AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **56** (2013), Pages 77–84

Revisiting the spreading and covering numbers

Ben Babcock   Adam Van Tuyl

Department of Mathematical Sciences
Lakehead University
Thunder Bay, ON P7B 5E1
Canada
bababcoc@lakeheadu.ca   avantuyl@lakeheadu.ca

Abstract

We revisit the problem of computing the spreading and covering numbers. We show a connection between some of the spreading numbers and the number of non-negative integer 2×2 matrices whose entries sum to d, and we construct an algorithm to compute improved upper bounds for the covering numbers.

1 Introduction

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. For any non-negative integer d, let M_d be the set of all monomials of degree d in R. For any subset $W \subseteq M_d$, let

$$R_1W = \{x_i m \ : \ m \in W \ \text{ and } \ 1 \leq i \leq n\}.$$

For any $W \subseteq M_d$, we always have $|R_1W| \leq n|W|$ and $R_1W \subseteq M_{d+1}$.

We are interested in finding subsets W where either $|R_1W| = n|W|$ or $R_1W = M_{d+1}$. We define the *spreading number* to be

$$\alpha_n(d) = \max\{\ |W| \ : \ W \subseteq M_d \quad \text{and} \quad |R_1W| = n|W|\}.$$

The terminology is derived from the fact that the elements of R_1W are "spread" out in M_{d+1}. Similarly, the *covering number* is defined to be

$$\rho_n(d+1) = \min\left\{\ |W| \ : \ W \subseteq M_d \ \text{ and } \quad R_1W = M_{d+1}\right\}.$$

In this case the elements of R_1W "cover" the elements of M_{d+1}.

Geramita, Gregory, and Roberts introduced $\alpha_n(d)$ and $\rho_n(d+1)$ to study the Ideal Generation Conjecture for a set of generic points in $\mathbb{P}^n$ (see [3, Theorem 4.7]). When $n = 1$, it is trivial to show that $\alpha_1(d) = \rho_n(d+1) = 1$ for all d. When $n = 2$, $\alpha_2(d) = \lfloor \frac{d}{2} \rfloor + 1$ and $\rho_2(d+1) = \lceil \frac{d}{2} \rceil + 1$. Geramita, et al. gave exact values for

$\alpha_n(d)$ for all d when $n = 3$ or 4, and some scattered results and bounds on other values. Curtis [2] later found a formula for $\rho_3(d)$ for all d and an improved lower bound on $\rho_4(d)$. Using techniques from linear programming, Hulett and Will [4] improved these lower bounds on $\rho_4(d)$. Carlini, Hà, and the second author [1] later reformulated the problem by constructing simplicial complexes whose dimensions were related to either $\alpha_n(d)$ or $\rho_n(d+1)$.

Surprisingly, computing new exact values of $\alpha_n(d)$ and $\rho_n(d+1)$ remains elusive. However, we present two new contributions: 1) a new connection between the numbers $\alpha_4(d)$ and the number of integer matrices with a specific property; and 2) a new greedy algorithm which gives upper bounds on $\rho_n(d+1)$ that improves upon known bounds. Hopefully these observations will be of use for future attacks on computing $\alpha_n(d)$ and $\rho_n(d+1)$.

2 Preliminaries

We begin by translating our problem of computing $\alpha_n(d)$ and $\rho_n(d+1)$ into a graph theory problem. Fix positive integers n and d. Let $S_n(d)$ denote the graph whose vertex set is the set of monomials M_d in $R = k[x_1, \ldots, x_n]$, and two vertices m_i, m_j are adjacent if and only if $\deg\big(\operatorname{lcm}(m_i, m_j)\big) = d + 1$. We abuse notation and use M_d to denote both the vertices of $S_n(d)$ and the set of monomials of degree d in $R = k[x_1, \ldots, x_n]$. We denote the number of vertices of $S_n(d)$ by $v_d(n)$; it is clear that $v_d(n) = \binom{n+d-1}{d}$.

