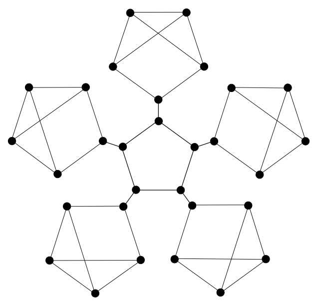

Figure 2: A graph of order 30 with $\chi'_{[14]} = 5$.

M_j. Then, by $|M_i| < n-1 < |M_j|$, there exists at least a connected component consisting of a path P of odd length starting and finishing with edges of M_j. The exchange of edges in P increases once $|M_i|$ and decreases once $|M_j|$. The iteration of this process furnishes a $[n-1]$-covering of G of size t. □

Proposition 3. *Let G be a 3-edge-colorable cubic graph of order $2n \geq 8$. Then $\chi'_{[n-1]}(G) = 4$.*

Proof. Clearly $\chi'_{[n-1]}(G) \geq \lceil |E(G)|/n-1 \rceil = \lceil 3n/n-1 \rceil = 4$. Let M_1, M_2 and M_3 be three pairwise disjoint perfect matchings of G. Let $M_4 \subset M_1$ be an arbitrary $[n-4]$-matching of G. The set $\{M_1, \ldots, M_4\}$ is a covering of G such that $\sum_{i=1}^{4} |M_i| = 4(n-1)$. The assertion follows from Lemma 1. □

There are only three cubic graphs of order $2n < 8$ which are 3-edge-colorable, namely, the complete graph K_4, the complete bipartite graph $K_{3,3}$ and the prism Y_3 on 6 vertices. By Proposition 1, $\chi'_{[n-1]}(K_4) = 6$ (as $n-1 = 1$ and K_4 has 6 edges), $\chi'_{[n-1]}(K_{3,3}) = \chi'_{[n-1]}(Y_3) = 5$ (as $n-1 = 3$, $K_{3,3}$ and Y_3 have 9 edges).

Lemma 2. *Let G be a cubic graph of order $2n$, with $2n \geq 8$. If there exist a perfect matching M and a $[n-1]$-matching N of G with empty intersection, then $\chi'_{[n-1]}(G) = 4$.*

Proof. Clearly $\chi'_{[n-1]}(G) \geq \lceil |E(G)|/n-1 \rceil = \lceil 3n/n-1 \rceil = 4$. Denote by H the complementary subgraph of $M \cup N$ in G. The subgraph H has all vertices of degree one but two vertices, say u and v, of degree two. If the vertices u and v are adjacent in G, then $N \cup \{[u,v]\}$ is a perfect matching of G disjoint from M. Since a cubic graph with two disjoint perfect matchings is 3-edge-colorable, the assertion follows from Proposition 3. Now consider the case u and v non-adjacent in G: the set of edges with both endvertices of degree 1 in H is a $[n-3]$-matching, say L, of G. Denote by e_1, e_2 (respectively f_1,f_2) the edges of H incident u (respectively v). Set $L_1 = L \cup \{e_1, f_1\}$ and $L_2 = (L \cup \{e_2, f_2\}) \setminus \{e\}$, where e is an arbitrary edge of L (such an edge does exist by the assumption on the order of G). The set $\{M, N, L_1, L_2\}$ satisfies Lemma 1, hence the assertion follows. □

4 3-graphs

We recall that an r-graph is an r-regular graph G of even order such that every edge-cut which separates $V(G)$ into two sets of odd cardinality has size at least r. This notion was introduced in [16]. A cubic graph is a 3-graph if and only if it is bridgeless. An r-graph G is 1-extendable (every edge of G is contained in a perfect matching), hence $\chi'_{[n-1]}(G) < \infty$.

A well-known conjecture of Berge and Fulkerson [9] can be stated as follows:

Conjecture 1 (Berge-Fulkerson). *Let G be a 3-graph. There exist six perfect matchings of G such that each edge of G belongs to exactly two of them.*

It was recently proved by the second author (see [12]) that such a conjecture can be easily stated in term of the excessive $[n]$-index.

Conjecture 2 (Berge-Fulkerson). *Let G be a 3-graph of order $2n$. Then $\chi'_{[n]}(G) \leq 5$.*

In the next proposition, we show how the Berge-Fulkerson conjecture implies that also the excessive $[n-1]$-index of a 3-graph is bounded by a constant.

