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Groups with right-invariant multiorders

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Abstract

A *Cayley object* for a group G is a structure on which G acts regularly as a group of automorphisms. The main theorem asserts that a necessary and sufficient condition for the free abelian group G of rank m to have the generic n-tuple of linear orders as a Cayley object is that $m > n$. The background to this theorem is discussed. The proof uses Kronecker's Theorem on diophantine approximation.

1 Cayley objects and homogeneous structures

The *regular representation* of a group G is the representation of the group acting on itself by right multiplication. A *Cayley object* for G is a structure on G admitting the regular representation as a group of automorphisms. The name comes from the fact that a Cayley graph for G is precisely a Cayley object which happens to be a graph.

A Cayley object must admit a transitive automorphism group. There is some interest in investigating objects with a high degree of symmetry which are Cayley objects for a group, or (in the other direction) groups which have a given highly symmetric object as a Cayley object. This is the topic of [2]; I refer to that paper for further motivation.

All objects here will be relational structures, consisting of a set carrying a collection of relations of various arities. A substructure of a relational structure will always be the induced substructure on a subset, consisting of all instances of each relation such that all arguments lie within the subset.

A relational structure M is said to be *homogeneous* if any isomorphism between finite substructures can be extended to an automorphism of M. This will be our "strong symmetry condition".

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The *age* of a relational structure M is the class of all finite relational structures of the same type which can be embedded into M.

Fraïssé [4] gave a necessary and sufficient condition for a class $\mathcal{C}$ of finite structures to be the age of a countable homogeneous structure:

(a) $\mathcal{C}$ is closed under isomorphism;

(b) $\mathcal{C}$ is closed under taking substructures;

(c) $\mathcal{C}$ contains only countably many members up to isomorphism;

(d) $\mathcal{C}$ has the *amalgamation property*, that is, given $A, B_1, B_2 \in \mathcal{C}$ with embeddings $f_i : A \to B_i$ for $i = 1, 2$, there exists $C \in \mathcal{C}$ and embeddings $g_i : B_i \to C$ for $i = 1, 2$ such that the composite embeddings $g_1 f_1$ and $g_2 f_2$ agree.

Moreover, if these conditions hold, there is a unique countable homogeneous structure M with age $\mathcal{C}$ (up to isomorphism). Such a class $\mathcal{C}$ is called a *Fraïssé class*, and M is its *Fraïssé limit*.

We say that $\mathcal{C}$ has the *strong amalgamation property* if the amalgamation can be done without identifying points outside A: that is, if $b_1 \in B_2$ and $b_2 \in B_2$ satisfy $g_1(b_1) = g_2(b_2)$, then there exists $a \in A$ such that $b_i = f_i(a)$ for $i = 1, 2$.

For example, the class of all finite totally ordered sets is a Fraïssé class; its Fraïssé limit is the ordered set $\mathbb{Q}$, the unique countable dense ordered set without endpoints. I generalise this example in the next section.

The homogeneous structure M with age $\mathcal{C}$ is characterised by the following *extension property*:

> If $A, B \in \mathcal{C}$ with $A \subseteq B$ and $|B| = |A| + 1$, then every embedding of A into M can be extended to an embedding of B into M.

2 Multiorders

An *n-order* is a set with n linear orders. If we do not need to specify the number of orders, we refer to a *multiorder*.

The class of finite n-orders is a Fraïssé class. (More generally, if we take any finite number of Fraïssé classes, each of which has strong amalgamation, and consider the finite sets carrying a structure from each class, with no relationship between the different structures, we obtain a Fraïssé class.)

The Fraïssé limit of the class of finite n-orders will be called the *generic* (countable) n-order.

The case $n = 2$ arises in connection with the thriving field of *permutation patterns*. If a finite set X carries a 2-order, we can use the first order to enumerate X as $(x_1 <_1 x_2 <_1 \cdots <_1 x_k)$, and then the second order defines a permutation of the labels $\{1, 2, \ldots, k\}$. The notion of induced substructure coincides exactly with that used in the theory of permutation patterns. So, in a sense, the theory of permutation patterns is the theory of the age of the generic countable 2-order. Is there a similar theory for the generic n-orders with $n > 2$?

In this context, the countable homogeneous 2-orders were determined in [3].

Problem Determine all countable homogeneous n-orders, for $n > 2$.

The extension property characterising the generic n-order on a countable set X is the following:

> Given any k points $x_1, \ldots, x_k$ of X, for each $i \in \{1, \ldots, n\}$, let I_i be one of the $k+1$ intervals (including semi-infinite intervals) into which X is divided by $x_1, \ldots, x_k$ in the order $<_i$. Then $I_1 \cap \cdots \cap I_n \neq \emptyset$.

This is because adding a point to a finite totally ordered set involves putting it into one of the intervals defined by the set: i.e. before the first element, or between the ith and $(i+1)$st for $i = 1, \ldots, k-1$, or after the last element.

For ease of use, we give a simpler but equivalent condition.