Definition 2.1 *A subset $V \subseteq M_d$ is an* independent set *if any two distinct elements of V are not adjacent; V is a* maximal independent set *if it is not properly contained in any larger independent set.*

Definition 2.2 *A subset of M_d in which any two vertices are adjacent is called a* clique. *If $C_1, \ldots, C_t$ are cliques, we say they form a* clique cover *of $S_n(d)$ if $C_1 \cup \cdots \cup C_t = M_d$. For any monomial m of degree $d - 1$, an* upward clique *is the clique consisting of the vertices $mx_i \in M_d$ for all $x_i \in \{x_1, \ldots, x_n\}$.*

As shown in [3], $\alpha_n(d)$ and $\rho_n(d+1)$ are equivalent to an invariant of $S_n(d)$:

Lemma 2.3 *With the notation as above*

(i) $\alpha_n(d)$ is the cardinality of the largest maximal independent set of $S_n(d)$.

(ii) $\rho_n(d+1)$ is the minimum cardinality of an upward clique cover of the vertices of $S_n(d+1)$.

Example 2.4 *If we consider $S_3(3)$ (see Figure 1), then $x_1^2x_2$ and $x_1^2x_3$ are adjacent but $x_1^2x_2$ and $x_2^2x_3$ are not. The graph $S_3(3)$ has $\alpha_3(3) = 4$ because $\{x_1^3, x_2^3, x_3^3, x_1x_2x_3\}$ forms a maximal independent set. Also $\rho_3(2+1) = 4$ because $C_1 = \{x_1^3, x_1^2x_2, x_1^2x_3\}$, $C_2 = \{x_2^3, x_1x_2^2, x_3x_2^2\}$, $C_3 = \{x_3^3, x_1x_3^2, x_2x_3^2\}$, and $C_4 = \{x_1^2x_2, x_1x_2^2, x_1x_2x_3\}$ form a minimal upward clique cover.*

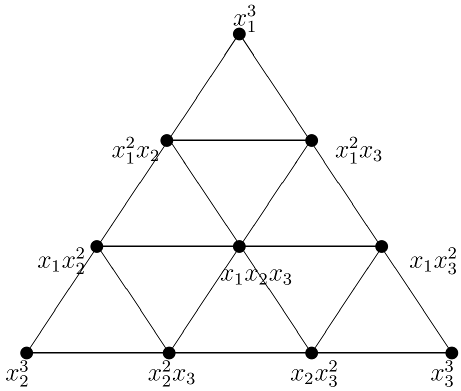

Figure 1: The graph $S_3(3)$

Computing the size of a largest maximal independent set or a minimum clique cover of a graph are both NP-hard problems. This explains, in part, why computing $\alpha_n(d)$ and $\rho_n(d+1)$ is so difficult.

3 A053307

We demonstrate a relation between the sequence $\alpha_4(d)$ and a known integer sequence which is denoted in the OEIS as A053307 [6]. In [3] we find the explicit formula:

$$\alpha_4(d) = \begin{cases} \frac{v_4(d)}{4} & \text{for } d \text{ odd} \\ \frac{v_4(d)}{4} + \frac{3d+6}{8} & \text{for } d \text{ even.} \end{cases}$$

Theorem 3.1 *For all $d \geq 0$, $\alpha_4(d)$ equals the number of non-negative integer 2×2 matrices with sum of entries equal to d, under row and column permutations.*

PROOF: Recalling that $v_4(d) = \binom{d+3}{3}$, it follows that

$$\alpha_4(2d+1) = \frac{\binom{2d+4}{3}}{4} = \frac{(2d+4)(2d+3)(2d+2)}{3!(4)} = \frac{(d+2)(2d+3)(d+1)}{6}.$$

Similarly,

$$\alpha_4(2d) = \frac{\binom{2d+3}{3}}{4} + \frac{6d+6}{8} = \frac{8d^3+24d^2+40d+24}{24} = \frac{(d+1)^3+2(d+1)}{3}.$$

The OEIS reveals that $\alpha_4(2d+1) = \text{A000330}(d+1)$; the sequence A000330 is the sequence whose i-th term is given by $0^2+1^2+\cdots+i^2$. Similarly, $\alpha_4(2d) = \text{A006527}(d+1)$, the sequence whose i-th term is $(i^3+2i)/3$. So, $\alpha_4(d)$ is an interleaved sequence.

Let $a(d)$ be the number of non-negative integer 2×2 matrices with sum of entries equal to d, under row and column permutations. The OEIS lists this sequence as A053307, and contains a comment, attributed to Paul Barry, that the integer sequence A053307 is also the interleaved sequence of A000330 and A006527, i.e., A053307$(2d+1)$ = A000330$(d+1)$ and A053307$(2d)$ = A006527$(d+1)$. The conclusion follows from this observation.

