

The left cells with a -values 5,6 in the affine Weyl group \widetilde{E}_8^*

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Abstract

We describe all the left cells of a -values 5, 6 in the affine Weyl group \widetilde{E}_8 in this paper. More precisely, we show that each of those left cells is left-connected, verifying a conjecture of Lusztig in our case. We find all the distinguished involutions in those left cells which occur as the vertices of the corresponding distinguished involution graphs. We also obtain all the corresponding left cell graphs.

1 Introduction

1.1

Let W be a Coxeter group with S its Coxeter generator set. Kazhdan and Lusztig introduced the concepts of left, right and two-sided cells in W in order to construct representations of W and the associated Hecke algebra \mathcal{H} in [3]. Lusztig further introduced the function $a : W \rightarrow \mathbb{N} \cup \{\infty\}$, he proved that when W is either a Weyl group or an affine Weyl group, the function a is constant on any two-sided cell of W and that each left cell of W contains a unique distinguished involution (see [6]). Distinguished involutions play an important role in the representation theory of W and \mathcal{H} . Thus this yields a big project to describe all the left cells of a Coxeter group W , and to find all the distinguished involutions of W when W is either a Weyl group or an affine Weyl group.

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1.2

Now we consider the cells of the affine Weyl group \widetilde{E}_8 . All the cells Γ of \widetilde{E}_8 with $a(\Gamma)$ either equal to 120 or less than or equal to 4 have been described explicitly (see [4], [9], [12], [13], [19]). In the present paper, we describe all the cells Γ of \widetilde{E}_8 with $a(\Gamma) \in \{5, 6\}$.

Let \leq be the Bruhat-Chevalley order and ℓ the length function on W . For any two-sided cell Ω and any left cell Γ of W , define $F(\Omega) = \{w \in \Omega \mid a(sw), a(wt) < a(w), \forall s, t \in S \text{ with } sw, wt < w\}$ and $E_{\min}(\Gamma) = \{w \in \Gamma \mid \ell(w) \leq \ell(y), \forall y \in \Gamma\}$. In [18], the second-named author introduced an algorithm for finding a representative set of left cells (or an l.c.r. set for short) of W in a two-sided cell Ω with the starting set $F(\Omega)$. On the other hand, when W is an affine Weyl group of simply-laced type and when Γ is a left cell of W with $a(\Gamma) \leq 6$, the second-named author described the distinguished involution in Γ in terms of elements of $E_{\min}(\Gamma)$ (see [17, Theorem A]). By applying these results, we find all the distinguished involutions d (which occur as the vertices of certain distinguished involution graphs) of \widetilde{E}_8 and all the left cell graphs in the two-sided cells Ω of \widetilde{E}_8 with $a(d), a(\Omega) \in \{5, 6\}$.

A subset K of W is *left-connected*, if for any $x, y \in K$, there exists a sequence $x_0 = x, x_1, \dots, x_r = y$ in K with some $r \geq 0$ such that $x_{i-1}x_i^{-1} \in S$ for $1 \leq i \leq r$. Lusztig conjectured in [1] that any left cell Γ of an affine Weyl group is left-connected. The conjecture is supported by all the existing data. In the present paper, we prove that all the left cells Γ of \widetilde{E}_8 with $a(\Gamma) \in \{5, 6\}$ are left-connected, verifying Lusztig's conjecture in our case.

1.3

The contents of the paper are organized as follows. Section 2 is served as preliminaries, we collect some concepts, notation and known results there. In Section 3, we show that all the left cells Γ of \widetilde{E}_8 with $a(\Gamma) \in \{5, 6\}$ are left-connected. In Section 4, we introduce distinguished involution graphs and left cell graphs for any two-sided cell Ω of \widetilde{E}_8 with $a(\Omega) \in \{5, 6\}$. We display all the elements of $E(\Omega) = \{w \in \Omega \mid a(sw) < a(w), \forall s \in \mathcal{L}(w)\}$, all the distinguished involution graphs and all the left cell graphs for Ω in Appendices of the electronic version of the paper (see [2]) and only attach a small portion of them to the present paper.

2 Preliminaries

2.1

Let $W = (W, S)$ be a Coxeter group with S its Coxeter generator set. Let $\mathcal{A} = \mathbb{Z}[u, u^{-1}]$ be the ring of all Laurent polynomials in an indeterminate u with integer coefficients. The Hecke algebra \mathcal{H} of W over \mathcal{A} has two \mathcal{A} -bases $\{T_x \mid x \in W\}$ and $\{C_w \mid w \in W\}$ which satisfy the relations

$$\begin{cases} T_w T_{w'} = T_{ww'}, & \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ (T_s - u^{-1})(T_s + u) = 0, & \text{for } s \in S, \end{cases}$$

and

$$C_w = \sum_{y \leq w} u^{\ell(w) - \ell(y)} P_{y,w}(u^{-2}) T_y,$$

where $P_{y,w} \in \mathbb{Z}[u]$ satisfies $P_{w,w} = 1$ for $w \in W$, $P_{y,w} = 0$ if $y \not\leq w$ and $\deg P_{y,w} \leq (1/2)(\ell(w) - \ell(y) - 1)$ if $y < w$. The $P_{y,w}$'s are called *Kazhdan-Lusztig polynomials* (see [3]).

2.2

For $y, w \in W$ with $\ell(y) < \ell(w)$, denote by $\mu(y, w)$ or $\mu(w, y)$ the coefficient of $u^{(1/2)(\ell(w) - \ell(y) - 1)}$ in $P_{y,w}$. The elements y and w are called *jointed*, written $y \text{ --- } w$, if $\mu(y, w) \neq 0$. To any $x \in W$, we associate two subsets of S :

$$\mathcal{L}(x) = \{s \in S \mid sx < x\} \quad \text{and} \quad \mathcal{R}(x) = \{s \in S \mid xs < x\}.$$

2.3

Let \leq_L (respectively, \leq_R, \leq_{LR}) be the preorder on W defined as in [3], and let \sim_L (respectively, \sim_R, \sim_{LR}) be the equivalence relation on W determined by \leq_L (respectively, \leq_R, \leq_{LR}). The corresponding equivalence classes of W are called the *left* (respectively, *right, two-sided*) *cells* of W . The preorder \leq_L (respectively, \leq_R, \leq_{LR}) on W induces a partial order on the set of left (respectively, right, two-sided) cells of W .

If $x \leq_L y$ (respectively, $x \leq_R y$), then $\mathcal{R}(x) \supseteq \mathcal{R}(y)$ (respectively, $\mathcal{L}(x) \supseteq \mathcal{L}(y)$). In particular, If $x \sim_L y$ (respectively, $x \sim_R y$), then $\mathcal{R}(x) = \mathcal{R}(y)$ (respectively, $\mathcal{L}(x) = \mathcal{L}(y)$) (see [3, Proposition 2.4]). So we may define $\mathcal{R}(\Gamma)$ (respectively, $\mathcal{L}(\Gamma)$) for a left (respectively, right) cell Γ of W to be the common set $\mathcal{R}(x)$ (respectively, $\mathcal{L}(x)$) for all $x \in \Gamma$.

2.4

Define $h_{x,y,z} \in \mathcal{A}$ by

$$C_x C_y = \sum_z h_{x,y,z} C_z$$

for any $x, y, z \in W$. In [5], Lusztig defined a function $a : W \rightarrow \mathbb{N} \cup \{\infty\}$ by setting

$$a(z) = \min\{k \in \mathbb{N} \mid u^k h_{x,y,z} \in \mathbb{Z}[u], \forall x, y \in W\} \quad \text{for any } z \in W$$

with the convention that $a(z) = \infty$ if $\{k \in \mathbb{N} \mid u^k h_{x,y,z} \in \mathbb{Z}[u], \forall x, y \in W\} = \emptyset$.

2.5

An affine Weyl group W_a is a Coxeter group which can be realized geometrically as follows. Let G be a connected, adjoint reductive linear algebraic group over the complex number field \mathbb{C} . Fix a maximal torus T of G , let X be the character group of T and let $\Phi \subset X$ be the root system of G with $\Delta = \{\alpha_1, \dots, \alpha_l\}$ a choice of simple root system. Then $E = X \otimes_{\mathbb{Z}} \mathbb{R}$ is a Euclidean space with an inner product $\langle \cdot, \cdot \rangle$ such that the Weyl group (W_0, S_0) of G with $W_0 = N_G(T)/T$ acts naturally on E and preserves its inner product, where S_0 is the set of simple reflections s_i corresponding to the simple roots α_i , $1 \leq i \leq l$. Denote by N the group of all the translations $T_\lambda : x \mapsto x + \lambda$ on E with λ ranging over X . Then the semidirect product $W_a = N \rtimes W_0$ of W_0 with N is an *affine Weyl group*. Let K be the type dual to the type of G . Then the type of W_a is \tilde{K} . In the case where there is no danger of confusion, W_a is denoted simply by its type \tilde{K} . Let $w \mapsto \bar{w}$ be the canonical homomorphism from W_a to $W_0 \cong W_a/N$.

The following properties of the function a on (W_a, S) were proved by Lusztig:

(1) If $x \underset{LR}{\leq} y$ then $a(x) \geq a(y)$. In particular, if $x \underset{LR}{\sim} y$ then $a(x) = a(y)$. So we may define the value $a(\Gamma)$ for a left (respectively, right, two-sided) cell Γ of W_a to be the common value $a(x)$ for all $x \in \Gamma$ (see [5, Theorem 5.4]).

(2) If $a(x) = a(y)$ and $x \underset{L}{\leq} y$ (respectively, $x \underset{R}{\leq} y$) then $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$) (see [5, Corollary 6.3], [6, Corollary 1.9]).

(3) Let $\delta(z) = \deg P_{e,z}$ for $z \in W_a$, where e is the identity of the group W_a . Define, for any $i \in \mathbb{Z}$,

$$\mathcal{D}_i = \{w \in W_a \mid \ell(w) - 2\delta(w) - a(w) = i\}.$$

Then Lusztig proved that $\mathcal{D}_i \neq \emptyset$ only if $i \geq 0$, that \mathcal{D}_0 is a finite set of involutions (following Lusztig, we call elements of \mathcal{D}_0 *distinguished involutions* of W_a) and that each left (respectively, right) cell of W_a contains a unique element of \mathcal{D}_0 (see [6, Subsection 1.3, Proposition 1.4 and Theorems 1.10, 2.2]).

(4) For any $I \subset S$, let w_I be the longest element in the subgroup W_I of W_a generated by I (note that W_I is always finite if W_a is irreducible). Then $w_I \in \mathcal{D}_0$ and $a(w_I) = \ell(w_I)$ (see [5, Proposition 2.4 and Corollary 6.3]).

(5) If $W_{(i)} := \{w \in W_a \mid a(w) = i\}$ contains an element of the form w_I for some $I \subset S$, then the set $\{w \in W_{(i)} \mid \mathcal{R}(w) = I\}$ forms a single left cell of W_a by (1), (2) and (4).

Call $s \in S$ *special* if the group $W_{S \setminus \{s\}}$ has the maximum possible order among all the standard parabolic subgroups W_I , $I \subset S$, of W_a . For $s \in S$, let

$$Y_s = \{w \in W_a \mid \mathcal{R}(w) \subseteq \{s\}\}.$$

Then Lusztig and Xi proved the result (6) below.

(6) Let $s \in S$ be special. Then $\Omega \cap Y_s$ is non-empty and forms a single left cell of W_a for any two-sided cell Ω of W_a (see [8, Theorem 1.2]).