Proposition 2.1 *An n-order $(<_1, \ldots, <_n)$ on a countable set X is generic if and only if, for any choice of x_i and y_i (for $i = 1, \ldots, n$) with $x_i <_i y_i$ (possibly $x_i = -\infty$ or $y_i = \infty$), there is a point $z \in X$ satisfying $x_i <_i z <_i y_i$ for $i = 1, \ldots, n$.*

Proof It is clear that the condition in the Proposition implies that in the extension property. Suppose that the condition in the extension property is true, and assume the hypotheses of the Proposition. Take the finite set $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. For each i, the set of points z satisfying $x_i < z < y_i$ is a union of intervals defined by this finite set; pick one of them. By the extension property, the intersection of the chosen intervals is non-empty. $\square$

We will be interested in the case where all the orders are right-invariant for a transitive group. This means that none of them have end-points, and we don't need to worry about the "semi-infinite" intervals.

3 Dense right orders on groups

To say that a group G has a totally ordered set which is a Cayley object means that there is a total order $<$ on G which is *right-invariant*, that is, if $x < y$, then $xg < yg$ for any $g \in G$. If we have such an order, let $P = \{g \in G : 1 < g\}$; then

(a) G is the disjoint union of $\{1\}$, P and P^{-1};

(b) $P^2 \subseteq P$.

Conversely, if we have a set P satisfying these two conditions then, setting $x < y$ if $y = px$ for some $p \in P$ defines a right-invariant order on G. Moreover, the order is dense if and only if (b) is replaced by the stronger condition

(bb) $P^2 = P$.

For if $x < y$, then $x = py$ for some $p \in P$. If $P = P^2$, then write $p = qr$ for some $q, r \in P$; then $x < rx < qrx = y$.

A group is said to be *right-orderable* if it has a right-invariant order (sometimes called a *right order* for short). A great deal is known about right-orderable groups (see Chapter VII of [1] for a survey, and note that since a right order of an abelian group is also a left order, the results of the whole book apply in the case of abelian groups). Less attention has been paid to groups with a dense right order. Here is the example which will be important to us.

Theorem 3.1 *Let $\mathbb{Z}^m$ denote the free abelian group of rank $m > 1$. Suppose that $<$ is a right order on G. Then there is a non-zero vector $c \in \mathbb{R}^m$ such that $x < y$ if $c \cdot x < c \cdot y$, where the dot denotes the usual inner product. Moreover, if the components of c are linearly independent over $\mathbb{Q}$, then the order is dense, and $x < y$ if and only if $c \cdot x < c \cdot y$.*

Note that, if the components of c are not linearly independent over $\mathbb{Q}$, then there are non-zero elements z of $\mathbb{Z}^m$ which satisfy $c \cdot z = 0$, forming a subgroup A which is free abelian of smaller rank; to complete the specification of the order, we have to choose a right order of A. Note that the order is *non-archimedean* in this case; if a, b are positive elements with $a \in A$ and $b \notin A$, then $a^n < b$ for all positive n.

For example, $\mathbb{Z}$ has just two right orders (the usual order and its reverse), neither of which is dense. For $\mathbb{Z}^2$, using a vector of the form $c = (1, \alpha)$ gives a dense order if α is irrational. However, if α is rational, or if $c = (0, 1)$, then we do not yet have enough information to define the order, since the points $z \in \mathbb{Z}^2$ which satisfy $c \cdot z = 0$ will form a subgroup whose order is not yet specified. This subgroup has rank 1, and so (as before) has just two orders.

I have not found a convenient exposition of the proof of this theorem, so here is a sketch. By factoring out the subgroup of "small" elements, we may assume that the ordering is archimedean. Then a theorem of Hölder [7] shows that there is an isomorphism to an additive subgroup of $\mathbb{R}$, which clearly has the form given. See also [1, Theorem 1.3.4] or [5, p. 62].

4 The main theorem

The main result of this paper is the first known class of groups admitting homogeneous right multiorders. This result was conjectured in [2].

Theorem 4.1 *Let m and n be positive integers. The free abelian group $\mathbb{Z}^m$ of rank m has a right-invariant generic n-tuple of orders if and only if $m > n$.*

The proof of the theorem requires a number of lemmas. First we show that, if $m > n$, then there is a $\mathbb{Z}^m$-invariant generic n-tuple of orders. We note first that it suffices to show the result when $m = n + 1$, since dropping some orders from a G-invariant generic multiorder yields a G-invariant generic multiorder.

The proof uses an important result of Kronecker [8] on diophantine approximation, for which several proofs are given in Chapter XXIII of Hardy and Wright [6].

Theorem 4.2 *Let m be a positive integer, and let $c \in \mathbb{R}^m$ be a vector whose components are linearly independent on $\mathbb{Q}$. Then, given any $\epsilon > 0$, any line in $\mathbb{R}^m$ with direction vector c passes within distance ϵ of some lattice point in $\mathbb{Z}^m$.*

We also need an existence result for a certain kind of matrix.

