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Hamilton cycle decompositions of k-uniform k-partite hypergraphs

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Abstract

Let $m \geq 2$ and $k \geq 2$ be integers. We show that $K^{(k)}_{k\times m}$ has a decomposition into Hamilton cycles of Kierstead-Katona type if $k \mid m$. We also show that $K^{(3)}_{3\times m} - T$ has a decomposition into Hamilton cycles where T is a 1-factor if and only if $3 \nmid m$ and $m \neq 4$. We introduce a notion of symmetry and comment on the existence of symmetric Hamilton cycle decompositions of $K^{(k)}_{k\times m}$.

1 Introduction

Let $G = G(V, E)$ be a graph whose vertex set V has n vertices and an edge set E. A *decomposition* of G is a partition of E. A *Hamilton cycle decomposition* of G is a decomposition of G into Hamilton cycles. A graph G must necessarily have even regularity for a Hamilton cycle decomposition of G to exist. The existence of Hamilton cycle decompositions for families of such graphs like K_n (n odd) and $K_n - F$ (n even and F a 1-factor) was classified in the late 19th century by Walecki [6]. Furthermore, the bipartite graphs $K_{n,n}$ (n even) and $K_{n,n} - F$ (n odd and F a bipartite 1-factor) have Hamilton cycle decompositions.

Let G be a k-uniform hypergraph with $V = \{v_0, \ldots, v_{n-1}\}$. Berge [3] generalized the definition of a Hamilton cycle H as a sequence of vertices and hyperedges

$$H = (v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_0),$$

where v_i and v_{i-1} are incident with e_i (modulo n) and $e_1, \dots, e_n$ are distinct hyperedges. A classification of the existence of Hamilton cycle decompositions of complete 3-uniform hypergraphs (also minus a 1-factor) of this type was completed in 1994 by Verrall [9].

Kierstead and Katona [5] introduced an alternative generalization of a Hamilton cycle; a Hamilton cycle in a k-uniform hypergraph is represented by a sequence of vertices of G

$$H = (v_0, v_1, \dots, v_{n-1}, v_0)$$

where the hyperedge $\{v_i, v_{i+1}, \dots, v_{i+k-1}\}$ (modulo n) is contained in H for each $i \in \mathbb{Z}_n$. Meszka and Rosa [7], along with Bailey and Stevens [2], investigated the existence of a Hamilton cycle decomposition of the complete k-uniform hypergraph $K_n^{(k)}$ for various n and k using the K-K definition of a Hamilton cycle. We will use this definition for the duration of the paper. For ease of notation, a *decomposition* will mean a K-K type Hamilton cycle decomposition of a hypergraph, unless otherwise specified. Furthermore, all indices for vertices and partitions are taken from quotient rings of integers (e.g. $\mathbb{Z}_m$) and any index arithmetic takes place in the specified ring.

Let $K_{k\times m} = K_{m,\dots,m}$ denote the complete k-partite graph in which each part contains m vertices. Let $V^0, \dots, V^{k-1}$ denote the parts of the vertex set V, where $V^i = \{0^i, 1^i, \dots, (m-1)^i\}$. The use of a superscript (or bars in the $k = 3$ case) indicates a vertex in a corresponding part of V, whereas the absence of such indicates the value of the vertex in $\mathbb{Z}_m$. Denote $K_{k\times m}^{(k)}$ as the complete k-uniform k-partite graph whose edge set E is defined as

$$E = \left\{ \{v_0^0, v_1^1, \dots, v_{k-1}^{k-1}\} : \ v_i^i \in V^i \text{ (and thus } v_i \in \mathbb{Z}_m) \right\}$$

and hence $|E| = m^k$. In this way, we associate E with $\mathbb{Z}_m^k$, where

$$\{v_0^0, \dots, v_{k-1}^{k-1}\} \longleftrightarrow (v_0, v_1, \dots, v_{k-1}).$$

Furthermore, a necessary condition for the existence of a Hamilton cycle decomposition of $K_{k\times m}^{(k)}$ is that $km \mid m^k$ or $k \mid m^{k-1}$.