(7) By the notation $x = y \cdot z$ ($x, y, z \in W_a$), we mean $x = yz$ and $\ell(x) = \ell(y) + \ell(z)$. In this case, we have $x \underset{L}{\leq} z$ and $x \underset{R}{\leq} y$ and hence $a(x) \geq a(y), a(z)$ by (1). In particular, if $I = \mathcal{R}(x)$ (respectively, $I = \mathcal{L}(x)$), then $a(x) \geq \ell(w_I)$ by (4).

Lusztig also proved the following

2.6 Theorem

(see [7, Theorems 4.8 and 8.1]) *Let G and W_a be the algebraic group and the corresponding affine Weyl group given in 2.5 respectively. Then there exists a bijective map $\mathbf{u} \mapsto c(\mathbf{u})$ from the set $\mathfrak{U}(G)$ of unipotent conjugacy classes in G to the set $\text{Cell}(W_a)$ of two-sided cells in W_a , which satisfies the equation $a(c(\mathbf{u})) = \dim \mathcal{B}_u$, where u is any element in \mathbf{u} , and $\dim \mathcal{B}_u$ is the dimension of the variety of all the Borel subgroups of G containing u .*

2.7

In the remaining part of the section, we assume that (W_a, S) is an irreducible affine Weyl group of simply-laced type, that is, the order $o(st)$ of the product st is not greater than 3 for any $s, t \in S$, or equivalently, W_a is of type \tilde{A}, \tilde{D} or \tilde{E} .

Given $s, t \in S$ with $o(st) = 3$, a set of the form $\{sy, tsy\}$ (respectively, $\{ys, yst\}$) is called a *left* (respectively, *right*) $\{s, t\}$ -string (or a *left* (respectively, *right*) string for short), if $\mathcal{L}(y) \cap \{s, t\} = \emptyset$ (respectively, $\mathcal{R}(y) \cap \{s, t\} = \emptyset$).

2.8 Proposition

(See [3, Theorem 4.2], [16, Proposition 4.6].) *For $s, t \in S$ with $o(st) = 3$, let $\{x_1, x_2\}$ and $\{y_1, y_2\}$ be two left (respectively, right) $\{s, t\}$ -strings. Then*

- (1) $x_1 \text{ --- } y_1$ if and only if $x_2 \text{ --- } y_2$;
- (2) $x_1 \underset{R}{\sim} y_1$ (respectively, $x_1 \underset{L}{\sim} y_1$) if and only if $x_2 \underset{R}{\sim} y_2$ (respectively, $x_2 \underset{L}{\sim} y_2$).

2.9

Two elements $x, y \in W_a$ form a (*left*) primitive pair, if there exist two sequences $x_0 = x, x_1, \dots, x_r$ and $y_0 = y, y_1, \dots, y_r$ in W_a such that the following conditions are satisfied:

(a) For every $1 \leq i \leq r$, there exist some $s_i, t_i \in S$ with $o(s_i t_i) = 3$ such that both $\{x_{i-1}, x_i\}$ and $\{y_{i-1}, y_i\}$ are left $\{s_i, t_i\}$ -strings.

(b) $x_i \text{ --- } y_i$ for some (and then for all by Proposition 2.8 (1)) i with $0 \leq i \leq r$.

(c) Either $\mathcal{L}(x) \not\subseteq \mathcal{L}(y)$ and $\mathcal{L}(y_r) \not\subseteq \mathcal{L}(x_r)$, or $\mathcal{L}(y) \not\subseteq \mathcal{L}(x)$ and $\mathcal{L}(x_r) \not\subseteq \mathcal{L}(y_r)$ hold.

2.10 Proposition

(See [14, Subsection 3.3]) $x \sim_L y$ if $x, y \in W_a$ form a left primitive pair.

2.11

We say that x is obtained from w by a left (respectively, right) $\{s, t\}$ -star operation (or a left (respectively, right) star operation for short), if $\{x, w\}$ is a left (respectively, right) $\{s, t\}$ -string. Note that the resulting element x for a left (respectively, right) $\{s, t\}$ -star operation on w is always unique whenever it exists.

Let d be an involution of W_a , i.e., $d^2 = e$. Suppose that there exist some $s, t \in S$ with $o(st) = 3$ such that $|\mathcal{L}(d) \cap \{s, t\}| = 1$ (the notation $|X|$ stands for the cardinality of a set X). We say that $d' \in W_a$ is obtained from d by the two-sided $\{s, t\}$ -star operations, if d' can be obtained from d by a left $\{s, t\}$ -star operation followed by a right $\{s, t\}$ -star operation. In this case, there must exist some $t_1, t_2 \in \{s, t\}$ such that $d' = t_1 d t_2$, where $\{d, t_1 d\}$ is a left $\{s, t\}$ -string and $\{t_1 d, d'\}$ is a right $\{s, t\}$ -string. It is easily seen that d' is also an involution. Hence we also have $d' = t_2 d t_1$ with $\{d, d t_1\}$ a right $\{s, t\}$ -string and $\{d t_1, d'\}$ a left $\{s, t\}$ -string. Moreover, we have the following result.

2.12 Proposition.

(See [15, Proposition 5.12]) Let $d \in \mathcal{D}_0$. Suppose that there exist some $s, t \in S$ with $o(st) = 3$ such that $|\mathcal{L}(d) \cap \{s, t\}| = 1$. If d' is obtained from d by the two-sided $\{s, t\}$ -star operations, then $d' \in \mathcal{D}_0$.

2.13

In the present paper, when we mention a graph, we always mean a set M of vertices together with a set E of edges, where $M \subseteq W_a$ and, each element of E is a certain two-element subset of M . We usually denote a graph by a capital calligraphic letter (e.g., $\mathcal{M} = (M, E)$). By a vertex-labeled graph (or a v-l graph for short), we mean a graph $\mathcal{M} = (M, E)$ with each $x \in M$ labeled by some subset of S (say $\mathcal{V}(x)$, we usually take $\mathcal{V}(x)$ to be $\mathcal{R}(x)$ or $\mathcal{L}(x)$). By a vertex-edge-labeled graph (or a v-e-l graph for short), we mean a v-l graph each of whose edges is labeled by either some element or some subset in S .

By a path in a graph $\mathcal{M} = (M, E)$, we mean a sequence z_0, z_1, \dots, z_r in M with some $r > 0$ such that $\{z_{i-1}, z_i\} \in E$ for every $1 \leq i \leq r$.

Two edges $\{x, y\} \neq \{x', y'\}$ of a graph $\mathcal{M} = (M, E)$ are said parallel, if there are two paths $x_0 = x, x_1, \dots, x_r = x'$ and $y_0 = y, y_1, \dots, y_r = y'$ in \mathcal{M} with some $r \geq 1$ such that $\{x_i, y_i\} \in E$ for every $0 \leq i \leq r$.

For any two v-l (respectively, v-e-l) graphs $\mathcal{M} = (M, E)$ and $\mathcal{M}' = (M', E')$, we write $\mathcal{M} \simeq \mathcal{M}'$ (respectively, $\mathcal{M} \cong \mathcal{M}'$), if there exists a bijective map $\phi : M \rightarrow M'$ satisfying the conditions (a)–(b) (respectively, (a)–(c)) below.

- (a) $\mathcal{V}(x) = \mathcal{V}(\phi(x))$ for any $x \in M$.
- (b) For any $x, y \in M$, we have $\{x, y\} \in E$ if and only if $\{\phi(x), \phi(y)\} \in E'$.
- (c) For any $\{x, y\} \in E$, the label of $\{x, y\} \in E$ is equal to that of $\{\phi(x), \phi(y)\}$.

2.14

For any $x \in W_a$, denote by $M(x)$ the set of all $y \in W_a$ such that there is a sequence $x_0 = x, x_1, \dots, x_r = y$ in W_a with some $r \geq 0$ such that $\{x_{i-1}, x_i\}$ is a right string for every $1 \leq i \leq r$.

Define the graph $\mathcal{M}(x)$ associated to $x \in W_a$ to be a v-l graph with $M(x)$ its vertex set and E its edge set, where E consists of all such two-element subsets $\{y, z\}$ in $M(x)$ that $\{y, z\}$ is a right string, and each $y \in M(x)$ is labeled by the set $\mathcal{R}(y)$.

The *left cell graph* associated to $x \in W_a$, written $\mathcal{M}_L(x)$, is by definition a v-l graph, whose vertex set $M_L(x)$ consists of all the left cells Γ of W_a with $\Gamma \cap M(x) \neq \emptyset$; $\{\Gamma, \Gamma'\} \subseteq M_L(x)$ is an edge of $\mathcal{M}_L(x)$, if there are some $y \in M(x) \cap \Gamma$ and some $y' \in M(x) \cap \Gamma'$ with $\{y, y'\}$ an edge of $\mathcal{M}(x)$; each $\Gamma \in M_L(x)$ is labeled by the set $\mathcal{R}(\Gamma)$ (see 2.3).

2.15

Let $d \in \mathcal{D}_0$. Denote by $D(d)$ the set of all such $d' \in W_a$ that there is a sequence $d_0 = d, d_1, \dots, d_r = d'$ in W_a with some $r \geq 0$, where d_i is obtained from d_{i-1} by the two-sided $\{s_i, t_i\}$ -star operations with some $s_i, t_i \in S$, $o(s_i t_i) = 3$, for every $1 \leq i \leq r$.

By Proposition 2.12, we have $D(d) \subseteq \mathcal{D}_0$ for any $d \in \mathcal{D}_0$, so $D(d)$ is a finite set by 2.5 (3).

Define a graph $\mathcal{D}(d)$ associated to $d \in \mathcal{D}_0$ to be a v-e-l graph with $D(d)$ its vertex set, its edge set consists of all the two-element subsets $\{d_1, d_2\}$ in $D(d)$, where d_1 is obtained from d_2 by the two-sided $\{s, t\}$ -star operations with some $s, t \in S$, $o(st) = 3$; in this case, we label the edge $\{d_1, d_2\}$ by the pair $t_1, t_2 \in \{s, t\}$ if $d_2 = t_1 d_1 t_2$. When $t_1 = t_2$, we denote the pair t_1, t_2 simply by t_1 for simplifying the notation. Each $d' \in D(d)$ is labeled by the set $\mathcal{R}(d')$. Call $\mathcal{D}(d)$ the *distinguished involution graph* associated to d .

By the definition, we have $D(d') = D(d)$ and $\mathcal{D}(d) = \mathcal{D}(d')$ for any $d' \in D(d)$.

2.16

Two elements $x, x' \in W_a$ are said to have *the same (right) generalized τ -invariants*, if for any path $z_0 = x, z_1, \dots, z_r$ in $\mathcal{M}(x)$, there is a path $z'_0 = x', z'_1, \dots, z'_r$ in $\mathcal{M}(x')$ with $\mathcal{R}(z'_i) = \mathcal{R}(z_i)$ for every $0 \leq i \leq r$, and if the above condition remains valid when the roles of x and x' are interchanged.

Then the following result is known.

2.17 Proposition

(See [16, Proposition 4.2], [19, Proposition 2.6], [20]) (a) *If $x \sim_L y$ in W_a , then x, y have the same right generalized τ -invariants.*

(b) *If $x, y \in W_a$ satisfy either $x \sim_L y$ or $y \in M(x)$, then the left cell graphs $\mathcal{M}_L(x)$ and $\mathcal{M}_L(y)$ are the same.*

2.18

Keep the notation in 2.5. Let $-\alpha_0$ be the highest short root in Φ . Denote $s_0 = s_{\alpha_0}T_{-\alpha_0}$, where s_{α_0} is the reflection in E with respect to α_0 . Then $S = S_0 \cup \{s_0\}$ forms a Coxeter generator set of W_a .

