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A note on uniformly resolvable decompositions of K_v and $K_v - I$ into 2-stars and 4-cycles

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Abstract

In this paper we consider the uniformly resolvable decomposition of the complete graph K_v, or the complete graph minus a 1-factor $K_v - I$, into two graphs such that each resolution class contains only blocks isomorphic to the same graph. We completely solve the case in which the resolution classes are either all 2-stars or 4-cycles.

1 Introduction and Definitions

Given a collection of graphs $\mathcal{H}$, an *$\mathcal{H}$-decomposition* of K_v is a decomposition of the edges of K_v into isomorphic copies of graphs in $\mathcal{H}$. The copies of $H \in \mathcal{H}$ in the decomposition are called *blocks*. Such a decomposition is called *resolvable* if it is possible to partition the blocks into *classes* $\mathcal{P}_i$ such that every point of K_v appears exactly once in some block of each $\mathcal{P}_i$.

A resolvable *$\mathcal{H}$-decomposition* of K_v is sometimes also referred to as an *$\mathcal{H}$-factorization of K_v*, a class can be called an *$\mathcal{H}$-factor of K_v*. The case where $\mathcal{H}$ is a single edge (K_2) is known as a 1-*factorization of K_v*, these are well known to exist for $G = K_v$ if and only if v is even [1]. A single class of a 1-*factorization*, a pairing of all points, is also known as a 1-*factor*. In many cases we wish to place further constraints on the classes. For example, a class is called *uniform* if every block of the class is isomorphic to the same graph from $\mathcal{H}$. Of particular note is the result of Rees [6] which finds necessary and sufficient conditions for the existence of uniform $\{K_2, K_3\}$-decompositions of K_v. Uniformly resolvable decompositions of K_v have also been studied in [2], [3], [8] and [9]. A 2-star (or $K_{1,2}$) is the simple graph with two edges. In what follows we will denote the 4-cycle C_4 having vertices

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$\{a_1, a_2, a_3, a_4\}$ and edges $\{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_4, a_1\}\}$ by (a_1, a_2, a_3, a_4) and the path $K_{1,2}$ having vertices $\{a_1, a_2, a_3\}$ and edges $\{\{a_1, a_3\}, \{a_1, a_2\}\}$, by $(a_1; a_2, a_3)$. Let $J_{URD}(v; r, s; K_{1,2}, C_4)$ denote the set of all pairs (r, s) such that there exists a uniformly resolvable decomposition of $K_v - I$ (the complete undireceted graph minus a 1-factor) into r classes of 2-stars and s classes of 4-cycles. Given $v \equiv 0 \pmod{12}$, define

$$\begin{aligned} J(v) &= \{(3x, 1 + \tfrac{v-4}{2} - 2x) \mid x = 0, \dots, \tfrac{v-4}{4}\} \quad \text{and} \\ \hat{J}(v) &= \{(3x, 1 + \tfrac{v-4}{2} - 2x) \mid x = 1, \dots, \tfrac{v-4}{4}\}. \end{aligned}$$

In this paper we study the existence of a uniformly resolvable decomposition of K_v and $K_v - I$ into r classes of 2-stars and s classes of 4-cycles. Our main results in this paper are collected in the following theorem.

Main Theorem.

(i) *A uniformly resolvable decomposition of $K_v - I$ into $r > 0$ classes of 2-stars and $s > 0$ classes of 4-cycles exists if and only if $v \equiv 0 \pmod{12}$ and $J_{URD}(v; r, s; K_{1,2}, C_4) = \hat{J}(v)$.*

(ii) *A uniformly resolvable decomposition of K_v into r classes of 2-stars and s classes of 4-cycles exists if and only if $v \equiv 9 \pmod{12}$ and $s = 0$.*

2 Preliminaries and necessary conditions

In this section we will introduce some useful definitions and give necessary conditions for the existence of a uniformly resolvable decomposition of K_v and $K_v - I$ into 2-stars and 4-cycles. For missing terms or results that are not explicitly explained in the paper, the reader is referred to [1] and its online updates. A (resolvable) $\mathcal{H}$-decomposition of the complete multipartite graph with u parts each of size g is known as a (resolvable) group divisible design $\mathcal{H}$-(R)GDD of type g^u, the parts of size g are called the groups of the design. When $\mathcal{H} = K_n$ we will call it an n-(R)GDD. A 3-RGDD of type g^u exists if and only if $g(u-1)$ is even and $gu \equiv 0 \pmod{3}$, except when $(g, u) \in \{(2, 6), (2, 3), (6, 3)\}$ [7]. When $\mathcal{H} = \{K_{1,2}, C_4\}$ a $(K_{1,2}, C_4)$-URGDD(r, s) of type g^u is a uniformly resolvable decomposition of complete multipartite graph with u parts each of size g into r classes consisting entirely of 2-stars and s classes consisting entirely of 4-cycles.

