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Group connectivity of semistrong product of graphs

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Abstract

For a 2-edge-connected graph G, the *group connectivity number* of G is defined as $\Lambda_g(G) = \min\{k : G$ is A-connected for every abelian group with $|A| \geq k\}$. Let $G \bullet H$ denote the semistrong product of two graphs G and H. In this paper, we extend the result of Yan et al. [*Int. J. Algebra* 4 (2010), 1185–1200] on group connectivity in Cartesian product of graphs to semistrong products and make a slightly stronger conclusion: $\Lambda_g(G \bullet H) \leq 5$ for two nontrivial connected simple graphs G and H, where equality holds if and only if either $G \bullet H \cong T \bullet K_2$, where T is a tree, or $G \bullet H \cong K_2 \bullet K_{1,m}$, where $m \geq 2$.

1 Introduction

Graphs considered in this paper are finite, loopless, and may have multiple edges. Terminology and notation not defined here can be found in [1].

Let G be a graph. For a vertex $v \in V(G)$, define $E_G(v)$ to be the set of all edges which are incident with v in G. An edge cut X of G is *trivial* if $X = E_G(v)$ for some $v \in V(G)$. For an edge subset $E' \subseteq E(G)$, we use $G[E']$ to denote the subgraph of

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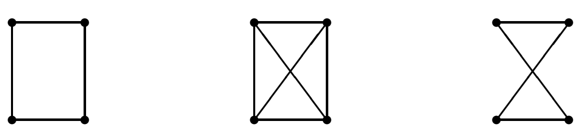
G induced by E'. A *cycle* is a connected 2-regular graph. We denote by C_n and P_n a cycle of length n and a path of length n, respectively. For simplicity, we say an n-cycle for a cycle C_n, where $n \geq 3$.

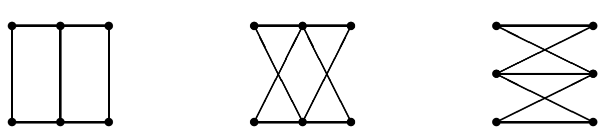
Let G and H be two graphs. The *Cartesian product* of G and H, denoted by $G \square H$, is defined to be the graph with vertex set $V(G) \times V(H)$, and $(u_1, v_1)(u_2, v_2) \in E(G \square H)$ if and only if either $v_1 = v_2$ and $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. The *strong product* $G \boxtimes H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, where two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \boxtimes H$, if either $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $u_1u_2 \in E(G)$ and $v_1 = v_2$ or $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$. The *semistrong product* $G \bullet H$ of G and H is the graph with vertex set $V(G) \times V(H)$ and $(u_1, v_1)(u_2, v_2) \in E(G \bullet H)$ if and only if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

From the definitions, the edge set of a strong product is the union of the edge sets of a Cartesian product and a semistrong product. So the condition for a strong product is much stronger than semistrong products and Cartesian products. For example: $K_2 \boxtimes K_2$ is a K_4, while $K_2 \bullet K_2 \cong K_2 \square K_2$ is a 4-cycle, which can be seen in Figure 1. Let $e_1 \in E(G)$ and $e_2 \in E(H)$. Then $G[\{e_1\}] \square H[\{e_2\}] \cong K_2 \square K_2$, and $G[\{e_1\}] \bullet H[\{e_2\}] \cong K_2 \bullet K_2$. Hence every edge of $G \bullet H$ is also in a 4-cycle just like every edge of $G \square H$. Yet, $K_2 \bullet K_2$ is a 'twisted' 4-cycle so that $G \bullet H$ has a different structure from $G \square H$ and $G \bullet H$ is not isomorphic to $H \bullet G$. We take $K_2 \bullet P_2$, $P_2 \bullet K_2$ and $K_2 \square P_2$ for example which are depicted in Figure 2.



$K_2 \square K_2$ $\quad$ $K_2 \boxtimes K_2$ $\quad$ $K_2 \bullet K_2$

Figure 1: Three different products of K_2 and K_2



$K_2 \square P_2 \cong P_2 \square K_2$ $\quad$ $K_2 \bullet P_2$ $\quad$ $P_2 \bullet K_2$

Figure 2: Semistrong product and Cartesian product of K_2 and P_2

Let D be an orientation of a graph G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let *tail*$(e) = u$ and *head*$(e) = v$. For a vertex $v \in V(G)$, let $E^+(v)$ denote the set of edges with tail at v and $E^-(v)$ the set of edges with head at v. Let A denote an (additive) abelian group with the identity element 0 and let A^* denote the set of nonzero elements of A. We define $F(G, A)$ to be the set of labelings of $E(G)$ using elements of A and define $F^*(G, A)$ to be the set of labelings of $E(G)$ using nonzero elements of A.