In Section 2, we show that $K_{3\times m}^{(3)}$ has a Hamilton cycle decomposition when the necessary condition of $3 \mid m$ is satisfied and relate this to triomino tilings of an $m \times m$ grid on a torus. In Section 3, we classify when $K_{3\times m}^{(3)} - T$ has a Hamilton cycle decomposition, where $3 \nmid m$ and T is a 1-factor. In Section 4, we generalize this result to show that if $k \mid m$, then $K_{k\times m}^{(k)}$ has a Hamilton cycle decomposition. We also give an example showing that $k \mid m$ is not a necessary condition, and we conjecture that the necessary numerical condition above is sufficient. In Section 5, we summarize these results and relate them to symmetric Hamilton cycle decompositions.

2 Decompositions of $K_{3\times m}^{(3)}$

For this section, let $G = K_{3\times m}^{(3)}$. Since a necessary condition for the existence of a Hamilton cycle decomposition of G is $3m \mid m^3$, we may assume $3 \mid m$.

For convenience, we use W, $\overline{W}$, and $\overline{\overline{W}}$ instead of V^0, V^1, and V^2 to denote the vertices of G, where

$$\begin{aligned} W &= \{0, 1, \ldots, m-1\}, \\ \overline{W} &= \{\overline{0}, \overline{1}, \ldots, \overline{m-1}\}, \text{ and} \\ \overline{\overline{W}} &= \{\overline{\overline{0}}, \overline{\overline{1}}, \ldots, \overline{\overline{m-1}}\}. \end{aligned}$$

Let $(a, b, c) \in \mathbb{Z}_m^3$ denote the edge $\{a, \overline{b}, \overline{\overline{c}}\} \in E(G)$ for each $a \in W$, $\overline{b} \in \overline{W}$, and $\overline{\overline{c}} \in \overline{\overline{W}}$. A Hamilton cycle H of G is necessarily of the form

$$H = (a_0, \overline{b_0}, \overline{\overline{c_0}}, a_1, \overline{b_1}, \overline{\overline{c_1}}, \ldots, a_{m-1}, \overline{b_{m-1}}, \overline{\overline{c_{m-1}}}, a_0). \tag{1}$$

A Hamilton cycle H is called *cyclic* if there exists a *difference* $d \in \mathbb{Z}_m$ such that $a_i - a_{i+1} = b_i - b_{i+1} = c_i - c_{i+1} = d$ for each $i \in \mathbb{Z}_m$. Note this is possible only when d is a unit of $\mathbb{Z}_m$. We now make some useful observations.

Observation 2.1. Let H be a cyclic Hamilton cycle with difference d as given in (1). Then $b_i - a_i = b_j - a_j$ and $c_i - b_i = c_j - b_j$ for each $i, j \in \mathbb{Z}_m$.

The *difference type* of an edge (a, b, c) is the ordered pair $(b-a, c-b)$. Note there are exactly m edges in G which have a specified difference type (x, y), and these edges form a 1-factor:

$$\{(a, a+x, a+x+y) : \ a \in \mathbb{Z}_m\}.$$

From these definitions, we have the following lemma.

Lemma 2.2. *Let H be a cyclic Hamilton cycle as given in* (1) *with difference d. Then there exists a difference type (x, y) for which H is the union of all edges of difference type (x, y), $(x+d, y)$ and $(x, y+d)$.*

Proof. Let H be a cyclic Hamilton cycle with difference d. Define $x = b_0 - a_0$, $y = c_0 - b_0$, and let $i \in \mathbb{Z}_m$. Using Observation 2.1, we compute the difference type of edge (a_i, b_i, c_i) to be

$$(b_i - a_i, c_i - b_i) = (b_0 - a_0, c_0 - b_0) = (x, y).$$

The sequence b_i, c_i, a_{i+1} in H implies that (a_{i+1}, b_i, c_i) is an edge in H, which has difference type

$$(b_i - a_{i+1}, c_i - b_i) = (b_i - (a_i - d), c_i - b_i) = (x+d, y).$$

Similarly, (a_{i+1}, b_{i+1}, c_i) is an edge in H, which has difference type

$$(b_{i+1} - a_{i+1}, c_i - b_{i+1}) = (x, c_i - (b_i - d)) = (x, y+d).$$

Since there are m edges of each difference type and $3m$ edges in H, it follows that H is the union of all edges of these three difference types. $\square$