Proposition 4. *The Berge-Fulkerson conjecture implies $\chi'_{[n-1]}(G) \leq 5$ for each 3-graph G of order greater than 4.*

Proof. Let G be a 3-graph. If G is 3-edge-colorable, then the result follows from Proposition 3. Assume that G is not 3-edge-colorable. By assuming that Conjecture 1 is true, we have 5 perfect matchings of G covering the edge-set of G such that each edge belongs to at most two perfect matchings. The intersection of each pair of these perfect matchings is non empty, since the existence of two disjoint perfect matchings implies the existence of a 3-edge-coloring. Hence, intersecting in pairs the 5 perfect matchings, we find at least 10 distinct edges of G belonging to exactly two of the five perfect matchings. We can delete five of these edges once from the perfect matchings. We obtain a covering of cardinality $5n - 5$ and the assertion follows from Lemma 1. □

By using an analogous argument, it is possible to prove that cubic graphs of order $2n$ with excessive $[n]$-index 4 have excessive $[n-1]$-index equals to 4 too. It is proved in [8] that some classical families of snarks (i.e. non 3-edge-colorable cubic graph with girth at least 5 and cyclically 4-connected), such as Flower snarks and Blanusa snarks, have excessive $[n]$-index 4. Hence, we can state that the excessive $[n-1]$-index is 4 for snarks of these families.

A generation of all snarks up to 36 vertices was obtained by Brinkmann, Goedgebeur, Hägglund and Markström (see [4]). Janos Hägglund verified that all snarks with at most 32 vertices have two perfect matchings M_1 and M_2 such that their intersection is a unique edge e. Since M_1 and $M_2 \setminus \{e\}$ satisfy the hypothesis of Lemma 2 we can state the following proposition:

Proposition 5. *Let G be a snark of order $2n \leq 32$. Then, $\chi_{[n-1]}(G) = 4$.*

In what follows we will construct examples of 2-connected and 3-connected cubic graphs having $\chi'_{[n-1]}(G) > 4$. Furthermore we will give some evidence to the fact that the excessive $[n-1]$-index of a cyclically 4-connected cubic graph could be at most 4.

4.1 2-connected

In this section, we furnish an example of a 3-graph G of order $2n$ with $\chi'_{[n-1]}(G) > 4$. Let G be the graph obtained in the following way: consider nine copies P_i, for $i = 0, \ldots, 8$, of the graph obtained by removing from the Petersen graph an edge $[u_i, v_i]$. Add the edges $[v_i, u_{i+1}]$, indices taken modulo 9 (see Figure 3).

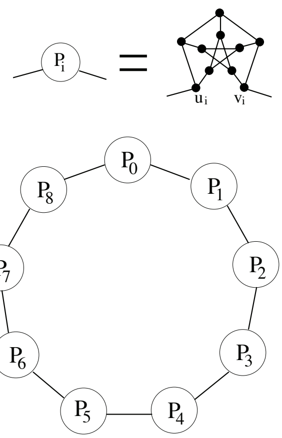

Figure 3: A 2-connected cubic graph G of order 90 with $\chi'_{[44]}(G) > 4$

Suppose there exists a $[n-1]$-covering $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ of G of size four. Each M_i leaves two vertices of G uncovered, so we have at most eight vertices which are uncovered at least once. It follows that in at least one of the nine P_i's, which without loss of generality we can suppose to be P_0, all vertices are covered by each $[n-1]$-matching. Starting from P_0, we obtain a copy of the Petersen graph by removing the two edges $[u_0, v_8]$ and $[v_0, u_1]$ and adding the edge $[u_0, v_0]$. Consider each $[n-1]$-matching M_j: both the edges $[u_0, v_8]$ and $[u_0, v_1]$ belong to M_j or neither one does. In the first case, we add to $M_j \cap P_0$ the edge $[u_0, v_0]$ in order to obtain a perfect matching of P, in the latter case $M_j \cap P_0$ is a perfect matching of P. This implies that we have a covering of the Petersen graph with four perfect matchings. Since it is well-known that $\chi'_{[n]}(P) = 5$ (see for instance [3]), we have a contradiction. Then the excessive $[n-1]$-index of G is at least 5 (one can check that it is actually 5).