We need the following definition. Let (s_1, t_1) and (s_2, t_2) be two pairs of non-negative integers. Define $(s_1, t_1) + (s_2, t_2) = (s_1 + s_2, t_1 + t_2)$. If X and Y are two sets of pairs of non negative integers, then $X + Y$ denotes the set $\{(s_1, t_1) + (s_2, t_2) : (s_1, t_1) \in X, (s_2, t_2) \in Y\}$. If X is a set of pairs of non negative integers and h is a positive integer, then $h * X$ denotes the set of all pairs of non-negative integers which can be obtained by adding any h elements of X together (repetitions of elements of X are allowed).

Lemma 2.1. *A uniformly resolvable decomposition of K_v into r classes of 2-stars and $s > 0$ classes of 4-cycles does not exist for any $v \geq 4$.*

Proof. The case $r = 0$ corresponds to a uniformly resolvable decomposition of K_v into 4-cycles which is known not to exist [5]. Suppose that there exists a uniformly resolvable decomposition D of K_v into r classes of 2-stars and s classes of 4-cycles with $r > 0$ and $s > 0$. Then by resolvability follows $v \equiv 0 \pmod{12}$. Counting the edges of K_v that appear in D we obtain

$$r\frac{v}{3}2 + s\frac{v}{4}4 = \frac{v(v-1)}{2}.$$

Hence $4r + 6s = 3(v-1)$, which is a contradiction, because $4r + 6s$ cannot be odd for any $r, s > 0$. □

Lemma 2.2. *If there exists a uniformly resolvable decomposition of $K_v - I$ into r classes of 2-stars and $s > 0$ classes of 4-cycles then $v \equiv 0 \pmod{12}$ and $(r, s) \in J(v)$.*

Proof. The condition $v \equiv 0 \pmod{12}$ is trivial. Now let D be a decomposition of $K_v - I$ into r classes of 2-stars and s classes of 4-cycles. Counting the edges of $K_v - I$ that appear in D we obtain $r\frac{v}{3}2 + s\frac{v}{4}4 = \frac{v(v-2)}{2}$ and hence

(1) $2r + 3s = 3\frac{(v-2)}{2}$.

(1) implies that $r \equiv 0 \pmod 3$ and $3s \equiv 3\frac{v-2}{2} \pmod 2$; that is $s \equiv 1 \pmod 2$. The equation (1), for $r = 3x$ and $s = 1 + 2y$ is equivalent to

(2) $x + y = \frac{v-4}{4}$.

Thus $x = 0, 1, \ldots, \frac{v-4}{4}$ and $y = \frac{v-4}{4} - x$ and hence $(r, s) \in \{(3x, 1 + \frac{v-4}{2} - 2x) \mid x = 0, 1, \ldots, \frac{v-4}{4}\}$. This completes the proof of the lemma. □

3 Small cases

Lemma 3.1. *A $(K_{1,2}, C_4)$-URGDD(r, s) of type 4^3 exists for $(r, s) \in \{(0, 4), (3, 2),$ $(6, 0)\}$.*

Proof. The case $(0, 4)$ corresponds to a uniformly resolvable decomposition of type 4^3 into 4 parallel classes of 4-cycles which is known to exist [2]. The rest of the cases are given explicitly below.

- $(3, 2)$.
  Take the groups to be $\{0, 1, 2, 3\}, \{4, 5, 6, 7\}, \{8, 9, a, b\}$ and the classes listed below:
  $\{(0; 6, 8), (1; 9, a), (2; 5, 7), (3; 4, b)\}$, $\{(9; 0, 7), (8; 1, 6), (4; 2, b), (5; 3, a)\}$,
  $\{(7; 0, 8), (6; 9, 2), (b; 1, 5), (a; 3, 4)\}$, $\{(0, 4, 1, 5), (6, a, 7, b), (2, 9, 3, 8)\}$
  $\{(5, 8, 4, 9), (1, 6, 3, 7), (2, b, 0, a)\}$.

- $(6, 0)$.
  Take the groups to be $\{a_1, \ldots, a_4\}$, $\{b_1, \ldots, b_4\}$, $\{c_1, \ldots, c_4\}$ and the classes listed below:
  $\{(b_i; c_i, a_i,) \mid i \in Z_4\}$, $\{(b_i; a_{i+1}, c_{i+1}) \mid i \in Z_4\}$, $\{(c_i; a_{i+1}, b_{i+1}) \mid i \in Z_4\}$, $\{(c_i; a_{i+3}, b_{i+2}) \mid i \in Z_4\}$, $\{(a_i; c_i, b_{i+1}) \mid i \in Z_4\}$, $\{(a_i; c_{i+2}, b_{i+2}) \mid i \in Z_4\}$.

□

Lemma 3.2. $J_{URD}(12; r, s; K_{1,2}, C_4)$=$\{(0, 5), (3, 3), (6, 1)\}$.