The *alcove form* of an element $w \in W_a$ is, by definition, a Φ -tuple $(k(w, \alpha))_{\alpha \in \Phi}$ over \mathbb{Z} determined by the following conditions.

- (a) $k(w, -\alpha) = -k(w, \alpha)$ for any $\alpha \in \Phi$;
- (b) $k(e, \alpha) = 0$ for any $\alpha \in \Phi$, where e is the identity element of W_a ;
- (c) If $w' = ws_i$ ($0 \leq i \leq l$), then

$$k(w', \alpha) = k(w, (\alpha)\bar{s}_i) + \varepsilon(\alpha, i)$$

with

$$\varepsilon(\alpha, i) = \begin{cases} 0 & \text{if } \alpha \neq \pm\alpha_i; \\ -1 & \text{if } \alpha = \alpha_i; \\ 1 & \text{if } \alpha = -\alpha_i, \end{cases}$$

where $\bar{s}_i = s_i$ if $1 \leq i \leq l$, and $\bar{s}_0 = s_{\alpha_0}$.

By condition (a), we can also denote the alcove form of $w \in W_a$ by a Φ^+ -tuple $(k(w, \alpha))_{\alpha \in \Phi^+}$, where Φ^+ is the positive root system of Φ containing Δ .

Condition (c) defines a set of operators $\{s_i \mid 0 \leq i \leq l\}$ on the alcove forms of elements w of W_a :

$$s_i : (k(w; \alpha))_{\alpha \in \Phi} \mapsto (k(w; (\alpha)\bar{s}_i) + \varepsilon(\alpha, i))_{\alpha \in \Phi}.$$

2.19

Let $w' = x \cdot w$ with $w', x, w \in W_a$. In this case, we call w' a *left* (respectively, *right*) *extension* of w (respectively, x), and call w (respectively, x) a *left* (respectively, *right*) *retraction* of w' . When $w' = x \cdot w \cdot y$ with $w', x, y, w \in W_a$, we call w' an *extension* of w and call w a *retraction* of w' .

Then the following results were shown by the second-named author:

2.20 Proposition

(See [10, Proposition 4.3], [11], [12, Proposition 2.1]) *Let $w \in W_a$.*

(1) $\ell(w) = \sum_{\alpha \in \Phi^+} |k(w, \alpha)|$, where the notation $|x|$ stands for the absolute value of $x \in \mathbb{Z}$;

(2) $\mathcal{R}(w) = \{s_i \mid k(w, \alpha_i) < 0, 0 \leq i \leq l\}$;

(3) w' is a left extension of w if and only if the inequalities $k(w', \alpha)k(w, \alpha) \geq 0$ and $|k(w', \alpha)| \geq |k(w, \alpha)|$ hold for any $\alpha \in \Phi^+$.

Remark. The alcove forms of the elements of the affine Weyl group \widetilde{E}_8 , together with the above results in terms of alcove forms, will be used in proceeding various algorithms of the paper by computer programmes.

3 Left cells and distinguished involutions of \widetilde{E}_8 in $W_{(i)}$, $i = 5, 6$

In the present section, we assume that W_a is an irreducible affine Weyl group of simply-laced type.

3.1

Consider the Condition (C) on $w \in W_a$ below:

(C) $w = x \cdot w_J \cdot y$ for some $x, y \in W_a$ and some $J \subseteq S$ with $\ell(w_J) = a(w)$.

For any left cell Γ and any two-sided cell Ω of W_a , recall the notation $E_{\min}(\Gamma)$ and $F(\Omega)$ defined in 1.2. Define $E(\Gamma) = \{w \in \Gamma \mid a(sw) < a(w), \forall s \in \mathcal{L}(w)\}$. Let $\mathcal{R}_i = \{J \subset S \mid a(w_J) \leq i - 1\}$, $i \geq 1$.

3.2

Theorem. (See [17, Theorem A]) *Suppose that Condition (C) holds on all elements in a left cell Γ of W_a .*

(1) *If $w = w_J \cdot x \in E_{\min}(\Gamma)$ with $J = \mathcal{L}(w)$, then the distinguished involution in Γ has the expression $x^{-1} \cdot w_J \cdot x$.*

(2) *If $E_{\min}(\Gamma) = E(\Gamma)$, then Γ is left-connected.*

3.3

Theorem. (See [17, Theorem B]) *Let Γ be a left cell of W_a with $m := a(\Gamma) \leq 6$.*

(1) *Any $w \in \Gamma$ satisfies Condition (C).*

(2) *Any $w \in E(\Gamma)$ has the form $w = w_J \cdot y$ for some $y \in W_a$ and some $J \subseteq S$ with $\ell(w_J) = m$.*

3.4

Theorem. (See [17, Proposition 2.4]) *For $w \in W_a$, write $w = w_J \cdot w_1$ with $J = \mathcal{L}(w)$ for some $w_1 \in W_a$. If w satisfies $a(w) > \ell(w_J) \leq 5$, then there exists some $s \in J$ such that $sw \underset{L}{\sim} w$.*

The following algorithm is for finding the set $E(\Omega)$ from $F(\Omega)$.

3.5

Algorithm. (See [18, Algorithm 3.4])

(1) Set $Y_0 = F(\Omega)$.

Let $k \geq 0$. Suppose that the set Y_k has been found.

(2) If $Y_k = \emptyset$, then the algorithm terminates.

(3) If $Y_k \neq \emptyset$, then find the set $Y_{k+1} = \{xs \mid x \in Y_k; s \in S \setminus \mathcal{R}(x); xs \in E(\Omega)\}$.

3.6

Theorem. (see [18, Theorem 3.5]) For a two-sided cell Ω of W_a , let Y_j , $j > 0$, be obtained from the set $F(\Omega)$ by Algorithm 3.5.

(1) There exists some $t \in \mathbb{N}$ such that $Y_j \neq \emptyset$ and $Y_h = \emptyset$ for $0 \leq j \leq t < h$;

(2) $E(\Omega) = \bigcup_{k=0}^t Y_k$.

In the step (3) of Algorithm 3.5, it is not so easy in general to determine if the element xs with $x \in Y_k$ and $s \in S \setminus \mathcal{R}(x)$ is in the set $E(\Omega)$. The following result provides a criterion for doing so.

3.7

Theorem. Let Ω be a two-sided cell of W_a with $0 < a(\Omega) = i \leq 6$. Suppose $w \in E(\Omega)$, and $s \in S \setminus \mathcal{R}(w)$. Then the element ws is in $E(\Omega)$ if and only if any right retraction w_1 of tw satisfies $\mathcal{R}(w_1) \in \mathcal{R}_i$ for any $t \in \mathcal{L}(ws)$.

Proof. If $ws \in E(\Omega)$ then $w \underset{R}{\sim} ws$, and for any $t \in \mathcal{L}(ws)$, we have $a(tws) < a(ws) = i$. Hence any right retraction w_1 of tw satisfies $J = \mathcal{R}(w_1) \in \mathcal{R}_i$ by the fact that $a(w_j) \leq a(w_1) \leq a(tws) < i$.

Now assume that any right retraction w_1 of tw satisfies $\mathcal{R}(w_1) \in \mathcal{R}_i$ for any $t \in \mathcal{L}(ws)$. By this assumption on ws , to show that $ws \in E(\Omega)$, we need only to prove that the element ws is in Ω . By 2.5 (2), this is equivalent to proving that $a(ws) = i$.

Let us argue by contrary. If $a(ws) \neq i$, then $a(ws) > i$ by 2.5 (7). Let $ws = w_2 \cdot w_J$ with $J = \mathcal{R}(ws)$ for some $w_2 \in W_a$. If $w_2 = e$, the identity element of W_a , then $ws = w_J$. Hence $s \in \mathcal{L}(ws)$, w^{-1} is a right retraction of $sws = w^{-1}$, but $\mathcal{R}(w^{-1}) \notin \mathcal{R}_i$, a contradiction. This implies $w_2 \neq e$. Let $t \in \mathcal{L}(w_2) \subseteq \mathcal{L}(ws)$, then $J = \mathcal{R}(tw_2 \cdot w_J) = \mathcal{R}(tws) \in \mathcal{R}_i$, hence $a(ws) > \ell(w_J) \leq 5$. Then there exists a sequence s_1, s_2, \dots, s_n in S such that $s_k \in \mathcal{R}(wss_1s_2 \cdots s_{k-1}) \in \mathcal{R}_i$, $wss_1s_2 \cdots s_k \underset{R}{\sim} wss_1s_2 \cdots s_{k-1}$, $1 \leq k \leq n$ and $\mathcal{R}(wss_1s_2 \cdots s_n) \notin \mathcal{R}_i$ by Theorem 3.4. Hence $a(w_I) \geq i$ with $I = \mathcal{R}(wss_1s_2 \cdots s_n)$. Let $y_1 = s_n s_{n-1} \cdots s_1$, then $ws = x_1 \cdot w_I \cdot y_1$ for some $x_1 \in W_a$ and $x_1 \cdot w_I \underset{R}{\sim} ws$. If $x_1 \neq e$, let $t \in \mathcal{L}(x_1) \subseteq \mathcal{L}(ws)$, then $\mathcal{R}(tx_1 \cdot w_I) \notin \mathcal{R}_i$, a contradiction. If $x_1 = e$, then $I \subseteq \mathcal{L}(ws)$. If there exists some

$t \in \mathcal{L}(ws) \setminus \mathcal{L}(w)$, then $tws = w$ and $a(tws) = i$. By Theorem 3.3 (1), $tws = x_2 \cdot w_J \cdot y_2$ for some $x_2, y_2 \in W_a$ and some $J \subset S$ with $\ell(w_J) = i$. Since $x_2 \cdot w_J$ is a right retraction of tws satisfying $\mathcal{R}(x_2 \cdot w_J) \notin \mathcal{R}_i$, contradicting our assumption. This implies that $I \subseteq \mathcal{L}(w)$. Then $a(ws) = a(w_I) \leq a(w)$, a contradiction. Hence $a(ws) = i$, our proof is completed. \square

3.8 Proposition.

Suppose that an element $x \in W_a$ and a left cell Γ of W_a satisfy $|M(x) \cap \Gamma| = n$ for some $1 \leq n \leq \infty$. Let $\Gamma' \in M_L(x)$ and $y \in \Gamma'$.

- (1) $\mathcal{M}_L(y) = \mathcal{M}_L(x)$.
- (2) $|M(x) \cap \Gamma'| = n$.
- (3) $|M(x)|/|M_L(x)| = n$.

Proof. (1) follows by the assumption $y \in \Gamma' \in M_L(x)$ and Proposition 2.17 (b). For (2), we need only to consider the case where $1 \leq n < \infty$. For, suppose that (2) has been proved in the case of $1 \leq n < \infty$. Now assume $|M(x) \cap \Gamma| = \infty$. If (2) in the present case is false, then there exists some $\Gamma' \in M_L(x)$ with $|M(x) \cap \Gamma'| = m < \infty$. Since $\Gamma \in M_L(x)$, this would imply that $|M(x) \cap \Gamma| = m < \infty$ by our assumption with Γ', m, Γ in the place of Γ, n, Γ' , respectively, contradicting our assumption. So (2) is proved in the case of $n = \infty$. Now assume $1 \leq n < \infty$. In the proof of (2), we may assume without loss of generality that $\{\Gamma, \Gamma'\}$ is an edge of $\mathcal{M}_L(x)$.