Definition 2.3. The cyclic Hamilton cycle H from Lemma 2.2 with difference d is *centered* at (x, y), and denoted as $H = h^d_{x,y}$. If $d = 1$, we abbreviate H as $h_{x,y}$ or $h^+_{x,y}$. Similarly, we use $h^-_{x,y}$ when $d = -1$.

A set of three difference types which are of the form (x, y), $(x+d, y)$, and $(x, y+d)$ is called a *permissible triple.*

Example 2.4. Let $m = 3$. The 9 difference types of edges from $K^{(3)}_{3\times 3}$ can be partitioned into 3 permissible triples:

$$\{(0, 0), (1, 0), (0, 1)\},\ \{(1, 1), (2, 1), (1, 2)\},\ \text{and } \{(2, 2), (0, 2), (2, 0)\}.$$

These give the cyclic Hamilton cycles

$$\begin{aligned}
h_{0,0} &= (0, \overline{0}, \overline{\overline{0}}, 2, \overline{2}, \overline{\overline{2}}, 1, \overline{1}, \overline{\overline{1}}, 0),\\
h_{1,1} &= (0, \overline{1}, \overline{\overline{2}}, 2, \overline{0}, \overline{\overline{1}}, 1, \overline{2}, \overline{\overline{0}}, 0), \text{ and}\\
h_{2,2} &= (0, \overline{2}, \overline{\overline{1}}, 2, \overline{1}, \overline{\overline{0}}, 1, \overline{0}, \overline{\overline{2}}, 0).
\end{aligned}$$

Each Hamilton cycle corresponds to a permissible triple in a 3×3 grid representing $\mathbb{Z}^2_3$. The collection of permissible triples with $d = 1$ is equivalent to a covering of $\mathbb{Z}^2_3$ by triominoes on a torus. See Figure 2.

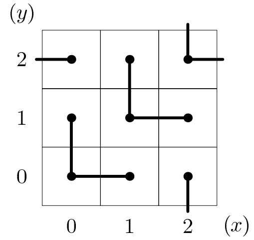

Figure 1: A triomino tiling of $\mathbb{Z}^2_3$

We now generalize Example 2.4 to find a decomposition of G in which each Hamilton cycle is cyclic by partitioning the m^2 difference types into $\frac{m^2}{3}$ permissible triples.

Let $i \in \mathbb{Z}_3$. Define $\mathcal{A}_i$ as

$$\mathcal{A}_i = \{(x, y) \in \mathbb{Z}^2_m : \ x - y \equiv i \bmod 3\}.$$

Note this is well-defined since $3 \mid m$.

Suppose that $(x, y) \in \mathcal{A}_0$ and d is a unit of $\mathbb{Z}_m$. Then $x \equiv y \bmod 3$, so

$$(x + d) - y \equiv d \bmod 3,\ \text{and } x - (y + d) \equiv 2d \bmod 3.$$

Thus every permissible triple contains a difference type from each $\mathcal{A}_i$, for each $i \in \mathbb{Z}_3$. Hence, $|\mathcal{A}_i| = \frac{m^2}{3}$ for each $i \in \mathbb{Z}_3$. Using this construction, we prove the following theorem.

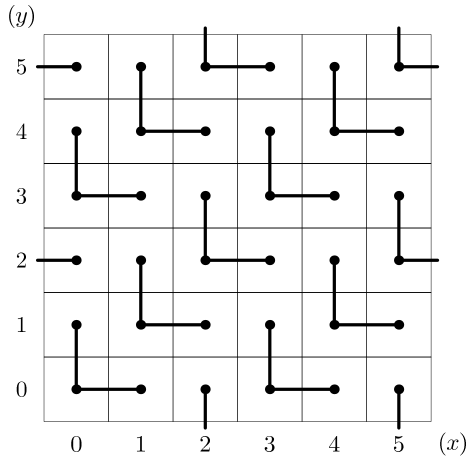

Figure 2: A triomino tiling of $\mathbb{Z}_6^2$.