Given a function $f \in F(G, A)$, define $\partial f : V(G) \to A$ as

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. The value $\partial f(v)$ is known as the *net flow out of v under f.*

For a graph G, a function $b : V(G) \to A$ is an *A-valued zero-sum function* on G if $\sum_{v \in V(G)} b(v) = 0$. The set of all A-valued zero-sum functions on G is denoted by $\mathcal{Z}(G, A)$. Given $b \in \mathcal{Z}(G, A)$, a function $f \in F^*(G, A)$ is an *(A, b)-nowhere-zero flow* if G has an orientation D such that $\partial f = b$. A graph G is *A-connected* if for every $b \in \mathcal{Z}(G, A)$, G admits an (A, b)-nowhere-zero flow. A *nowhere-zero A-flow* is an $(A, 0)$-nowhere-zero flow, where 0 denotes the function on $V(G)$ that is identically zero. More specifically, a *nowhere-zero k-flow* is a nowhere-zero Z_k-flow, where Z_k is the cyclic group of order k. Tutte [13] proved that G admits a nowhere-zero A-flow with $|A| = k$ if and only if G admits a nowhere-zero k-flow.

The concept of *group connectivity* was first introduced by Jaeger et al. [7] as a generalization of nowhere-zero flows. Here *group connectivity* is referred to the general properties of a graph being A-connected for some particular A. It observed in [7] that the *group connectivity* of G is independent of the orientation of G and every A-connected graph is 2-edge-connected.

For a 2-edge-connected graph G, the *group connectivity number* of G is defined as

$$\Lambda_g(G) = \min\{k : G \text{ is } A\text{-connected for every abelian group with } |A| \geq k\}.$$

Many researchers have devoted to the study of group connectivity in products of two graphs. Imrich et al. [6] proved that the strong product of two nontrivial connected simple graphs is Z_3-flow contractible (Proposition 1.2 in [5] shows that Z_3-flow contractible is equivalent to Z_3-connected) if and only if it is not a K_4-tree (the strong product $T \boxtimes K_2$ of a tree T and K_2 is called a K_4-tree). Yan et al. [14] studied the group connectivity in strong product of graphs in a different way. We state it in the following form.

Theorem 1.1 *Let G and H be two nontrivial connected simple graphs. Then $\Lambda_g(G \boxtimes H) \leq 4$, where equality holds if and only if $G \boxtimes H \cong T \boxtimes K_2$, where T is a tree.*

Yan et al. also examined the group connectivity number of Cartesian product of graphs.

Theorem 1.2 ([14, 11]) *Let G and H be two nontrivial connected simple graphs. Then $\Lambda_g(G\Box H) \leq 5$, where equality holds if and only if either $G\Box H \cong T\Box K_2$, where T is a tree or $G\Box H \cong K_{1,m}\Box K_{1,n}$, where $n, m \geq 2$.*

In this paper, we extend the result on Cartesian product in Theorem 1.2 to semistrong product. The following theorem is our main conclusion.

Theorem 1.3 *Let G and H be two nontrivial connected simple graphs. Then $\Lambda_g(G\bullet H) \leq 5$, where equality holds if and only if either $G\bullet H \cong T\bullet K_2$, where T is a tree or $G\bullet H \cong K_2\bullet K_{1,m}$, where $m \geq 2$.*

2 Preliminaries

For a subset $X \subseteq E(G)$, the contraction G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting all loops generated by this process. Note that even if G is simple, G/X may have multiple edges. For simplicity, we write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G, then G/H denotes $G/E(H)$.

Let $O(G)$ denote the set of vertices of odd degree in G. In [2] Catlin introduced collapsible graphs. A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph H_R of G with $O(H_R) = R$. Clearly, the trivial graph K_1 is collapsible. Lai [9] proved that a collapsible graph is A-connected with $|A| = 4$. Some useful results on collapsible graphs are summarized as follows.