Proof. The case $(0, 5)$ corresponds to a uniformly resolvable decomposition of $K_{12} - I$ into 5 parallel classes of 4-cycles which is known to exist [2]. Take a $(K_{1,2}, C_4)$-URGDD(r, s) of type 4^3 with $(r, s) \in \{(0, 4), (3, 2), (6, 0)\}$. Fill in each of the groups of size 4 with the same uniformly resolvable decomposition of $K_4 - I$ into 1 4-cycle. The result is a uniformly resolvable decomposition of $K_{12} - I$ into r classes of 2-stars and s classes of 4-cycles for each $(r, s) \in \{(0, 5), (3, 3), (6, 1)\}$. □

Lemma 3.3. *There exists a* $(K_{1,2}, C_4)$*-URGDD*(r, s) *of type* 12^2 *with* $(r, s) \in \{(0, 6), (9, 0)\}$.

Proof. The case $(0, 6)$ corresponds to a C_4-URGDD of type 12^2 with 6 parallel classes of 4-cycles which is known to exist [2]. The case $(9, 0)$ corresponds to a $K_{1,2}$-URGDD of type 12^2 with 9 parallel classes of 2-stars which is known to exist [10].

□

4 Conclusion

We are now able to prove our main result.

Theorem 4.1. *For every* $v \equiv 0 \pmod{24}$, $J_{URD}(v; r, s; K_{1,2}, C_4)$= $\hat{J}(v)$

Proof. The necessity follows from Lemma 2.2. Now let $v = 24k$, $k \geq 1$. Start with a 1-factorization $\mathcal{F}$=$\{F_1, \ldots, F_{2k-1}\}$ of the complete graph K_{2k} [1]. Give weight 12 to all points of K_{2k} and place on each edge of a given 1-factor of $\mathcal{F}$ the same $(K_{1,2}, C_4)$-URGDD(r_1, s_1) of type 12^2 with $(r_1, s_1) \in \{(0, 6), (9, 0)\}$, which comes from Lemma 3.3. Fill in each of the groups of size 12 with the same uniformly resolvable decomposition of $K_{12} - I_i$, $i = 1, 2, \ldots, 2k$, into r_2 classes of 2-stars and s_2 classes of 4-cycles with $(r_2, s_2) \in \{(0, 5), (3, 3), (6, 1)\}$, which comes from Lemma 3.2.

The result is a uniformly resolvable decomposition of $K_{24k} - I$ into r classes of 2-stars and s classes of 4-cycles for each $(r, s) \in \{(0, 5), (3, 3), (6, 1)\} + (2k - 1) * \{(0, 6), (9, 0)\}$ and $I = \bigcup_{i=1}^{2k} I_i$. Since $(2k-1) * \{(0, 6), (9, 0)\} = \{(9y, 12k-6-6y), y = 0, 1, \ldots, (2k-1)\}$, it is easy to see that $\{(0, 5), (3, 3), (6, 1)\} + (2k-1) * \{(0, 6), (9, 0)\} = J(24k)$. This completes the proof.

□

Theorem 4.2. *For every* $v \equiv 12 \pmod{24}$, $J_{URD}(v; r, s; K_{1,2}, C_4) = \hat{J}(v)$

Proof. The necessity follows from Lemma 2.2. The case $v = 12$ follows from Lemma 3.2. Now let $v = 12 + 24k$, $k \geq 1$, and start with a 3-RGDD of type 3^{1+2k}. Give weight 4 to each point of this 3-RGDD. Place on each block of a given resolution class of this 3-RGDD the same $(K_{1,2}, C_4)$-URGDD(r_1, s_1) of type 4^3 with $(r_1, s_1) \in \{(0,4),(3,2),(6,0)\}$, which comes from Lemma 3.1. Fill in each of the groups of size 12 with the same uniformly resolvable decomposition of $K_{12} - I_i$, $i = 1, 2, \ldots, 1+2k$, into r_2 classes of 2-stars and s_2 classes of 4-cycles with $(r_2, s_2) \in \{(0,5),(3,3),(6,1)\}$, which come from Lemma 3.2.

The result is a uniformly resolvable decomposition of $K_{24k} - I$ into r classes of 2-stars and s classes of 4-cycles for each $(r, s) \in \{(0,5),(3,3),(6,1)\} + (3k) * \{(0,4),(3,2),(6,0)\}$ and $I = \cup_{i=1}^{1+2k} I_i$. Since $(3k) * \{(0,4),(3,2),(6,0)\} = \{(3y, 12k - 2y), y = 0, 1, \ldots, 6k\}$, it easy to see that

$$\{(0,5),(3,3),(6,1)\} + (3k) * \{(0,4),(3,2),(6,0)\} = J(12 + 24k).$$

This completes the proof.

□

Theorem 4.3. *A uniformly resolvable decomposition of* K_v *into* r *classes of* 2*-stars and* s *classes of* 4*-cycles exists if and only if* $v \equiv 9 \pmod{12}$ *and* s*=0.*

Proof. The necessity follows from Lemma 2.1. For $v \equiv 9 \pmod{12}$, the existence of a uniformly resolvable decomposition of K_v into $\frac{v(v-1)}{6}$ classes of 2-stars follows from [4]. □

Combining Theorems 4.1, 4.2 and 4.3 gives the main theorem of this paper.

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