Let $M(x) \cap \Gamma = \{x_1, x_2, \dots, x_n\}$. Then there are some $1 \leq k \leq n$ and some $z \in M(x) \cap \Gamma'$ such that $\{x_k, z\}$ is an edge of $\mathcal{M}(x)$. Since $x_i, i = 1, \dots, n$, have the same right generalized τ -invariants by Proposition 2.17 (a), there is a sequence of pairwise different elements z_1, \dots, z_n in $M(x)$ such that $\mathcal{R}(z_i) = \mathcal{R}(\Gamma')$ and $\{x_i, z_i\}$ is an edge of $\mathcal{M}(x)$ for every $1 \leq i \leq n$, where $z_k = z$. Then $z_1, \dots, z_n \in \Gamma'$ by Proposition 2.8 (2). If there is some $z' \in (M(x) \cap \Gamma') \setminus \{z_1, \dots, z_n\}$, then there is some $x' \in (M(x) \cap \Gamma) \setminus \{x_1, \dots, x_n\}$ with $\{x', z'\}$ an edge of $\mathcal{M}(x)$ by Propositions 2.8 (2) and 2.17 (a), contradicting the assumption of $|M(x) \cap \Gamma| = n$. This proves the equation $|M(x) \cap \Gamma'| = n$ and hence (2) is proved.

Finally, (3) follows immediately by (2). \square

Next we state an algorithm for obtaining the set $M(x)$ from any $x \in W_a$.

3.9

Algorithm. (1) Set $M_0 = \{x\}$.

Let $k \geq 0$. Suppose that the set M_j has been found for any $j \leq k$.

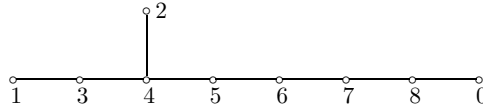
(2) If $M_k = \emptyset$, then the algorithm terminates.

(3) If $M_k \neq \emptyset$, then find the set M_{k+1} of all $ws \in W_a$ with $w \in M_k$ and $s \in S$ such that $ws \notin M_i, \forall i \leq k$ and $\{w, ws\}$ is a right string.

Then $M(x) = \cup_{k \geq 0} M_k$. If $M(x)$ is a finite set, then we can get the set $M(x)$ by this algorithm in a finite number of steps.

3.10

From now on, we concentrate on the affine Weyl group \widetilde{E}_8 . Let $s_i, 0 \leq i \leq 8$, be the Coxeter generators of \widetilde{E}_8 whose labels coincide with the nodes in the following Coxeter graph.



In the subsequent discussion, we denote by \mathbf{i} (boldfaced) the generator s_i in \widetilde{E}_8 for simplifying the notation.

In \widetilde{E}_8 , the set $W_{(5)}$ is a single two-sided cell, while $W_{(6)}$ is a union of two two-sided cells (say Ω_6^1, Ω_6^2) by Theorem 2.6, where we assume $\mathbf{131257} \in \Omega_6^1$. Denote $E(5) = E(W_{(5)})$ and $E(6)^i = E(\Omega_6^i), i = 1, 2$ (see 1.2).

Let

$$P(5) = \{07523, 07521, 13125, 13126, 13127, 13128, 13120, 13157, 13158, 13150, 13168, 13160, 13170, 24216, 24217, 24218, 24210, 24268, 24260, 24270, 34368, 34360, 34370, 45417, 45418, 45410, 45470, 56512, 56518, 56510, 56523, 56528, 56520, 56538, 56530, 67601, 67602, 67603, 67604, 67612, 67614, 67623, 78712, 78714, 78715, 78723, 78725, 78735, 80812, 80814, 80815, 80816, 80823, 80825, 80826, 80835, 80836, 80846\}.$$

$$P(6)^1 = \{131257, 131258, 131250, 131268, 131260, 131270, 131570, 131565, 131676, 131787, 131808, 242168, 242160, 242170, 242676, 242787, 242808, 343676, 343787, 343808, 454170, 454787, 454808, 565128, 565120, 565238, 565230, 565808, 676120, 676140, 676230, 787125, 787235, 808125, 808126, 808235, 808236, 808146\}.$$

$$P(6)^2 = \{143143, 234234, 254254, 354354, 465465, 576576, 687687, 708708\}.$$

The set $P(5)$ (respectively, $P(6)^1 \cup P(6)^2$) consists of all the elements of \widetilde{E}_8 in $W_{(5)}$ (respectively, $W_{(6)}$) of the form w_I for some $I \subset S$. Let \overline{S} be the power set of S . Define $\mathcal{L}(6)^i = \{J \in \overline{S} \mid w_J \in P(6)^i\}, i = 1, 2$. For any $X \subseteq \widetilde{E}_8$, define $R(X) = \{\mathcal{R}(y) \in \overline{S} \mid y \in X\}$. Let Γ_w be the left cell of \widetilde{E}_8 containing w for $w \in \widetilde{E}_8$. Our computational results show that $\mathcal{L}(6)^1 \subseteq R(M(\mathbf{131257}))$ and $\mathcal{L}(6)^2 \subseteq R(M(\mathbf{143143}))$. This implies that $\{\Gamma_{w_J} \mid w_J \in P(6)^1\} \subseteq M_L(\mathbf{131257})$ and

$\{\Gamma_{w_J} \mid w_J \in P(6)^2\} \subseteq M_L(\mathbf{143143})$ by 2.5 (5). So each of the sets $P(6)^i$, $i = 1, 2$, is contained in a two-sided cell of \tilde{E}_8 . Since $\mathbf{131257} \in \Omega_6^1 \cap P(6)^1$ by our assumption, we have $P(6)^1 \subseteq \Omega_6^1$. Since $F(\Omega_6^2) \cap (P(6)^1 \cup P(6)^2) \neq \emptyset$ by Theorem 3.3 (2), we must have $\Omega_6^2 \cap P(6)^2 \neq \emptyset$. So $P(6)^2 \subseteq \Omega_6^2$. Hence $F(W_{(5)}) = P(5)$ and $F(\Omega_6^i) = P(6)^i$, $i = 1, 2$, by Theorem 3.3 (2).

3.11

We can get the set $E(5)$ (respectively, $E(6)^i$, $i = 1, 2$) from $Y_0 = P(5)$ (respectively, $Y_0 = P(6)^i$, $i = 1, 2$) by applying Algorithm 3.5. The most difficult part in the preceding the algorithm is the third step, i.e., finding the set Y_{k+1} from Y_k . Theorem 3.7 provides us with a powerful technical tool in doing so, since, by the computer programme MATLAB, we can easily check, for any ws with $w \in Y_k$ and $s \in S \setminus \mathcal{R}(w)$, the condition that any right retraction w_1 of tw_s satisfies $\mathcal{R}(w_1) \in \mathcal{R}_5$ (respectively, $\mathcal{R}(w_1) \in \mathcal{R}_6$) for every $t \in \mathcal{L}(ws)$.

By Theorem 3.6, the algorithm must terminate after a finite number of steps. We get the set $E(5) = \bigcup_{k=0}^n Y_k$ for some $n \in \mathbb{N}$ (respectively, $E(6)^1 = \bigcup_{k=0}^m Y_k$, $E(6)^2 = \bigcup_{k=0}^r Y_k$ for some $m, r \in \mathbb{N}$) (see [2, Appendix I]).

3.12

In \tilde{E}_8 , only the generator \mathbf{o} is special (see 2.5 (5)). Denote by Γ^0 (respectively, Γ^1, Γ^2) the unique left cell Γ of \tilde{E}_8 in $W_{(5)}$ (respectively, Ω_6^1, Ω_6^2) with $\mathcal{R}(\Gamma) = \{\mathbf{o}\}$. Define

$$R(5) = \{\{2, 3, 5, 7, \mathbf{o}\}, \{\mathbf{o}\}, \{1, 2, 3, 5\}\},$$

$$R(6)^1 = \{\{\mathbf{o}\}, \{1, 3, 5, 6\}, \{1, 3, 5, 7, \mathbf{o}\}\},$$

$$R(6)^2 = \{\{\mathbf{o}\}, \{1, 3, 4\}\};$$

$$U = \{\Gamma_{23570}, \Gamma^0, \Gamma_{13125}, \Gamma^1, \Gamma_{131565}, \Gamma_{131570}, \Gamma^2, \Gamma_{143143}\}.$$

For any $X \subseteq \tilde{E}_8$ and any $n \in \mathbb{N}$, define $X_n = \{x \in X \mid |M(x)| = n\}$.

Define $\hat{E} = \{E(5)_n, E(6)_i^1, E(6)_j^2 \mid n \in \{50, 135, 1650\}, i \in \{50, 135, 990, 1030\}, j \in \{50, 135, 1735\}\}$, $E' = \hat{E} \setminus \{E(6)_{50}^i \mid i = 1, 2\}$.

The following computational results are concerned with the elements x in $E(5) \cup E(6)^1 \cup E(6)^2$ and the related sets $M(x)$, $R(M(x))$.

3.13 Proposition.

(a) For any $x \in E(5)$, we have $|M(x)| \in \{50, 135, 1650\}$.

(a1) If $|M(x)| = 50$, then $R(M(x)) \cap R(5) = \{\{2, 3, 5, 7, \mathbf{o}\}\}$ and there exists a unique $y \in M(x)$ with $\mathcal{R}(y) = \{2, 3, 5, 7, \mathbf{o}\}$.

(a2) If $|M(x)| = 135$, then $R(M(x)) \cap R(5) = \{\{\mathbf{o}\}\}$ and there exists a unique $y \in M(x)$ with $\mathcal{R}(y) = \{\mathbf{o}\}$.

(a3) If $|M(x)| = 1650$, then $R(M(x)) \cap R(5) = \{\{1, 2, 3, 5\}\}$ and there exist exactly two elements (say y, z) in $M(x)$ such that $\mathcal{R}(y) = \mathcal{R}(z) = \{1, 2, 3, 5\}$.

- (b) For any $x \in E(6)^1$, we have $|M(x)| \in \{50, 135, 990, 1030\}$.
 - (b1) If $|M(x)| = 50$, then $R(M(x)) \cap R(6)^1 = \emptyset$.
 - (b2) If $|M(x)| = 135$, then $R(M(x)) \cap R(6)^1 = \{\{0\}\}$ and there exists a unique $y \in M(x)$ with $\mathcal{R}(y) = \{0\}$.
 - (b3) If $|M(x)| = 990$, then $R(M(x)) \cap R(6)^1 = \{\{1, 3, 5, 6\}\}$ and there exist exactly two elements (say y, z) in $M(x)$ such that $\mathcal{R}(y) = \mathcal{R}(z) = \{1, 3, 5, 6\}$.
 - (b4) If $|M(x)| = 1030$, then $R(M(x)) \cap R(6)^1 = \{\{1, 3, 5, 7, 0\}\}$ and there exists a unique $y \in M(x)$ with $\mathcal{R}(y) = \{1, 3, 5, 7, 0\}$.
- (c) For any $x \in E(6)^2$, we have $|M(x)| \in \{50, 135, 1735\}$.
 - (c1) If $|M(x)| = 50$, then $R(M(x)) \cap R(6)^2 = \emptyset$.
 - (c2) If $|M(x)| = 135$, then $R(M(x)) \cap R(6)^2 = \{\{0\}\}$ and there exists a unique $y \in M(x)$ with $\mathcal{R}(y) = \{0\}$.
 - (c3) If $|M(x)| = 1735$, then $R(M(x)) \cap R(6)^2 = \{\{1, 3, 4\}\}$ and there exists a unique $y \in M(x)$ with $\mathcal{R}(y) = \{1, 3, 4\}$.