Lemma 4.3 *Let m be a positive integer. Then there exists a $m \times m$ real matrix A having the properties*

(a) A is invertible;

(b) each row of A has components which are linearly independent over $\mathbb{Q}$;

(c) the last row of A is orthogonal to all the others.

Proof The construction here is due to Robin Chapman (personal communication), and replaces a non-constructive existence proof given in the first version of the paper.

Start with a matrix B whose jth row has $m - 1$ in position j and -1 in all other positions, for $j < m$, and whose last row has every entry 1. Then B satisfies conditions (a) and (c). Condition (c) is clear; for (a), replacing the last column by the sum of the columns shows that $\det(B) = m \det(B')$, where B' is the principal $(m-1) \times (m-1)$ submatrix of B, and is non-singular since it has the form $mI - J$, where J the all-1 matrix.

Now choose a transcendental real number θ; for $j < m$ and $k \leq m$, multiply the element in row j and column k by θ^{k-1}, and for $k \leq m$ multiply the element in row m and column k by $\theta^{-(k-1)}$. The result is the required matrix A. □

Now we give the construction. As remarked earlier, we assume that $m = n + 1$, and we need to find a generic n-order on $\mathbb{Z}^m$. Let A be a matrix having the properties of Lemma 4.3. Use the first $m - 1$ rows to define $m - 1$ right-invariant total orders $<_1, \ldots, <_{m-1}$ on $\mathbb{Z}^m$. We claim that this $(m - 1)$-tuple is generic. Note that each order is dense, by Theorem 3.1.

An interval in the ith order consists of the vectors lying between two parallel hyperplanes perpendicular to the ith row of the matrix. Since the matrix is invertible, the intersection of $m-1$ intervals (one for each order) is a cylinder with parallelepiped cross-section in a direction orthogonal to the first $m-1$ rows of the matrix, hence (by condition (c)) parallel to the mth row. By Kronecker's Theorem, there is a lattice point arbitrarily close to this line, and in particular close enough that it lies in the cylinder defined by the intervals. So this intersection is non-empty in the lattice $\mathbb{Z}^m$, and we are done.

Now we turn to the non-existence proofs.

Since we may omit some orders from a generic multiorder and it remains generic, proving the non-existence for $n = m$ will yield the result for all $n \geq m$; so assume that $m = n$. Our proof is by induction on n; it is split into three cases, of which only the second case requires the induction hypothesis.

Let $(<_1, \ldots, <_n)$ be an n-tuple of right-invariant linear orders on $\mathbb{Z}^n$. We have to prove that this tuple is not generic. Let $c_1, c_2, \ldots, c_n$ be vectors defining the top section of the ordering, as in Theorem 3.1. We use this notation for the remainder of the proof.

Lemma 4.4 *If $c_1, \ldots, c_n$ are linearly dependent, then the n-tuple of orders is not generic.*

Proof Suppose that c_k is a linear combination of $c_1, \ldots, c_{k-1}$, say $c_k = a_1c_1 + \cdots + a_{k-1}c_{k-1}$. By reversing some of the orders if necessary, we may assume that all the coefficients are non-negative. Now choose intervals $x_i \leq c_i \cdot z \leq y_i$ in the group. Any point z lying in all these intervals must also lie in the interval

$$\sum_{j=1}^{k-1} a_j x_j \leq c_k \cdot z \leq \sum_{j=1}^{k-1} a_j y_j.$$

So the interval $x_k \leq c_k \cdot z \leq y_k$ does not meet the intersection of these $k-1$ intervals if we choose, say, $y_k < \sum_{j=1}^{k-1} a_j x_j$. □

Lemma 4.5 *If $c_1, \ldots, c_n$ are linearly independent and at least one of them has linearly dependent components over $\mathbb{Q}$, then the n-tuple of orders is not generic.*

Proof Without loss of generality, we may assume that c_1 has linearly dependent components over $\mathbb{Q}$. Then $A = \{z \in \mathbb{Z}^n : c_1 \cdot z = 0\}$ is a non-zero subgroup of $\mathbb{Z}^m$, and contains an interval I_1 in the order $<_1$. So the restrictions of the other orders to A form an $(n-1)$-tuple of orders on an abelian group of rank at most $n-1$. By the inductive hypothesis, they cannot be generic, so some intersection of intervals in these orders is disjoint from A, and hence from I_1. So the original order is not generic. □

Lemma 4.6 *If $c_1, \ldots, c_n$ are linearly independent and all of them have components which are linearly independent over $\mathbb{Q}$, then the n-tuple of orders is not generic.*

Proof Each of the orders $<_1, \ldots, <_n$ is dense; an interval in $<_i$ consists of the lattice points lying between two parallel hyperplanes perpendicular to c_i, and these hyperplanes may be arbitrarily close together. So the intersection of the n intervals is a parallelepiped whose volume can be made arbitrarily small. This parallelepiped tiles the Euclidean space, so if we make its volume less than 1 we can find a translate containing no lattice point. □

These three lemmas complete the proof of the theorem. □

Problem Find further examples of groups with generic right multiorders.

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