Theorem 2.5. *Let $G = K^{(3)}_{3\times m}$. Then G has a decomposition if and only if $3 \mid m$.*

Proof. Suppose G has a decomposition. Since G has m^3 edges and each Hamilton cycle contains $3m$ edges, it follows that $3m \mid m^3$ and hence $3 \mid m$.

Now, suppose $3 \mid m$. Let $h_{x,y}$ and $h_{x',y'}$ be cyclic Hamilton cycles for which (x, y) and (x', y') are distinct elements of $\mathcal{A}_0$. These are disjoint Hamilton cycles provided their three difference types are distinct. Suppose not, and two difference types were identical. Those would then belong to the same $\mathcal{A}_i$. So either $(x+1, y) = (x'+1, y')$ or $(x, y+1) = (x', y'+1)$, which is not possible. Therefore, $h_{x,y}$ and $h_{x',y'}$ are disjoint Hamilton cycles.

Define $\mathcal{H} = \{h_{x,y} : \ (x, y) \in \mathcal{A}_0\}$. Then $\mathcal{H}$ is a partition of the edges of G into $\frac{m^2}{3}$ Hamilton cycles, giving a decomposition of G.

$\square$

Example 2.6. Let $m = 6$. Using the argument from Theorem 2.5, we can partition $\mathbb{Z}_6^2$ into permissible triples with $d = 1$. See Figure 2. The 36 difference types are partitioned into 12 permissible triples, which give rise to the following 12 Hamilton cycles:

$$\begin{aligned}
h_{0,0} &= (0, \overline{0}, \overline{\overline{0}}, 5, \overline{5}, \overline{\overline{5}}, 4, \overline{4}, \overline{\overline{4}}, 3, \overline{3}, \overline{\overline{3}}, 2, \overline{2}, \overline{\overline{2}}, 1, \overline{1}, \overline{\overline{1}}, 0),\\
h_{1,1} &= (0, \overline{1}, \overline{\overline{2}}, 5, \overline{0}, \overline{\overline{1}}, 4, \overline{5}, \overline{\overline{0}}, 3, \overline{4}, \overline{\overline{5}}, 2, \overline{3}, \overline{\overline{4}}, 1, \overline{2}, \overline{\overline{3}}, 0),\\
h_{2,2} &= (0, \overline{2}, \overline{\overline{4}}, 5, \overline{1}, \overline{\overline{3}}, 4, \overline{0}, \overline{\overline{2}}, 3, \overline{5}, \overline{\overline{1}}, 2, \overline{4}, \overline{\overline{0}}, 1, \overline{3}, \overline{\overline{5}}, 0),\\
h_{3,3} &= (0, \overline{3}, \overline{\overline{0}}, 5, \overline{2}, \overline{\overline{5}}, 4, \overline{1}, \overline{\overline{4}}, 3, \overline{0}, \overline{\overline{3}}, 2, \overline{5}, \overline{\overline{2}}, 1, \overline{4}, \overline{\overline{1}}, 0),\\
h_{4,4} &= (0, \overline{4}, \overline{\overline{2}}, 5, \overline{3}, \overline{\overline{1}}, 4, \overline{2}, \overline{\overline{0}}, 3, \overline{1}, \overline{\overline{5}}, 2, \overline{0}, \overline{\overline{4}}, 1, \overline{5}, \overline{\overline{3}}, 0),\\
h_{5,5} &= (0, \overline{5}, \overline{\overline{4}}, 5, \overline{4}, \overline{\overline{3}}, 4, \overline{3}, \overline{\overline{2}}, 3, \overline{2}, \overline{\overline{1}}, 2, \overline{1}, \overline{\overline{0}}, 1, \overline{0}, \overline{\overline{5}}, 0),
\end{aligned}$$