The next result is concerned with the sets $M(x)$ and $M_L(x)$ for $Y \in \widehat{E}$ (in particular, $Y \in E'$) and $x \in Y$.

3.14 Proposition.

- (a) Let $Y \in E'$ and $x, y \in Y$. Then $U \cap M_L(x) = U \cap M_L(y)$ and $|U \cap M_L(x)| = 1$. We have $|M(x) \cap \Gamma| = |M(y) \cap \Gamma| \in \{1, 2\}$ for any $\Gamma \in U \cap M_L(x)$. Further, we have $\mathcal{M}_L(x) = \mathcal{M}_L(y)$. For any $z \in E(6)_{50}^1 \cup E(6)_{50}^2$, we have $U \cap M_L(z) = \emptyset$.
- (b) Let x, y be in $E(5)$ (respectively, $E(6)^1, E(6)^2$) with $|M(x)| \neq |M(y)|$. Then x, y have different right generalized τ -invariants. Hence $x \not\sim_L y$. For any $Y \in \widehat{E}$ and $z \in Y$, we have $E(\Gamma_z) \subset Y$ and $Y = \bigcup_{w \in Y} E(\Gamma_w)$.
- (c) Let $Y \in E'$ and $x \in Y$. If $|M(x)| \in \{50, 135, 1030, 1735\}$, then $|M_L(x)| = |M(x)|$; if $|M(x)| \in \{1650, 990\}$, then $|M_L(x)| = \frac{1}{2}|M(x)|$.
- (d) Let $Y \in E'$ and $x \in Y$. Then $\{\Gamma_w \mid w \in Y\} = M_L(x)$. Further, $|\{E(\Gamma_w) \mid w \in Y\}| = |M_L(x)|$.

Proof: (a) is a consequence of 2.5 (5) (6) and Propositions 3.13 and 3.8. Then (b) follows by Propositions 3.13 and 2.17 (a). We get (c) from (a) by Propositions 3.8 and 3.13. Now it remains to prove (d). We have $\{\Gamma_w \mid w \in Y\} \subseteq M_L(x)$ by (a). Take $y \in E(\Gamma_z)$ for any $z \in M(x)$. Then $R(M(y)) = R(M(z)) = R(M(x))$ by Proposition 2.17 (a). So $y \in Y$ by Proposition 3.13. Then $\Gamma_z \in \{\Gamma_w \mid w \in Y\}$. Hence $M_L(x) \subseteq \{\Gamma_w \mid w \in Y\}$.

3.15

Let $Y \in E'$. By Proposition 3.14 (a), (d), there is a unique left cell (say Γ) of \widetilde{E}_8 in $U \cap \{\Gamma_w \mid w \in Y\}$. For any $x, y \in Y$, we have $M(x) \cap \Gamma = \{x_1, \dots, x_n\}$ and $M(y) \cap \Gamma = \{y_1, \dots, y_n\}$, for some $n \in \{1, 2\}$ by Proposition 3.14 (a). If $x \not\sim_L y$, then

x, y satisfy the Condition (*) below by Propositions 2.17 (a) and 3.13:

(*) There are paths $z_0 = x, z_1, \dots, z_r = x_k$ in $\mathcal{M}(x)$ and $z'_0 = y, z'_1, \dots, z'_r = y_j$ in $\mathcal{M}(y)$ with some k, j satisfying $1 \leq k, j \leq n$ such that $\mathcal{R}(z'_i) = \mathcal{R}(z_i)$ for every $0 \leq i \leq r$.

Conversely, if $x, y \in Y$ satisfy the Condition (*), then $x \underset{L}{\sim} y$ by Proposition 2.8 (2).

Next we state an algorithm to partition any set $Y \in E'$ into its subsets $E(\Gamma_w)$, $w \in Y$.

3.16

Algorithm.

(1) Take any $y \in Y$.

We find the set Y_1 of all $x \in Y$ such that x, y satisfy the Condition (*).

Let $k \geq 1$. Suppose that all the sets $Y_j, j \leq k$, have been found.

(2) If $Y \setminus (\bigcup_{j=1}^k Y_j) = \emptyset$, then the algorithm terminates.

(3) If $Y \setminus (\bigcup_{j=1}^k Y_j) \neq \emptyset$, then take any $z \in Y \setminus (\bigcup_{j=1}^k Y_j)$ and then find all such $x \in Y \setminus (\bigcup_{j=1}^k Y_j)$ that x, z satisfy the Condition (*). Let Y_{k+1} be the set of all such elements x .

The algorithm must terminate after a finite number of steps by Proposition 3.14 (b) and 3.15. Hence $Y = \bigcup_{j=1}^m Y_j$ and $\{Y_j \mid 1 \leq j \leq m\} = \{E(\Gamma_w) \mid w \in Y\}$ for some $m \in \mathbb{N}$.

3.17

Define

$$\begin{aligned} A &= \{E(\Gamma_y) \mid y \in E(5)_{50}\}, & B &= \{E(\Gamma_y) \mid y \in E(5)_{135}\}, \\ C &= \{E(\Gamma_y) \mid y \in E(5)_{1650}\}, & D &= \{E(\Gamma_y) \mid y \in E(6)_{1030}^1\}, \\ E &= \{E(\Gamma_y) \mid y \in E(6)_{990}^1\}, & F &= \{E(\Gamma_y) \mid y \in E(6)_{135}^1\}, \\ H &= \{E(\Gamma_y) \mid y \in E(6)_{1735}^2\}, & J &= \{E(\Gamma_y) \mid y \in E(6)_{135}^2\}. \end{aligned}$$

Let $X_{I'} = \{w \mid w \in X \text{ with } \mathcal{R}(w) = I'\}$ for any $I' \subset S$ and $X \subset \tilde{E}_8$.

Let $Y = E(6)_{50}^i$ and $x \in Y$ for some $i \in \{1, 2\}$. Then our computational results show that $\mathcal{M}(x) \simeq \mathcal{M}(\mathbf{07523})$ and $R(Y) = R(M(\mathbf{07523}))$. We have $\mathcal{M}(\mathbf{07523}) \simeq \mathcal{D}(\mathbf{07523})$, so $\mathcal{M}(x) \simeq \mathcal{D}(\mathbf{07523})$ (see 2.13). From the graph $\mathcal{D}(\mathbf{07523})$ (see 4.10), we have $|\mathcal{M}(x)_{I'}| = 1$ if $I' \in R(M(\mathbf{07523})) \setminus \{\{4, 7\}, \{4, 8\}\}$ and $|\mathcal{M}(x)_{I'}| = 2$ if $I' \in \{\{4, 7\}, \{4, 8\}\}$. Define

$$\begin{aligned} Y_{\{4,7\}^1} &= \{w \mid w \in Y_{\{4,7\}}, \mathcal{R}(ws) = \{4, 8\}, \text{ for some } s \in S\}, \\ Y_{\{4,7\}^2} &= Y_{\{4,7\}} \setminus Y_{\{4,7\}^1}, \\ Y_{\{4,8\}^1} &= \{w \mid w \in Y_{\{4,8\}}, \mathcal{R}(ws) = \{4, 7\}, \text{ for some } s \in S\} \text{ and} \\ Y_{\{4,8\}^2} &= Y_{\{4,8\}} \setminus Y_{\{4,8\}^1}. \end{aligned}$$

Define $G =$
 $\left\{ (E(6)_{50}^1)_{I'}, (E(6)_{50}^1)_{\{4,7\}^i}, (E(6)_{50}^1)_{\{4,8\}^i} \mid I' \in R(E(6)_{50}^1) \setminus \{\{4, 7\}, \{4, 8\}\}, i = 1, 2 \right\}$
 and $I =$
 $\left\{ (E(6)_{50}^2)_{I'}, (E(6)_{50}^2)_{\{4,7\}^i}, (E(6)_{50}^2)_{\{4,8\}^i} \mid I' \in R(E(6)_{50}^2) \setminus \{\{4, 7\}, \{4, 8\}\}, i = 1, 2 \right\}.$

Then $|G| = |I| = 50$. Let K be in G (respectively, I) and $w \in K$, we can see that $E(\Gamma_w) \subseteq K$ by Propositions 2.17 (a) and 3.14 (b).

We have

$$E(5) = \bigcup_{K \in A \cup B \cup C} K, \quad E(6)^1 = \bigcup_{K \in D \cup E \cup F \cup G} K, \quad E(6)^2 = \bigcup_{K \in H \cup I \cup J} K.$$

By Proposition 3.14 (c) (d), we see that the number of left cells of \tilde{E}_8 in $W_{(5)}$ is equal to $|A| + |B| + |C| = 50 + 135 + 825 = 1010$.

3.18

The elements of $E(5)$ and $E(6)^1$ are listed in [2, Appendix I]. More precisely, the elements in the sets $\bigcup_{K \in X} K$ are listed in [2, Appendix I (X)] for $X \in \{A, B, C, D, E, F, G\}$. To explain the related notation, we take Table 1 below as an example. Table 1 is a part of [2, Appendix I (B)], where $B = \{B_i \mid 1 \leq i \leq 135\}$ (see 3.17).

B_1	2357	5652342357	6762345423		
B_8	2350	565234237805	676234542803	787234654203	808234765423

Table 1

The set $\mathcal{R}(B_i)$, together with the elements in B_i , are listed after the symbol B_i in turn. Hence, for example, after the symbol B_1 , we should read “ 2357 ” as $\mathcal{R}(B_1) = \{2, 3, 5, 7\}$ and “ 5652342357 6762345423 ” as $B_1 = \{5652342357, 6762345423\}$; also, after the symbol B_8 , we read “ 2350 ” as $\mathcal{R}(B_8) = \{2, 3, 5, 0\}$ and “ 565234237805 676234542803 787234654203 808234765423 ” as $B_8 = \{565234237805, 676234542803, 787234654203, 808234765423\}$.

The following proposition follows directly by our computational results in [2, Appendix I (X)] for $X \in \{A, B, C, D, E, F, G\}$.

3.19 Proposition.

All the elements of K have the same length for any $K \in Y$ with $Y \in \{A, B, C, D, E, F, G\}$.

3.20

For any $w = w_J \cdot x \in \tilde{E}_8$ with $J = \mathcal{L}(w)$, define $b_w = x^{-1}w_Jx$.

3.21 Proposition.

$K = E(\Gamma_w)$ for any $w \in K$ with $K \in G$.

Proof: We have $K \supseteq E(\Gamma_w)$ in general by 3.17.

For any $y \in K$, we have $E_{\min}(\Gamma_y) = E(\Gamma_y)$ by Proposition 3.19 and hence b_y is the distinguished involution of Γ_y by Theorems 3.2 (1) and 3.3 (1). Since $b_y = b_w$ for any $y \in K$ by our computational results, the set K is contained in a left cell of \tilde{E}_8 by 2.5 (3). Hence $K \subseteq E(\Gamma_w)$. \square

3.22

By 3.17, Propositions 3.14 (c), (d) and 3.21, we see that the number of left cells of \tilde{E}_8 in Ω_6^1 is equal to $|D| + |E| + |F| + |G| = 1030 + 495 + 135 + 50 = 1710$.