4.2 3-connected

In this section, we construct an example of a 3-connected cubic graph G of order $2n$ with $\chi'_{[n-1]}(G) > 4$. Consider the graph H in Figure 4. The graph H is obtained

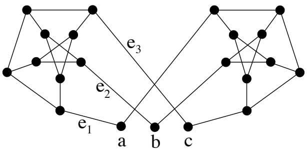

Figure 4: The graph H

starting from two copies of the Petersen graph P: in each copy of P delete a vertex, add three new vertices, say a, b, c, and construct the edges e_i, e'_i, $i = 1, 2, 3$. To construct the graph G, consider $2m$ copies of H, say $H_1, \ldots, H_{2m}$; for $1 \leq i \leq 2m$, label the vertices of degree 2 in H_i by a_i, b_i, c_i and add three new vertices u_i, v_i, w_i. Construct the edges $[a_i, u_i]$, $[b_i, v_i]$, $[c_i, w_i]$, with $1 \leq i \leq 2m$; $[v_i, u_{i+m}]$, $[v_i, w_{i+m}]$, with $1 \leq i \leq m$; $[u_i, w_{i-1}]$, with $1 \leq i \leq 2m$ (the subscripts are read modulo $2m$). One can verify that G is 3-edge-connected. In Figure 5, you can see a graph G of order $2n = 240$.

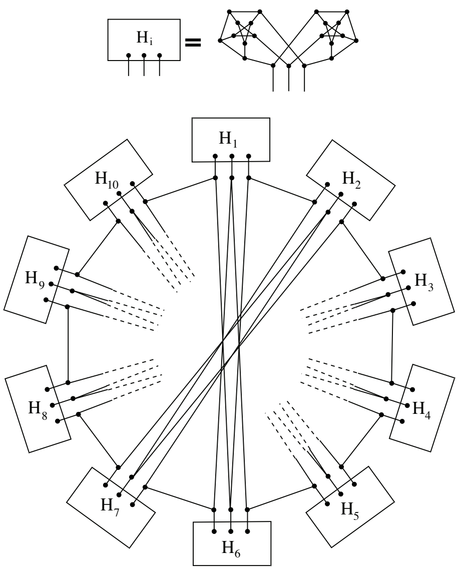

Figure 5: A 3-edge connected cubic graph G of order 240, with $\chi'_{119}(G) > 4$.

We show that $\chi'_{[n-1]}(G) > 4$. Suppose $\chi'_{[n-1]}(G) = 4$ and denote by $\{M_1, M_2, M_3, M_4\}$ a $[n-1]$-covering of G of size 4. Since $2m \geq 10$, there is (at least) one copy of H, say H_1, whose vertices are all covered in M_j, for every $1 \leq j \leq 4$. Furthermore, every M_j contains a copy of exactly one of the edges e_i's, otherwise H_1 has uncovered vertices in M_j. Therefore, we obtain an excessive factorization of size 4 for each copy of the Petersen graph in H_1, a contradiction by the fact that $\chi'_{[n]}(P) = 5$ again. □

4.3 Oddness 2 and 4

The oddness of a cubic graph G is the minimum number of odd circuits in a 2-factor of G. Obviously, the oddness of a cubic graph is an even number and it is 0 if and only if the graph is 3-edge-colorable. The next results hold for cubic graph of oddness 2 and 4.

Proposition 6. *Let G be a cubic graph of order $2n$ and oddness 2. Then*
$$\chi'_{[n-1]}(G) = 4.$$

Proof. Let F be a 2-factor of G having exactly 2 odd circuits. Let M be the complementary perfect matching of F in G. Let N be a $[n-1]$-matching of F. Since M and N are disjoint the assertion follows from Lemma 2. □

We recall that a permutation snark is a snark containing a 2-factor of exactly 2 odd circuits having no chords. The Petersen graph is a permutation snark. As a consequence of Proposition 6, the excessive $[n-1]$-index of every permutation snark of order $2n$ is 4.

In order to prove an equivalent result for cubic graphs of oddness 4, we need the following lemma proved in [10]. We refer to Schrijver's monograph [15] for the definition of fractional perfect matching and related topics.

Lemma 3. *If w is a fractional perfect matching in a cubic graph G and $c \in \mathbb{R}^E$, then G has a perfect matching M such that*
$$c \cdot \chi^M \geq c \cdot w$$
where $\cdot$ denotes the scalar product.

Proposition 7. *Let G be a cyclically 4-connected cubic graph of order $2n$ and oddness 4. Then, $\chi'_{[n-1]}(G) = 4$.*

Proof. Let M_1 be a perfect matching of G whose complementary 2-factor F has exactly 4 odd circuits. Since G is cyclically 4-connected, the function $\omega : E(G) \to \mathbb{R}$, defined by
$$\omega(e) = \begin{cases} \frac{1}{5} & \text{if } e \in M_1 \\ \\ \frac{2}{5} & \text{if } e \notin M_1 \end{cases}$$
for each $e \in E(G)$, is a fractional perfect matching of G (see for instance [10]).
Select a pair of incident edges in each odd circuit of F, say L the set of these eight edges of G. By Lemma 3, there exists a perfect matching M_2 such that
$$\chi^L \cdot \chi^{M_2} \geq \chi^L \cdot w.$$
The left hand side of the previous inequality is exactly the number of edges in $M_2 \cap L$ and the right one is equal to $8 \cdot \frac{2}{5} = \frac{16}{5}$. Whence, $|M_2 \cap L| = 4$. The subgraph $H = G \setminus (M_1 \cup (M_2 \cap L))$ consists of 4 paths of even length and, possibly, even cycles. Hence, the edges of H can be covered by two $[n-2]$-matchings, say N_1 and N_2. The set $\{M_1, M_2, N_1, N_2\}$ satisfies the hypothesis of Lemma 1 and the assertion follows. □