Theorem 2.1 ([2, 3]) *Each of the following holds.*

(i) Let G be a graph and H be a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible.

(ii) C_2, C_3 and $K_{3,3}$ are collapsible.

Let G be a graph with C_4 as a subgraph, and $\pi = \{X, Y\}$ the bipartition of $V(C_4)$ so that both X and Y are independent sets of C_4. Let G/π denote the graph obtained from G by identifying all vertices of X to form a single vertex x, identifying all vertices of Y to form a single vertex y, and then joining x, y with a new edge $e_\pi = xy$, so that

$$E(G) - E(C_4) = E(G/\pi) - \{e_\pi\}.$$

Either Example 1 and Corollary 1 in [2] or Theorem B in [12] implies the following theorem.

Theorem 2.2 *Let G/π be defined as above. If G/π is collapsible, then G is collapsible.*

Lemma 2.3 ([4, 7, 10]) *Each of the following holds.*

(i) $\Lambda_g(C_n) = n + 1$.

(ii) *For* $m \geq 2$, $\Lambda_g(K_{2,m}) = 5$.

Theorem 2.4 ([7]) *Let A be an abelian group with $|A| \geq 4$. Every graph which contains 2-edge-disjoint spanning trees is A-connected. In particular, every 4-edge-connected graph is A-connected.*

Lemma 2.5 ([8]) *Let T be a connected spanning subgraph of G. If for each edge $e \in E(T)$, G has a A-connected subgraph H_e with $e \in E(H_e)$, then G is A-connected.*

When H_1 and H_2 are two subgraphs of a graph G, we say that G is a *parallel connection* between H_1 and H_2, denoted by $H_1 \oplus H_2$, if $E(H_1) \cup E(H_2) = E(G)$, $|V(H_1) \cap V(H_2)| = 2$ and $|E(H_1) \cap E(H_2)| = 1$.

Lemma 2.6 ([14]) *Let C be a 4-cycle and A be an abelian group with $|A| = 4$. Let $G = H \oplus C$. Then H is A-connected if and only if G is A-connected.*

A graph G is *k-circuit connected* if G is connected and for any pair of edges $e_1, e_2 \in E(G)$, there exists a sequence of cycles $C_1, \cdots, C_m$ with $|E(C_i)| \leq k$ for $1 \leq i \leq m$, $k \geq 3$ such that $e_1 \in E(C_1)$, $e_2 \in E(C_m)$ and $E(C_j) \cap E(C_{j+1}) \neq \emptyset$ for $1 \leq j \leq m - 1$.

Lemma 2.7 ([14]) *Let G be 4-circuit connected and A be an abelian group with $|A| = 4$. Each of the following holds.*

(i) *If G has a nontrivial collapsible subgraph, then* $\Lambda_g(G) \leq 4$.

(ii) *If G has a nontrivial A-connected subgraph, then* $\Lambda_g(G) \leq 4$.

3 Proof of the main theorem

In this section, we investigate the group connectivity of semistrong product of two nontrivial connected simple graphs G and H. By the definition of semistrong product, we know that every edge of $G \bullet H$ is contained in a 4-cycle. Thus, $G \bullet H$ is 4-circuit connected. It follows from Lemmas 2.3(i) and 2.5 that $\Lambda_g(G \bullet H) \leq 5$. Here we mainly investigate the necessary and sufficient conditions on which equation holds.

Lemma 3.1 *Each of the following holds.*

(i) *Let G be a tree, then* $\Lambda_g(G \bullet K_2) = 5$.

(ii) $\Lambda_g(K_2 \bullet K_{1,m}) = 5$, *for* $m \geq 2$.

Proof. (i) Note that $\Lambda_g(G \bullet K_2) \leq 5$. So it is sufficed to prove that $G \bullet K_2$ is not Z_4-connected. We will proceed our proof by induction on $|V(G)|$.

If $|V(G)| = 2$, then $G \cong K_2$. Thus $G \bullet K_2 \cong C_4$. By Lemma 2.3(i), $G \bullet K_2$ is not Z_4-connected.