3.23

The elements of $E(6)^2$, together with the proof for the left-connectedness of left cells of \tilde{E}_8 containing them, are displayed in [2, Appendix I (H) (I) (J)].

Let $X \subseteq \tilde{E}_8$. For any $x, y \in X$, we denote $x \xrightarrow{X} y$ if there exists a sequence $x_0 = x, x_1, \dots, x_r = y$ in X such that $x_{i-1}x_i^{-1} \in S$ for every $1 \leq i \leq r$. This defines an equivalence relation in the set X . The corresponding equivalence classes of X are called *left-connected components* of X .

Given a left cell Γ of \tilde{E}_8 . It is easily seen that for any $x' \in \Gamma$, there is some $x \in E(\Gamma)$ such that $x' \xrightarrow{\Gamma} x$. So to show Γ is left-connected, one need only prove that any $x, y \in E(\Gamma)$ satisfy $x \xrightarrow{\Gamma} y$.

Now we take some examples to illustrate how our proof proceeds. Table 2 below is a part of [2, Appendix I (H)]. All the elements in the set $\bigcup_{i=1}^{1735} H_i$ are listed in [2, Appendix I (H)], where $H = \{H_1, H_2, \dots, H_{1735}\}$.

H_{227}	146	$w_1=2342341546$				
		$w_2=14314324546$				
		$2w_2=31w_1$				
		(1) w_2	134	(2)	$2w_2$	123
		$w_3=354354123456$				
		$421w_3=24531w_1$				
H_{233}	156	$w_1=23423415465$				
		$w_2=143143245465$				
		$2w_2=31w_1$				
		(1) w_3	345	(2)	$1w_3$	145
		(3) $21w_3$	125	(4)	$421w_3$	124
		\vdots				
H_{233}	156	$w_1=23423415465$				
		$w_2=143143245465$				
		$2w_2=31w_1$				
		(1) w_2	134	(2)	$2w_2$	123

$w_3 = \mathbf{3543541234565}$			
$\mathbf{421}w_3 = \mathbf{24531}w_1$			
(1) w_3	345	(2) $1w_3$	145
(3) $21w_3$	125	(4) $421w_3$	124
$w_4 = \mathbf{46546531423456}$			
$\mathbf{54132}w_4 = \mathbf{45624531}w_1$			
(1) w_4	456	(2) $2w_4$	256
(3) $32w_4$	2356	(4) $132w_4$	1256
(5) $4132w_4$	146	(6) $54132w_4$	145
(2) $2w_4$	256	(3) $32w_4$	2356
$22w_4$	456	$432w_4$	46
	⋮		

Table 2

The set $\mathcal{R}(H_i)$ occurs immediately after the symbol H_i , then the elements of the set H_i occur next, which are expressed as w_1, w_2, \dots , or u_1, u_2, \dots , etc. The other data displayed after the symbol H_i are for the proof of the left-connectedness of the left cell Γ of \tilde{E}_8 containing H_i . The proof is accomplished by showing that any two elements x, y of H_i satisfy $x \xrightarrow[\Gamma]{} y$. Let us take two examples to illustrate how this proceeds.

(i) The boldfaced numbers **146** occur immediately after the symbol H_{227} should be read as $\mathcal{R}(H_{227}) = \{\mathbf{1, 4, 6}\}$. Then the elements $w_1 = \mathbf{2342341546}$, $w_2 = \mathbf{14314324546}$, $w_3 = \mathbf{354354123456}$ occurring after **146** form the set H_{227} . The data

$$\text{“ } 2w_2 = 31w_1 \text{ ”}$$

$$(1) w_2 \quad \mathbf{134} \quad (2) 2w_2 \quad \mathbf{123} \text{ ”}$$

means that the element $2 \cdot w_2 = 31 \cdot w_1$ is a common left-extension of the elements w_1, w_2 and that $\mathcal{L}(w_2) = \{\mathbf{1, 3, 4}\}$, $\mathcal{L}(2w_2) = \{\mathbf{1, 2, 3}\}$. These data show that $\{w_2, 2w_2\}$ form a left $\{\mathbf{2, 4}\}$ -string and hence w_1, w_2 satisfy $w_1 \xrightarrow[\Gamma]{} w_2$ by the equations $2w_2 = 31w_1$, $a(w_1) = a(w_2) = 6$, the assertion 2.5 (2) and Proposition 2.8. On the other hand, the data

$$\text{“ } \mathbf{421}w_3 = \mathbf{24531}w_1 \text{ ”}$$

$$(1) w_3 \quad \mathbf{345} \quad (2) 1w_3 \quad \mathbf{145}$$

$$(3) 21w_3 \quad \mathbf{125} \quad (4) \mathbf{421}w_3 \quad \mathbf{124} \text{ ”}$$

means that the element $\mathbf{421} \cdot w_3 = \mathbf{24531} \cdot w_1$ is a common left-extension of the elements w_1, w_3 and that $\mathcal{L}(w_3) = \{\mathbf{3, 4, 5}\}$, $\mathcal{L}(1w_3) = \{\mathbf{1, 4, 5}\}$, $\mathcal{L}(21w_3) = \{\mathbf{1, 2, 5}\}$, $\mathcal{L}(\mathbf{421}w_3) = \{\mathbf{1, 2, 4}\}$. These data show that $\{w_3, 1w_3\}$, $\{1w_3, 21w_3\}$, $\{21w_3, \mathbf{421}w_3\}$ are three left strings and hence the elements w_1, w_3 satisfy $w_1 \xrightarrow[\Gamma]{} w_3$ by the equations $\mathbf{421} \cdot w_3 = \mathbf{24531} \cdot w_1$, $a(w_1) = a(w_3) = 6$, the assertion 2.5 (2) and Proposition 2.8.

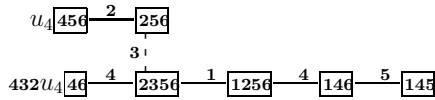
Since the set $H_{227} = E(\Gamma) = \{w_1, w_2, w_3\}$ is contained in a left-connected component of Γ , this proves that Γ is left-connected.

(ii) The boldfaced numbers **156** immediately after the symbol H_{233} should be read

as $\mathcal{R}(H_{233}) = \{1, 5, 6\}$. The elements $u_1 = 23423415465$, $u_2 = 143143245465$, $u_3 = 3543541234565$, $u_4 = 46546531423456$ occurring after 156 form the set H_{233} . From the following data

- “ $2u_2 = 31u_1$
 (1) u_2 134 (2) $2u_2$ 123
 $421u_3 = 24531u_1$
 (1) u_3 345 (2) $1u_3$ 145
 (3) $21u_3$ 125 (4) $421u_3$ 124
 $54132u_4 = 45624531u_1$
 (1) u_4 456 (2) $2u_4$ 256
 (3) $32u_4$ 2356 (4) $132u_4$ 1256
 (5) $4132u_4$ 146 (6) $54132u_4$ 145
 (7) $432u_4$ 46 ”

we can observe the relations $u_1 \xrightarrow[\Gamma]{} u_2$, $u_1 \xrightarrow[\Gamma]{} u_3$ and $u_1 \xrightarrow[\Gamma]{} u_4$ hold, where Γ is the left cell of \tilde{E}_8 containing the set H_{233} . The relations $u_1 \xrightarrow[\Gamma]{} u_2$ and $u_1 \xrightarrow[\Gamma]{} u_3$ can be observed in the way similar to that in (i). Now we consider the relation between u_1 and u_4 . The above data concerning this pair can be interpreted by the following graph,



where the vertex (say v) labeled by $\boxed{456}$ represents the element u_4 with $\mathcal{L}(u_4) = \{4, 5, 6\}$, all the elements represented by the other vertices (say v') of the graph can be written out according to the labels of the path connecting v and v' . For example, the element labeled by $\boxed{145}$ is $54132 \cdot u_4$ with $\mathcal{L}(54132 \cdot u_4) = \{1, 4, 5\}$ since the labels of the edges on the path connecting these two vertices are 2, 3, 1, 4, 5 in turn. We see that each of the pairs $\{u_4, 2 \cdot u_4\}$, $\{32 \cdot u_4, 132 \cdot u_4\}$, $\{132 \cdot u_4, 4132 \cdot u_4\}$, $\{4132 \cdot u_4, 54132 \cdot u_4\}$ are left strings and that the pair $\{2 \cdot u_4, 32 \cdot u_4\}$ is a left primitive pair, the latter can be observed from the subgraph with the vertices labeled by $\boxed{456}$, $\boxed{256}$, $\boxed{2356}$, $\boxed{46}$, respectively. This implies that $u_4 \xrightarrow[\Gamma]{} 54132 \cdot u_4$ by Proposition 2.10. Since $u := 54132 \cdot u_4$ is a common left-extension of the elements u_4 and u_1 by the equation $54132 \cdot u_4 = 45624531 \cdot u_1$, this implies $u_1 \xrightarrow[\Gamma]{} u_4$ by the fact $u, u_1, u_4 \in \Gamma$ and by 2.5 (2) and Proposition 2.8.

Since $H_{233} = E(\Gamma) = \{u_1, u_2, u_3, u_4\}$ is contained in a left-connected component of Γ , this proves that Γ is left-connected.

3.24

By our computational results (see [2, Appendix I (H) (I) (J)]), we observe that for any K in $H \cup I \cup J$, the set K contains a unique shortest element (say x). By the data in [2, Appendix I (H) (I) (J)] and the arguments similar to that in 3.23 (i), we see that for any $y \in K \setminus \{x\}$, there exists a sequence $x_0 = x, x_1, \dots, x_r = y$ in $W_{(6)}$ such that $x_i x_{i-1}^{-1} \in S$ for $1 \leq i \leq r$. This implies that K is contained in a left-connected component of $W_{(6)}$. We know that any left-connected component of $W_{(6)}$ is contained in a left cell (denote it by Γ_x) of the group. Then we have $x \xrightarrow[\Gamma_x]{} y$.

3.25 Proposition.

Let x the unique shortest element of K with $K \in I$. Then $K = E(\Gamma_x)$.

Proof: This is because $K \supseteq E(\Gamma_x)$ in general by 3.17 and $K \subseteq E(\Gamma_x)$ by 3.24. \square

3.26

By 3.17, Propositions 3.14 (c), (d) and 3.25, we can see that the number of left cells in Ω_6^2 is equal to $|H| + |I| + |J| = 1735 + 50 + 135 = 1920$.

Lusztig conjectured in [1] that if W is an affine Weyl group then any left cell of W is left-connected. The conjecture was supported by all the existing data. Now we have:

3.27 Theorem.

Any left cell Γ of \tilde{E}_8 in $W_{(i)}$, $i = 5, 6$, is left-connected.

Proof: First assume $\Gamma \subset W_{(5)} \cup \Omega_6^1$. Then $E(\Gamma) \in A \cup B \cup C \cup D \cup E \cup F \cup G$ by 3.17 and Proposition 3.21. We have $E(\Gamma) = E_{\min}(\Gamma)$ by Proposition 3.19. Then Γ is left-connected by Theorems 3.3 (1) and 3.2 (2).