Since no proof is given for Barry's comment, we sketch out why this is indeed the case. The generating function for A053307 is listed in the OEIS as

$$\frac{t^2-t+1}{(1-t^2)^2(1-t)^2}.$$

Multiplying the top and bottom of this expression by $(1+t)^2$ gives

$$\frac{(t^2-t+1)(1+t)^2}{(1-t^2)^2(1-t)^2(1+t)^2} = \frac{t^4+1}{(1-t^2)^4} + \frac{t(t^2+1)}{(1-t^2)^4}.$$

It follows that A053307$(2d)$ equals the coefficient of t^{2d} of $\frac{t^4+1}{(1-t^2)^4}$, and A053307$(2d+1)$ equals the coefficient of t^{2d+1} in the other rational function. Now the rational function $\frac{(t^2+1)}{(1-t)^4}$ is the generating function of A006527 (this is slightly different than what is listed in the OEIS because we want the sequence to start with 1, not 0, so we have dropped the extra multiple t). Replacing t with t^2 gives the first rational function on the right hand side, which means A006527$(d+1)$ = A053307$(2d)$. A similar analysis using $\frac{t(t+1)}{(1-t)^4}$, the generating function of A000330, will complete the proof. □

Even though the sequence A053307 and $\alpha_4(d)$ are related, it is not immediately apparent why they are linked, thus suggesting the following question:

Question 3.2 *Is there an explicit bijection between the the maximal independent sets of $S_4(d)$ and the number of non-negative integer 2×2 matrices with sum of entries equal to d, under row and column permutations?*

The correspondence may be a result of the two interleaved sequences that make up A053307. Explaining the relationship between $\alpha_4(d)$ and A053007 could open up new techniques for computing the spreading and covering numbers.

4 A Greedy Algorithm for bounding $\rho_n(d)$

We use the symmetry of the graph $S_n(d)$ to describe a greedy algorithm that bounds from above $\rho_n(d)$. We give evidence that our algorithm improves on known bounds.

4.1 The Algorithm

By Lemma 2.3, $\rho_n(d)$ is the cardinality of the minimum upward clique cover of $S_n(d)$. We give a greedy algorithm that constructs an upward clique cover. Roughly speaking, at each step, the algorithm picks an upward clique for any vertex that has not been covered. The number of upward cliques in our cover forms our bound on $\rho_n(d)$.

We begin with some observations. By definition, every upward clique is uniquely identified with a monomial from M_{d-1}. For vertices of $S_n(d)$ that consist of more than one indeterminate, many factorizations into a degree $d-1$ monomial and a variable are possible; e.g., $x_1x_2^2$ can be written as $(x_1)x_2^2$ or $(x_1x_2)x_2$. However, for monomials of the form x_i^d, there is one such factorization, that is $(x_i^{d-1})x_i$. Thus, x_i^d belongs only to the upward clique identified with x_i^{d-1}. These unique upward cliques containing each x_i^d must therefore be in any clique cover of $S_n(d)$, so we can use them as the our initial set.

Aside from our choice of initial members of the cover, we wish to take into account the symmetry of $S_n(d)$. Let $\mathrm{Sym}(n)$ denote the symmetric group on the set $\{1, 2, \ldots, n\}$. For any $\mathbf{x^a} = x_1^{a_1} \cdots x_n^{a_n} \in M_d$ and $\sigma \in \mathrm{Sym}(n)$, let $\sigma(\mathbf{x^a})$ be the monomial obtained by permuting the indices $1, \ldots, n$ according to the permutation σ. This operation preserves many properties of sets of vertices; e.g., independent sets and clique covers are both unaffected.

We use $\mathrm{Sym}(n)$ to create orbits of the vertices of $S_n(d)$; that is, for any $m \in M_d$, the *orbit of* m is the set $\{\sigma(m) \mid \sigma \in \mathrm{Sym}(n)\}$. Since elements of $\mathrm{Sym}(n)$ do not alter the exponents of a monomial, only the order of the exponents relative to the indeterminates, the orbit of m is also the set of all permutations of the exponents of m. By definition, the exponents of any $m \in M_d$ always sum to d, and therefore the orbits of $S_n(d)$ are in an one-to-one correspondence with the integer partitions of d of length at most n. We can write orbits as vectors in $\mathbb{N}^n$, and in this form it is easy to determine whether an orbit is an independent set, a clique, or neither by examining the entries in the vector. We will order our list of orbits with respect to the reverse lexicographical order, that is, if $\alpha, \beta \in \mathbb{N}^n$, then $\alpha \geq_{rlex} \beta$ if the last non-zero entry of $\alpha - \beta$ is negative. Iterating over the list of orbits of $S_n(d)$ in reverse lexicographical order will help us in computing an upper bound on $\rho_n(d)$.