$$
\begin{aligned}
h_{0,3} &= (0,\overline{0},\overline{\overline{3}},5,\overline{5},\overline{\overline{2}},4,\overline{4},\overline{\overline{1}},3,\overline{3},\overline{\overline{0}},2,\overline{2},\overline{\overline{5}},1,\overline{1},\overline{\overline{4}},0),\\
h_{1,4} &= (0,\overline{1},\overline{\overline{5}},5,\overline{0},\overline{\overline{4}},4,\overline{5},\overline{\overline{3}},3,\overline{4},\overline{\overline{2}},2,\overline{3},\overline{\overline{1}},1,\overline{2},\overline{\overline{0}},0),\\
h_{2,5} &= (0,\overline{2},\overline{\overline{1}},5,\overline{1},\overline{\overline{0}},4,\overline{0},\overline{\overline{5}},3,\overline{5},\overline{\overline{4}},2,\overline{4},\overline{\overline{3}},1,\overline{3},\overline{\overline{2}},0),\\
h_{3,0} &= (0,\overline{3},\overline{\overline{3}},5,\overline{2},\overline{\overline{2}},4,\overline{1},\overline{\overline{1}},3,\overline{0},\overline{\overline{0}},2,\overline{5},\overline{\overline{5}},1,\overline{4},\overline{\overline{4}},0),\\
h_{4,1} &= (0,\overline{4},\overline{\overline{5}},5,\overline{3},\overline{\overline{4}},4,\overline{2},\overline{\overline{3}},3,\overline{1},\overline{\overline{2}},2,\overline{0},\overline{\overline{1}},1,\overline{5},\overline{\overline{0}},0), \quad \text{and}\\
h_{5,2} &= (0,\overline{5},\overline{\overline{1}},5,\overline{4},\overline{\overline{0}},4,\overline{3},\overline{\overline{5}},3,\overline{2},\overline{\overline{4}},2,\overline{1},\overline{\overline{3}},1,\overline{0},\overline{\overline{2}},0).
\end{aligned}
$$

3 Decompositions of $K^{(3)}_{3\times m} - T$, $3 \nmid m$

A 1-factor in a k-uniform hypergraph G with n vertices is a collection of edges $\{e_1, \ldots, e_{n/k}\}$ such that no two edges have any vertices in common.

By Theorem 2.5, a decomposition of $K^{(3)}_{3\times m}$ is possible when $3 \mid m$. However, if $3 \nmid m$, the removal of a 1-factor T leaves a total of $m^3 - m$ edges, which is divisible by $3m$, making it numerically possible for $K^{(3)}_{3\times m} - T$ to have a decomposition. We begin with some small examples.

Example 3.1.

(a) Let $G = K^{(3)}_{3\times 2}$ and T be any 1-factor of G. Then G is a Hamilton cycle and hence has a decomposition. If T is the edge set of difference type $(0,0)$, we can express this decomposition as a tiling of $\mathbb{Z}_2^2 - \{(0,0)\}$. See Figure 3.

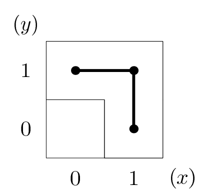

Figure 3: A tiling of $\mathbb{Z}_2^2 - \{(0,0)\}$.

(b) There does not exist a decomposition of $G = K^{(3)}_{3\times 4} - T$ for any 1-factor T.

Assume that $G = K^{(3)}_{3\times 4}$ has a decomposition into five Hamilton cycles $\{C_i : i \in [5]\}$ and a 1-factor T, and let $e = (a, b, c) \in T$. Let H be the generalized line graph of G, where $V(H) = E(G)$ and

$$E(H) = \{e_i e_j : \ e_i, e_j \in E(G) \text{ and } |e_i \cap e_j| = 2\}.$$

Then H is 9-regular and T is an independent set in H, implying that each edge of $N(e)$ belongs to one of the five Hamilton cycles. Thus $|N(e) \cap C_i| \le 2$ for each $i \in [5]$ and $\sum_{i=1}^{5} |N(e) \cap C_i| = 9$. It follows that four cycles intersect $N(e)$ in two vertices, say C_1, C_2, C_3, C_4. Then for each C_i, $i \in \{1, 2, 3, 4\}$, it follows that

$$
\begin{aligned}
&(a, b, c'), (a, b, c'') \in C_i \text{ for some } c', c'' \in (\mathbb{Z}_4)\backslash\{c\},\\
&(a, b', c), (a, b'', c) \in C_i \text{ for some } b', b'' \in (\mathbb{Z}_4)\backslash\{b\}, \text{ or}\\
&(a', b, c), (a'', b, c) \in C_i \text{ for some } a', a'' \in (\mathbb{Z}_4)\backslash\{a\}.
\end{aligned}
$$