Next assume $\Gamma \subset \Omega_6^2$. Then $E(\Gamma) \in H \cup I \cup J$ by 3.17 and Proposition 3.25. Hence $E(\Gamma)$ contains a unique shortest element w by 3.24. For any $x, y \in \Gamma$, there are some $x_0, y_0 \in E(\Gamma)$ such that $x \xrightarrow[\Gamma]{} x_0$ and $y \xrightarrow[\Gamma]{} y_0$. We have $w \xrightarrow[\Gamma]{} x_0$ and $w \xrightarrow[\Gamma]{} y_0$ by 3.24. So $x \xrightarrow[\Gamma]{} y$. This proves that Γ is left-connected. \square

4 The left cell graphs and the distinguished involution graphs of \tilde{E}_8 in $W_{(i)}$, $i = 5, 6$

4.1

A set $Q \subset \tilde{E}_8$ is called a *representative set for the left cells* (or an *l.c.r set* for short) of \tilde{E}_8 in a two-sided cell Ω if $Q \subseteq \Omega$ and $|Q \cap \Gamma| = 1$ for any left cell Γ of \tilde{E}_8 in Ω .

Now we state an algorithm for obtaining the set $D(d)$ from any $d \in \mathcal{D}_0$.

4.2

Algorithm.

(1) Let $D_0 = \{d\}$.

Let $k \geq 0$. Suppose that the set D_j has been found for any $0 \leq j \leq k$.

(2) If $D_k = \emptyset$, then the algorithm terminates.

(3) If $D_k \neq \emptyset$, then find the set D_{k+1} of all $d' \in \tilde{E}_8 \setminus \left(\bigcup_{i=0}^k D_i\right)$ such that d' can be obtained from some $d_1 \in D_k$ by the two-sided $\{s, t\}$ -star operations, where $s, t \in S$ satisfy $o(st) = 3$ and $|\{s, t\} \cap \mathcal{L}(d_1)| = 1$.

By Proposition 2.12 and by the recurrence procedure, we see that all the elements obtained from the above algorithm are distinguished involutions of \tilde{E}_8 and that $D(d) = \bigcup_{k \geq 0} D_k$.

Since $|D(d)| \leq |D_0| < \infty$ by 2.5 (3), the algorithm must terminate after a finite number of steps.

For any $d, d' \in \mathcal{D}_0$, we have either $D(d) = D(d')$ or $D(d') \cap D(d) = \emptyset$ by Proposition 2.12.

4.3

Denote $x_1 = 07523$, $x_2 = 5652342357$, $x_3 = 13125$, $y_1 = 131570$, $y_2 = 131565$, $y_3 = 13125742354654237$, $y_4 = 56523042357$, $z_1 = 143143$, $z_2 = 25425431436542765870$ and $z_3 = 143143254365427654318765423408765423$. Then x_1 (respectively, $x_2, x_3, y_1, y_2, y_3, y_4, z_1, z_2, z_3$) is the shortest element of $E(5)_{50}$ (respectively, $E(5)_{135}, E(5)_{1650}, E(6)_{1030}^1, E(6)_{990}^1, E(6)_{135}^1, E(6)_{50}^1, E(6)_{1735}^2, E(6)_{50}^2, E(6)_{135}^2$). The elements x_1, x_3, y_1, y_2, z_1 are in \mathcal{D}_0 by 2.5 (4). Any $w \in \{x_2, y_3, y_4, z_2, z_3\}$ is in $E_{\min}(\Gamma_w)$ by Proposition 3.14 (b). So $b_w \in \mathcal{D}_0$ by Theorems 3.2 (1) and 3.3 (1).

We have $|D(x_1)| = 50$, $|D(b_{x_2})| = 135$, $|D(x_3)| = 825$, $|D(y_1)| = 1030$, $|D(y_2)| = 495$, $|D(b_{y_3})| = 135$, $|D(b_{y_4})| = 50$, $|D(z_1)| = 1735$, $|D(b_{z_2})| = 50$ and $|D(b_{z_3})| = 135$ by our computational results in applying Algorithm 4.2. Let $D(5) = D(x_1) \sqcup D(b_{x_2}) \sqcup D(x_3)$, $D(6)^1 = D(y_1) \sqcup D(y_2) \sqcup D(b_{y_3}) \sqcup D(b_{y_4})$ and $D(6)^2 = D(z_1) \sqcup D(b_{z_2}) \sqcup D(b_{z_3})$. Then $|D(5)| = 1010$, $|D(6)^1| = 1710$ and $|D(6)^2| = 1920$ by our computational results. The set $D(5)$ (respectively, $D(6)^1, D(6)^2$) consists of all the distinguished involutions in the two-sided cell $W_{(5)}$ (respectively, Ω_6^1, Ω_6^2) and also forms an l.c.r set of it by 2.5 (3).

Let $\overline{D} = \{x_1, b_{x_2}, x_3, y_1, y_2, b_{y_3}, b_{y_4}, z_1, b_{z_2}, b_{z_3}\}$.

We have the following conclusion from our computational results.

4.4 Proposition.

If two edges of a distinguished involution graphs $\mathcal{D}(w)$ with $w \in \overline{D}$ are parallel (see

2.13) then they have the same label.

4.5 Remark.

By Proposition 4.4, we need only to display the label of some edge (say α) but to hide the labels of all the other edges in each parallel class (say \mathcal{X}) of edges in a displayed distinguished involution graph for simplifying the graph since those implicit labels of the edges in $\mathcal{X} \setminus \{\alpha\}$ can easily be recovered by Proposition 4.4.

4.6

We have $\mathcal{M}_L(w) \simeq \mathcal{D}(w)$ (see 2.13) for any $w \in \overline{\mathcal{D}}$. Thus we need only to draw all the distinguished involution graphs for the two-sided cells of a -values 5 and 6, then all the left cell graphs of those two-sided cells can be obtained from the corresponding distinguished involution graphs simply by forgetting all the labels of the edges.

From the graphs displayed in [2, Appendix II], we can see that $\mathcal{D}(x_1) \simeq \mathcal{D}(b_{y_4})$, $\mathcal{D}(x_1) \simeq \mathcal{D}(b_{z_2})$, $\mathcal{D}(b_{x_2}) \simeq \mathcal{D}(b_{y_3})$ and $\mathcal{D}(b_{x_2}) \simeq \mathcal{D}(b_{z_3})$. Although the above notation “ \simeq ” can’t be replaced by “ \cong ” (see 2.13) in general, the graphs in each of the above pairs can be obtained from one to another by changing only a few edge labels (see 4.7 below).

4.7

The distinguished involution graph $\mathcal{D}(x_1)$ (respectively, $\mathcal{D}(b_{y_4})$, $\mathcal{D}(b_{z_2})$) is displayed in FigA (respectively, FigG, FigI) of [2, Appendix II]. The graph $\mathcal{D}(b_{x_2})$ (respectively, $\mathcal{D}(x_3)$, $\mathcal{D}(y_1)$, $\mathcal{D}(y_2)$, $\mathcal{D}(b_{y_3})$, $\mathcal{D}(z_1)$, $\mathcal{D}(b_{z_3})$) is too big to be displayed in one page, hence it has to be taken into 2 (respectively, 6, 7, 3, 2, 9, 2) pieces, one page for each piece, being displayed in FigB1-FigB2 (respectively, FigC1-FigC6, FigD1-FigD7, FigE1-FigE3, FigF1-FigF2, FigH1-FigH9, FigJ1-FigJ2) of [2, Appendix II]. The following are the relations between some graphs:

- (i) $\mathcal{D}(b_{y_4})$ in FigG can be obtained from $\mathcal{D}(x_1)$ in FigA by replacing all the edge labels **13** by **1**.
- (ii) $\mathcal{D}(b_{z_2})$ in FigI can be obtained from $\mathcal{D}(x_1)$ in FigA by moving the extra label S from the vertex $\boxed{2357\mathbf{q}}$ to $\boxed{1257\mathbf{q}}$.
- (iii) It looks the same for FigB1 and FigF1, and also for FigB2, FigF2 and FigJ2.
- (iv) FigB1 can be obtained from FigJ1 by moving the extra label S from the vertex $\boxed{235\mathbf{s}}$ to $\boxed{2357}$ and by replacing all the edge labels **7s**, **80** by **s**, **o**, respectively.

4.8

Next we give some explanation for the distinguished involution graphs. As we know before, in a distinguished involution graph \mathcal{D} , once the expression of an element corresponding to one vertex (say v) of \mathcal{D} is known, the expressions of the elements

corresponding to all the other vertices (say v') of \mathcal{D} can be written out easily according to the labels on any path connecting v and v' . So we select one vertex by giving it an extra label S in each distinguished involution graph, the expression of the element corresponding to the selected vertex is given explicitly. More precisely, the selected vertex in $\mathcal{D}(x_1)$ (respectively, $\mathcal{D}(b_{y_4})$) is labeled by $\boxed{2357q}$ in FigA (respectively, FigG) with an extra label S nearby, the corresponding element is x_1 (respectively, b_{y_4}) (see 4.3). The selected vertex in $\mathcal{D}(b_{x_2})$ (respectively, $\mathcal{D}(x_3)$, $\mathcal{D}(y_1)$, $\mathcal{D}(y_2)$, $\mathcal{D}(b_{y_3})$, $\mathcal{D}(z_1)$, $\mathcal{D}(b_{z_2})$, $\mathcal{D}(b_{z_3})$) is located in FigB1 (respectively, FigC5, FigD5, FigE2, FigF1, FigH2, FigI, FigJ1), the corresponding element is b_{x_2} (respectively, x_3 , y_1 , y_2 , b_{y_3} , z_1 , b_{z_2} , b_{z_3}) (see 4.3 and 3.20).

Concerning the notation for the edge labels in a distinguished involution graph, we take the symbol $\boxed{235q} \xrightarrow{13} \boxed{125q}$ (respectively, $\boxed{235q} \xrightarrow{1} \boxed{125q}$) as an example: it means that an edge $\{y, z\}$ with $\mathcal{R}(y) = \{2, 3, 5, 8\}$ and $\mathcal{R}(z) = \{1, 2, 5, 8\}$ satisfies $z = 1y3$ (respectively, $z = 1y1$).

As mentioned in 4.7, some big graphs need to be divided into several smaller pieces such that each piece could be fit into one page. For example, the graph $\mathcal{D}(b_{z_3})$ is divided into two pieces FigJ1, FigJ2. There are two vertices (arranged as v_1, v_2) of $\mathcal{D}(b_{z_3})$, each of which occurs in both FigJ1 and FigJ2. For any $i \in \{1, 2\}$, we doubly box v_i and give it an extra label “ $i, 1$ ” in FigJ2 and “ $i, 2$ ” in FigJ1. For example, the vertex v_1 of $\mathcal{D}(b_{z_3})$ has the doubly boxed label $\boxed{\boxed{14q}}$, the one in FigJ2 has an extra label 1, 1, while that in FigJ1 has an extra label 1, 2.

Appendices

In Appendix I, we list all the sets $K \in X$ and $\mathcal{R}(K)$ for $X \in \{A, B, C, D, E, F, G, H, I, J\}$ (see 3.17) and some computational data involving K , by which one could easily prove the left-connectedness of the left cells containing K (see 3.23). Then in Appendix II, we display all the distinguished involution graphs for all the two-sided cell Ω of \tilde{E}_8 with $a(\Omega) \in \{5, 6\}$, by which one could easily get all the distinguished involutions d of \tilde{E}_8 in Ω and all the left cells graphs for Ω . The complete contents of Appendices I, II occupy almost two thousand pages and are included in the electronic version of our paper (see [2]). Owing to space limitations, we include only a small portion of the Appendices here, from which, the readers might get some concrete idea on our results.

Appendix I.