4.4 3*-connected graphs

The class of 3*-connected cubic graphs is first considered in [1]. A 3-graph is said to be 3*-connected if there exists a pair of vertices $a, b \in V(G)$ such that a, b are the endvertices of three openly disjoint paths Q_1, Q_2, Q_3 such that $V(G) = \bigcup_{i=1}^{3} V(Q_i)$.

It is natural to give an equivalent definition of the class of 3*-connected cubic graphs in our context. A 3*-connected cubic graph is a 3-graph having an $[n-1]$-matching M such that $G \setminus M$ is connected.

In the following proposition, we determine the excessive $[n-1]$-index of a 3*-connected cubic graph.

Proposition 8. *Let G be a 3^*-connected cubic graph of order $2n$, with $n \geq 4$. Then $\chi_{[n-1]}(G) = 4$.*

Proof. By the definition of 3*-connected cubic graph, there exist two vertices a, b and three openly disjoint paths $Q_1 = (a, u_1, \ldots, u_r, b)$, $Q_2 = (a, v_1, \ldots, v_s, b)$ and $Q_3 = (a, w_1, \ldots, w_t, b)$ spanning the graph G. Without loss of generality, we can assume r even and hence $s + t$ even. Let N be the complementary $[n-1]$-matching of $Q_1 \cup Q_2 \cup Q_3$.

Denote by C the $(s + t + 2)$-cycle $C = Q_2 \cup Q_3$ and by Q the subpath $Q = (u_1, u_2, \ldots, u_r)$ of Q_1.

As C has even length and Q has odd length (i.e. Q has an odd number of edges), we can color alternately the edges of C and those of Q obtaining a perfect matching M of G.

Since M and N are disjoint the assertion follows by Lemma 2. □

4.5 Circumference

The circumference of a graph G is the length of any longest circuit of G. In the next proposition we give a further support to the claim that large classes of cubic graphs have excessive $[n-1]$-index equals to 4.

Proposition 9. *Let G be a cubic graph of order $2n \geq 8$ and circumference at least $2n-2$. Then $\chi'_{[n-1]}(G) = 4$.*

Proof. If G has circumference $2n$, then G is 3-edge-colorable and the assertion follows from Proposition 3; if G has circumference $2n-1$, then G is 3*-connected and the assertion follows from Proposition 8.

We consider G with circumference $2n-2$. Denote by C a circuit of G of length $2n-2$ and by u, v the vertices of G not belonging to C. We color alternately the edges of C and obtain two $[n-1]$-matchings of G, say M_1 and N. We distinguish two cases according that u, v are adjacent or not.

If u, v are adjacent vertices in G, then $M = M_1 \cup \{[u, v]\}$ and N satisfy Lemma 2 and the assertion follows.

Consider u, v non-adjacent. The set L of chords of C is a $[n-4]$-matching of G (the vertices of C which are adjacent to u and v are uncovered in L). Denote by u_i (respectively, by v_i) the vertices of C adjacent to u (respectively, to v), with $i = 1, 2, 3$.

The subgraph $M_1 \cup L$ contains exactly three paths of odd length whose endvertices are the uncovered vertices of L. At least one of these three paths has one endvertex adjacent to u, say u_1, and the other adjacent to v, say v_1. The subgraph $H = M_1 \cup L \cup \{[u, u_i], [v, v_i] : i = 1, 2\}$ is 2-edge-colorable. Since exactly one connected component of H is a path of odd length, a color class of H is a perfect matching of G and the other one is a $[n-1]$-matching of G. Again, the assertion follows from Lemma 2. $\square$

5 Final Remarks

The main aim of this paper has been the study of the excessive $[n-1]$-index of cubic graphs. Among other results, we have given some evidence that each cyclically 4-connected cubic graph of order $2n$ can be covered with four $[n-1]$-matchings. Nevertheless, analogously to the perfect matchings case, we leave completely open the weaker problem of the existence of a constant k such that $\chi'_{[n-1]}(G)$ is at most k for every 3-graph G.

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