We now present our algorithm that returns a minimal upward clique cover; $\rho_n(d)$ is bounded above by the number of cliques in this cover.

Algorithm 4.1 *Compute an upper bound for $\rho_n(d)$.*

Input: n, d — *The number of variables and degree of monomials, respectively.*
Output: *A minimal upward clique cover of $S_n(d)$.*

Step 1 *Initialize our cover $\mathcal{C}$ with the set of upward cliques that contain x_i^d.*

Step 2 *Obtain a list, L, of the orbits of $S_n(d)$, where each orbit is represented as a vector in $\mathbb{N}^n$. Sort the list in reverse lexicographical order.*

Step 3 *Iterate over L. For each orbit $O \in L$, iterate over the vertices $v \in O$. If v is covered, continue. If not, iterate over the upward cliques containing v. Select the upward clique that contains the fewest number of vertices already covered, and add it to $\mathcal{C}$.*

Step 4 For each $v \in M_d$, compute its frequency, i.e., the number of upward cliques that contain it, in $\mathcal{C}$.

Step 5 Iterate over the elements of the $\mathcal{C}$. If an upward clique does not contain a vertex of frequency 1—all its vertices are represented by other cliques as well—then it is not essential to the cover, so discard it. Repeat this step until we complete an iteration without discarding any cliques.

Step 6 Return $\mathcal{C}$ as a minimal cover.

4.2 Comparison to Known Bounds

We compare the known bounds for $\rho_n(d)$ to the output of Algorithm 4.1. Geramita, Gregory, and Roberts proved:

Theorem 4.2 ([3, Theorem 5.2 and Proposition 5.9]) *For all $n \geq 2, d \geq 2$,*

$$\frac{v_n(d)}{n} \leq \alpha_n(d) \leq \rho_n(d) \leq \frac{v_n(d)}{n} + \frac{n-1}{n} v_{n-1}(d) \quad \text{where } v_n(d) = \binom{n+d-1}{d}.$$

Hulett and Will improved the bounds on $\rho_4(d)$:

Theorem 4.3 ([4, Theorems 4.1 and 4.2]) *For all $d \geq 5$,*

(i) if d is odd, $\rho_4(d) \leq (d^3 + 15d^2 - 61d + 261)/24$, or

(ii) if d is even, $\rho_4(d) \leq (d^3 + 15d^2 - 34d + 240)/24$.

We first consider the values of $\rho_4(d)$. In Table 1, **GGR** refers to the upper bound for $\rho_4(d)$ in Theorem 4.2 and **HW** refers to the bounds from Theorem 4.3, while **4.1** refers to the bounds found using Algorithm 4.1.

d	GGR	HW	4.1
5	30	19	19
6	42	33	29
7	57	38	40
8	75	60	55
9	97	69	74
10	121	100	96
11	150	114	122
12	182	155	147
13	219	175	185
14	260	227	223
15	306	254	275

Table 1: Comparison of upper bounds for $\rho_4(d)$.

d	GGR	4.1
6	110	61
7	162	94
8	231	142
9	319	209
10	429	285
11	565	392
12	728	515
13	924	671
14	1156	872

Table 2: Comparison of upper bounds for $\rho_5(d)$.

The output of Algorithm 4.1 is quite close to HW. In fact, it seems that for even d our bounds are equal or better, while the reverse is true for odd d. This pattern holds for at least $d \leq 24$, with the exception of $d = 22$. We are not certain why this is the case.

While the HW bound holds only for $n = 4$, our algorithm works for all $n \geq 2$. When tested against GGR for small values of d for $n = 5, 6$, Algorithm 4.1 consistently performs better. Refer to Table 2 for a comparison when $n = 5$. We hope this provides a useful example of how one can use the structure and symmetry of $S_n(d)$ along with a greedy algorithm to improve bounds on $\rho_n(d)$. It also suggests that the bounds of GGR are far from optimal.