Next we assume that for any tree G' with $3 \leq |V(G')| < |V(G)|$, $G' \bullet K_2$ is not Z_4-connected. Now we consider $G \bullet K_2$. Since G is a tree, there must be an edge, say xy, such that x has degree 1 in G. It follows that $G \bullet K_2 = (G - x) \bullet K_2 \oplus G[\{xy\}] \bullet K_2 \cong (G - x) \bullet K_2 \oplus C_4$. By the induction hypothesis, $(G - x) \bullet K_2$ is not Z_4-connected. So is $G \bullet K_2$ by Lemma 2.6.

(ii) It is clear that $K_2 \bullet K_{1,m} \cong K_{2,2m}$, where $m \geq 2$. By Lemma 2.3(ii), we get that $\Lambda_g(K_2 \bullet K_{1,m}) = 5$. □

Lemma 3.2 *If one of G and H is not a tree, then $\Lambda_g(G \bullet H) \leq 4$.*

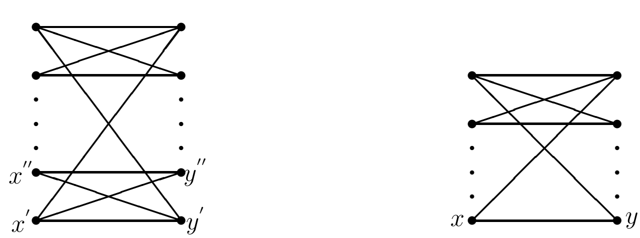
Proof. To prove this lemma, we need firstly prove the following two claims.

Claim 1. For $n \geq 3$, $C_n \bullet K_2$ is collapsible.

Proof of Claim 1. By induction on $|V(C_n)|$. For $n = 3$, $C_3 \bullet K_2 \cong K_{3,3}$. It follows from Theorem 2.1 that $C_3 \bullet K_2$ is collapsible.

We assume that for $4 \leq k < n$, $C_k \bullet K_2$ is collapsible. Then we consider the graph $C_n \bullet K_2$ depicted in Figure 3(a).

Let $\pi = \{X, Y\}$, where $X = \{x', x''\}$ and $Y = \{y', y''\}$. We get the graph $(C_n \bullet K_2)/\pi$ from $C_n \bullet K_2$ by identifying x' with x'' of X to form a single vertex x and identifying y' with y'' of Y to form a single vertex y. Fortunately $(C_n \bullet K_2)/\pi$ is isomorphic to $C_{n-1} \bullet K_2$ seen in Figure 3(b). By the induction hypothesis, $(C_n \bullet K_2)/\pi$ is collapsible. Theorem 2.2 shows that $C_n \bullet K_2$ is also collapsible.



(a) $C_n \bullet K_2$ (b) $C_{n-1} \bullet K_2$

Figure 3: The semistrong products $C_n \bullet K_2$ and $C_{n-1} \bullet K_2$

Claim 2. For $n \geq 3$, $K_2 \bullet C_n$ is A-connected with $|A| \geq 4$.

Proof of Claim 2. Note that $d((u,v)) = (d_G(u)+1) \times d_H(v)$, for any vertex $(u,v) \in V(G \bullet H)$. For $n \geq 3$, $K_2 \bullet C_n$ is a 4-regular graph seen in Figure 4. Next we will show that It is also 4-edge-connected.

Let X be a nontrivial edge cut of $K_2 \bullet C_n$. Note that each edge of $K_2 \bullet C_n$ is in a 'twisted' 4-cycle and a cycle is 2-edge-connected. Then $|X|$ is even and $|X| \geq 4$. Hence $K_2 \bullet C_n$ is 4-edge-connected. By Theorem 2.4, $K_2 \bullet C_n$ is A-connected with $|A| \geq 4$.

Now we are ready to prove our lemma. If G is not a tree, then G must contain a cycle. Thus $G \bullet H$ contains $C_n \bullet K_2$ as a subgraph. By Claim 1, $C_n \bullet K_2$ is collapsible. Note that $G \bullet H$ is 4-circuit connected. It follows from Lemma 2.7 that $\Lambda_g(G \bullet H) \leq 4$. Similarly if H is not a tree, then $G \bullet H$ contains $K_2 \bullet C_n$ as a subgraph. By Claim 2, $K_2 \bullet C_n$ is A-connected with $|A| \geq 4$. Lemma 2.7 also shows that $\Lambda_g(G \bullet H) \leq 4$. □