(1) $A = \{A_i \mid 1 \leq i \leq 50\}$. Each A_i is a set containing one element. Here are the sets A_i and $\mathcal{R}(A_i)$, $1 \leq i \leq 12$ (read “ $A_2 = \{075234\}$ with $\mathcal{R}(A_2) = \{4, 7, 0\}$ ”, etc):

A_1	23570	07523	A_2	470	075234	A_3	2360	075236	A_4	2358	075238
A_5	460	0752346	A_6	48	0752348	A_7	2368	0752368	A_8	50	07523465
A_9	468	07523468	A_{10}	237	07523687	A_{11}	58	075234658	A_{12}	47	075234687

(2) $B = \{B_i \mid 1 \leq i \leq 135\}$. Here are the sets B_i and $\mathcal{R}(B_i)$, $1 \leq i \leq 8$ (read “ $B_3 = \{56523423547, 67623454234\}$ with $\mathcal{R}(B_3) = \{4, 7\}$ ”, etc):

B_1	2357	5652342357	6762345423	B_2	1257	56523423157	67623454312
B_3	47	56523423547	67623454234				
B_4	2358	56523423785	67623454283	78723465423			
B_5	236	56523425763	67623454263	B_6	147	565234231547	676234543124
B_7	1258	565234231785	676234543182	787234654312			
B_8	2350	565234237805	676234542803	787234654203	808234765423		

(3) $C = \{C_i \mid 1 \leq i \leq 825\}$. Here are the sets C_i and $\mathcal{R}(C_i)$, $107 \leq i \leq 116$:

C_{107}	1238	13128	C_{108}	148	131284	C_{109}	348	1312843	2421834
C_{110}	158	1312845	C_{111}	358	13128435	24218345	C_{112}	168	13128456
C_{113}	458	131284354	242183454	454182345					
C_{114}	368	131284356	242183456	C_{115}	17	131284567			
C_{116}	258	1312843542	1315842345	2421834542	4541823425				

(4) $D = \{D_i \mid 1 \leq i \leq 1030\}$. Here are the sets D_i and $\mathcal{R}(D_i)$, $335 \leq i \leq 345$:

D_{335}	12370	131270	D_{336}	1470	1312704	D_{337}	1238	1312708
D_{338}	3470	13127043	24217034	D_{339}	1570	13127045		
D_{340}	148	13127048	D_{341}	3570	131270435	242170345		
D_{342}	348	131270438	242170348	D_{343}	160	131270456		
D_{344}	158	131270458	D_{345}	4570	1312704354	2421703454	4541702345	

(5) $E = \{E_i \mid 1 \leq i \leq 495\}$. Here are the sets E_i and $\mathcal{R}(E_i)$, $274 \leq i \leq 282$:

E_{274}	1367	131676	E_{275}	1467	1316764	3436761		
E_{276}	1368	1316768	1317876	E_{277}	1267	13167642	24267631	34367612
E_{278}	157	13167645	34367615					
E_{279}	1468	13167648	13178746	34367618	34378716			
E_{280}	1360	13167680	13178760	13180876				
E_{281}	1257	131676425	242676315	343676125				
E_{282}	1268	131676428	131787426	242676318	242787316	343676128	343787126	

(6) $F = \{F_i \mid 1 \leq i \leq 135\}$. Each F_i contains a single element. Here are the sets F_i and $\mathcal{R}(F_i)$, $1 \leq i \leq 6$:

F_1	2357	13125742354654237	F_2	1257	131257423546542317	F_3	47	131257423546542347
F_4	236	131257423546542763	F_5	2358	131257423546542783	F_6	147	1312574235465423147

(7) $G = \{G_i \mid 1 \leq i \leq 50\}$. Here are the sets G_i and $\mathcal{R}(G_i)$ for $1 \leq i \leq 6$:

G_1	23570	56523042357	67623045423	G_2	12570	565230423157	676230454312
G_3	470	565230423547	676230454234	G_4	2358	565230423785	676230454283
G_5	2360	565230425763	676230454263	G_6	1470	5652304231547	6762304543124

(8) $H = \{H_i \mid 1 \leq i \leq 1735\}$ and $I = \{I_i \mid 1 \leq i \leq 50\}$ and $J = \{J_i \mid 1 \leq i \leq 135\}$. Here are the sets X and $\mathcal{R}(X)$ for $X \in \{H_i, I_1, J_1 \mid 1 \leq i \leq 6\}$:

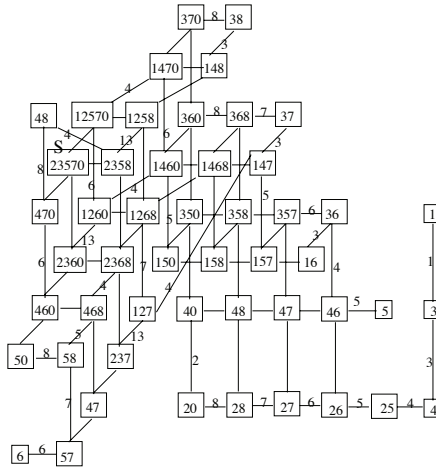
H_1	134	$w_1=143143$				
H_2	123	$w_1=1431432$ $w_2=23423413$ $1w_2=42w_1$				
		(1) w_2	234	(2) $1w_2$		124
H_3	135	$w_1=1431435$ $w_2=35435413$ $1w_2=45w_1$				
		(1) w_2	345	(2) $1w_2$		145
H_4	1235	$w_1=14314325$ $w_2=234234135$ $1w_2=42w_1$				
		(1) w_2	234	(2) $1w_2$		124
		$w_3=354354123$ $1w_3=45w_1$				
		(1) w_3	345	(2) $1w_3$		145
H_5	136	$w_1=14314356$ $w_2=354354136$ $1w_2=45w_1$				
		(1) w_2	345	(2) $1w_2$		145
		$w_3=4654653143$ $13w_3=5645w_1$				
		(1) w_3	456	(2) $3w_3$		356
		(3) $13w_3$	156			
H_6	14	$w_1=143143254$ $w_2=2342341354$ $1w_2=42w_1$				
		(1) w_2	234	(2) $1w_2$		124
		$w_3=3543541234$ $1w_3=45w_1$				
		(1) w_3	345	(2) $1w_3$		145

I_1	12570	$w_1=25425431436542765870$					
		$w_2=465465231431542765870$					
		$2w_2=56w_1$					
		(1)	w_2	456	(2)	$2w_2$	256
		$w_3=3543541243654231765870$					
		$642w_3=24563w_1$					
		(1)	w_3	345	(2)	$2w_3$	235
		(3)	$42w_3$	24	(4)	$642w_3$	246
		(5)	$442w_3$	235	(6)	$4642w_3$	236
		$w_4=5765764231435423165870$					
		$24w_4=6756w_1$					
		(1)	w_4	567	(2)	$4w_4$	467
		(3)	$24w_4$	267			
		$w_5=14314325423654231765870$					
		$63452w_5=43245613w_1$					
		(1)	w_5	134	(2)	$2w_5$	123
		(3)	$52w_5$	1235	(4)	$452w_5$	14
		(5)	$3452w_5$	34	(6)	$63452w_5$	346
		(7)	$6452w_5$	146	(8)	$652w_5$	1236
		$w_6=68768754231435465423170$					
		$245w_6=786756w_1$					
		(1)	w_6	678	(2)	$5w_6$	578
(3)	$45w_6$	478	(4)	$245w_6$	278		
$w_7=234234135423654231765870$							
$563451w_7=6543245613w_1$							
(1)	w_7	234	(2)	$1w_7$	124		
(3)	$51w_7$	125	(4)	$451w_7$	145		
(5)	$3451w_7$	35	(6)	$63451w_7$	36		
(7)	$563451w_7$	356	(8)	$6451w_7$	146		
(9)	$56451w_7$	156					
$w_8=708708654231435467654231$							
$2456w_8=80786756w_1$							
(1)	w_8	780	(2)	$6w_8$	680		
(3)	$56w_8$	580	(4)	$456w_8$	480		
(5)	$2456w_8$	280					
J_1	2358	$w_1=143143254365427654318765423408765423$					
		(1)	w_2	234	(2)	$1w_2$	124
		$w_3=3543541234365427654318765423408765423$					
		$1w_3=45w_1$					
		(1)	w_3	345	(2)	$1w_3$	145
		$w_4=46546523142345427654318765423408765423$					
$13w_4=5645w_1$							
(1)	w_4	456	(2)	$3w_4$	356		

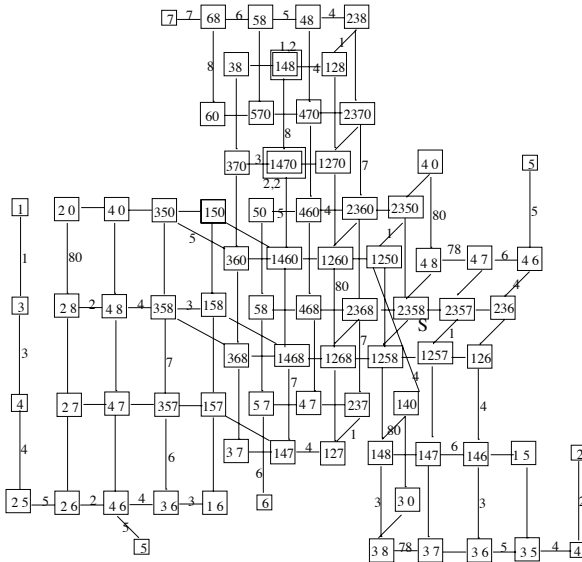
(3)	$13w_4$	156		
$w_5=254254314365423457654318765423408765423$				
$63413w_5=13456245w_1$				
(1)	w_5	245	(2)	$3w_5$ 235
(3)	$13w_5$	125	(4)	$413w_5$ 14
(5)	$3413w_5$	13	(6)	$63413w_5$ 136
(7)	$6413w_5$	146	(8)	$613w_5$ 126
$w_6=576576423143542345654318765423408765423$				
$134w_6=675645w_1$				
(1)	w_6	567	(2)	$4w_6$ 467
(3)	$34w_6$	367	(4)	$134w_6$ 167
$w_7=6876875423143542654231456765423408765423$				
$1345w_7=78675645w_1$				
(1)	w_7	678	(2)	$5w_7$ 578
(3)	$45w_7$	478	(4)	$345w_7$ 378
(5)	$1345w_7$	178		
$w_8=70870865423143542654376542314354678765423$				
$13456w_8=8078675645w_1$				
(1)	w_8	780	(2)	$6w_8$ 680
(3)	$56w_8$	580	(4)	$456w_8$ 480
(5)	$3456w_8$	380	(6)	$13456w_8$ 180

Appendix II.

Here are the distinguished involution graphs $\mathcal{D}(x_1)$ in FigA and $\mathcal{D}(b_{z_3})$ in FigJ1-FigJ2.



FigA



FigJ1

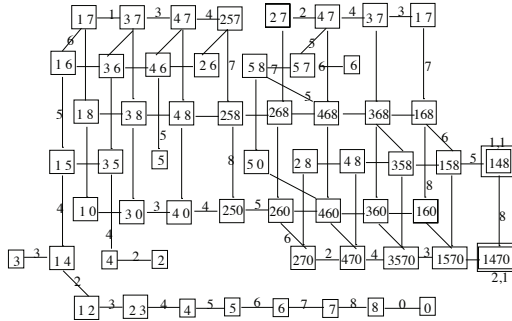


Fig12

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