4.3 Comments on implementation

Some of the computations were performed in *Macaulay2* 1.3.1 [5] with 4 GB of memory allocated to 1 CPU and 1 node on SHARCNET's Saw cluster.[1] Other computations ran on the Kraken cluster[2] and used up to 16 GB of memory in *Macaulay2* 1.4. Readers interested in our code can visit our websites[3]. The run times are taken from *Macaulay2*'s `time` function. The algorithm does not consume much memory, but as one might expect, as d increases the computational time increases significantly. As a result, we found it difficult to compute bounds beyond $d > 10$. When $n = 4$, the largest d for which we could compute a bound was for $\rho_4(24)$. In this case, the computation took 83051.40 seconds.

5 An additional (unsuccessful) attack

We end with a description of a theoretic approach for bounding the numbers $\alpha_n(d)$ using commutative algebra. While present computing power does not enable us to apply this approach, we record this method for future attacks.

Given a finite simple graph G with vertex set $V_G = \{z_1, \ldots, z_t\}$ and edge set E_G, the *edge ideal* of G is $I(G) = (z_i z_j \mid \{z_i, z_j\} \in E_G) \subseteq T = k[z_1, \ldots, z_t]$. Some of the graph invariants of G are encoded into the algebraic invariants of $I(G)$. For example, it is known (e.g., see [7]) that the Krull dimension of $T/I(G)$, denoted $\dim T/I(G)$, equals $\alpha(G)$, the *independence number* of G, that is, is the cardinality of the maximum independent set. When $G = S_n(d)$, it follows by Lemma 2.3 that $\alpha(S_n(d)) = \alpha_n(d)$ and thus $\alpha_n(d) = \dim T/I(S_n(d))$ where $T = k[z_m \mid m \in M_d]$.

To compute or bound $\alpha_n(d)$, it therefore suffices to compute or bound the dimension of a ring. One approach, therefore, is to make use of the following lemma:

---

[1] `https://www.sharcnet.ca/my/systems/show/41`

[2] `https://www.sharcnet.ca/my/systems/show/69`

[3] `https://github.com/tachyondecay/spreading-covering-numbers/`
`http://flash.lakeheadu.ca/ avantuyl/research/SpreadCover_Babcock_VanTuyl.html`

Lemma 5.1 *Let $L_1, \ldots, L_t$ be any t linear forms of T. Then*

$$\dim T/(I(S_n(d)), L_1, \ldots, L_t) + t \geq \dim T/I(S_n(d)) = \alpha_n(d).$$

Proof: This follows from the more general fact that for any homogeneous ideal I in T and linear form $L \in T$, then $\dim T/(I, L) \geq \dim T/I - 1$. □

A strategy to bound $\alpha_n(d)$ is to find linear forms $L_1, \ldots, L_t$ so that the computation of $\dim T/(I(S_n(d)), L_1, \ldots, L_t)$ is "easier" than that of $\dim T/I(S_n(d))$. We explored a number of ways one could pick the L_i's (e.g., picking forms at random, making use of the symmetry), but no method allowed us to improve existing bounds, even with our extensive computer resources.

Acknowledgements

We thank the anonymous referees for their useful suggestions. This work was made possible by the facilities of the Shared Hierarchical Academic Research Computing Network (SHARCNET:www.sharcnet.ca) and Compute/Calcul Canada. The first author was supported by an NSERC USRA and the second author by an NSERC Discovery Grant.

References

[1] E. Carlini, H. T. Hà and A. Van Tuyl, Computing the spreading and covering numbers, *Comm. Algebra* **29** (2001), 5687–5699.

[2] F. Curtis, A combinatorial problem involving monomial ideals, *J. Pure Appl. Algebra* **104** (1995), 161–167.

[3] A. V. Geramita, D. Gregory and L. Roberts, Monomial Ideals and Points in Projective Space, *J. Pure Appl. Algebra* **40** (1986), 33–62.

[4] H. Hulett and T. Will, Generating monomials in dimensions three and four, *J. Pure Appl. Algebra* **138** (1999), 139–150.

[5] D. R. Grayson and M. E. Stillman, Macaulay 2, a software system for research in algebraic geometry, `http://www.math.uiuc.edu/Macaulay2/`.

[6] The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/` (2010), sequences A000330, A006527, and A053307.

[7] R. H. Villarreal, *Monomial algebras*, Monographs and Textbooks in Pure and Applied Mathematics, **238**. Marcel Dekker, Inc., New York, 2001.

(Received 12 Apr 2012; revised 16 Nov 2012, 17 Jan 2013)