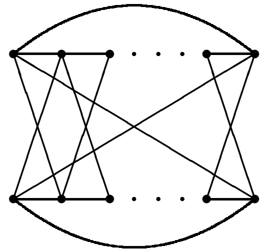


Figure 4: The graph $K_2 \bullet C_n$

Lemma 3.3 *Let G and H be trees and* $\min\{|V(G)|, |V(H)|\} \geq 3$*. Then*

$$\Lambda_g(G \bullet H) \leq 4.$$

Proof. Let G and H be trees. If $|V(G)| = 3$ and $|V(H)| = 3$, then $G \bullet H \cong P_2 \bullet P_2$. First we can prove that $P_2 \bullet P_2$ is collapsible.

In Figure 5(a), $x'y'x''y''x'$ is a 4-cycle of $P_2 \bullet P_2$. Let $\pi = \{X, Y\}$, where $X = \{x', x''\}$ and $Y = \{y', y''\}$. The graph $(P_2 \bullet P_2)/\pi$ is depicted in Figure 5(b). Contracting 2-cycles and 3-cycles generated by this process we get a graph K_1 which is collapsible. So it follows from Theorems 2.1 and 2.2 that $P_2 \bullet P_2$ is collapsible. Since $\min\{|V(G)|, |V(H)|\} \geq 3$, $G \bullet H$ contains $P_2 \bullet P_2$ as a collapsible subgraph. By Lemma 2.7, $\Lambda_g(G \bullet H) \leq 4$. □

(a) $P_2 \bullet P_2$ (b) $(P_2 \bullet P_2)/\pi$

Figure 5: Two graphs for Lemma 3.3

Lemma 3.4 *Let H be a tree which contains P_3 as an induced subgraph. Then* $\Lambda_g(K_2 \bullet H) \leq 4$.

Proof. As Lemma 3.3, we will first prove that $K_2 \bullet P_3$ is collapsible. The graph $K_2 \bullet P_3$ is depicted in Figure 6(a).

(a) $K_2 \bullet P_3$ (b) $(K_2 \bullet P_3)/\pi$

Figure 6: Two graphs for Lemma 3.4

It is clear that $x^{'}y^{'}x^{''}y^{''}x^{'}$ is a 4-cycle of $K_2 \bullet P_3$. Let $\pi = \{X, Y\}$, where $X = \{x^{'}, x^{''}\}$ and $Y = \{y^{'}, y^{''}\}$. The graph $(K_2 \bullet P_3)/\pi$ is depicted in Figure 6(b). Contracting all 2-cycles and 3-cycles generated by the process results in a graph K_1. It follows from Theorems 2.1 and 2.2 that $K_2 \bullet P_3$ is collapsible. Note that the graph $K_2 \bullet H$ contains $K_2 \bullet P_3$ as a subgraph. By Lemma 2.7, we obtained that $\Lambda_g(K_2 \bullet H) \leq 4$. □

Proof of Theorem 1.3. Let G and H be two nontrivial connected simple graphs. By Lemma 3.1, the adequacy of the theorem is obvious. Conversely, we suppose $\Lambda_g(G \bullet H) = 5$ to prove that either $G \bullet H \cong T \bullet K_2$, where T is a tree or $G \bullet H \cong K_2 \bullet K_{1,m}$, where $m \geq 2$.

If one of G and H contains a cycle, then by Lemma 3.2, $\Lambda_g(G \bullet H) \leq 4$. Therefore both G and H are trees.

Lemma 3.3 shows that if $\min\{|V(G)|, |V(H)|\} \geq 3$, then $\Lambda_g(G \bullet H) \leq 4$. So we also find that $|V(G)| = 2$ or $|V(H)| = 2$.

If $|V(H)| = 2$, then $H \cong K_2$. By Lemma 3.1, $G \bullet H \cong T \bullet K_2$, where T is a tree.

Next we consider that $|V(G)| = 2$ and $|V(H)| \neq 2$. It follows from Lemma 3.4 that if H contains P_3 as an induced subgraph, then $\Lambda_g(G \bullet H) \leq 4$. Thus $H \cong K_{1,m}$, where $m \geq 2$. By Lemma 3.1, $\Lambda_g(K_2 \bullet K_{1,m}) = 5$. Therefore $G \bullet H \cong K_2 \bullet K_{1,m}$, where $m \geq 2$. The proof is now complete.

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