So there must be two cycles, say C_1 and C_2, which intersect e at the same pair of vertices, say a and b. Suppose that the edges $(a,b,c'),(a,b,c'') \in C_1$ and $(a,b,d'),(a,b,d'') \in C_2$. Then $\{c,c',c'',d',d''\}$ is a set of distinct elements of $\mathbb{Z}_4$, which is impossible. Therefore, there is no decomposition of $K^{(3)}_{3\times 4} - T$ into Hamilton cycles for any 1-factor T.

(c) Let $G = K^{(3)}_{3\times 5} - T$ where T is the edge set of difference type $(2,2)$. Then

$$\mathcal{H} = \{h^+_{0,0}, h^+_{0,2}, h^+_{3,0}, h^+_{2,3}, h^-_{2,1}, h^-_{1,4}, h^-_{4,2}, h^-_{4,4}\}$$

is a decomposition of G. See Figure 4.

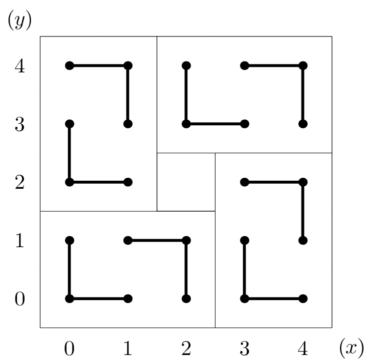

Figure 4: A tiling of $(\mathbb{Z}_5)^2 - \{(2,2)\}$.

(d) Let $G = K^{(3)}_{3\times 7} - T$, where T is the edge set of difference type $(3,3)$. Then

$$\begin{aligned}\mathcal{H} \; = \; & \{h^+_{0,0}, h^+_{3,0}, h^+_{5,0}, h^+_{0,2}, h^+_{4,3}, h^+_{0,4}, h^+_{2,4}, h^+_{4,5}\} \;\cup \\ & \{h^-_{2,1}, h^-_{4,2}, h^-_{6,2}, h^-_{2,3}, h^-_{6,4}, h^-_{1,6}, h^-_{3,6}, h^-_{6,6}\}\end{aligned}$$

is a decomposition of G. See Figure 5.

(e) Let $G = K^{(3)}_{3\times 10} - T$, where T is the edge set of difference type $(4,7)$. Then

$$\begin{aligned}\mathcal{H} \; = \; & \{h^+_{0,0}, h^+_{3,0}, h^+_{6,0}, h^+_{8,0}, h^+_{0,2}, h^+_{2,2}, h^+_{4,2}, h^+_{6,3}\} \;\cup \\ & \{h^+_{8,3}, h^+_{0,5}, h^+_{3,5}, h^+_{7,6}, h^+_{0,7}, h^+_{2,7}, h^+_{4,8}, h^+_{7,8}\} \;\cup \\ & \{h^-_{2,1}, h^-_{5,1}, h^-_{7,2}, h^-_{9,2}, h^-_{1,4}, h^-_{3,4}, h^-_{5,4}, h^-_{7,5}\} \;\cup \\ & \{h^-_{9,5}, h^-_{2,6}, h^-_{5,6}, h^-_{6,7}, h^-_{9,7}, h^-_{1,9}, h^-_{3,9}, h^-_{6,9}, h^-_{9,9}\}\end{aligned}$$

is a decomposition of G. See Figure 6.

We now make some observations that will allow us to inductively construct decompositions for $K^{(3)}_{3\times m} - T$ for some 1-factor T and arbitrary m with $3 \nmid m$.

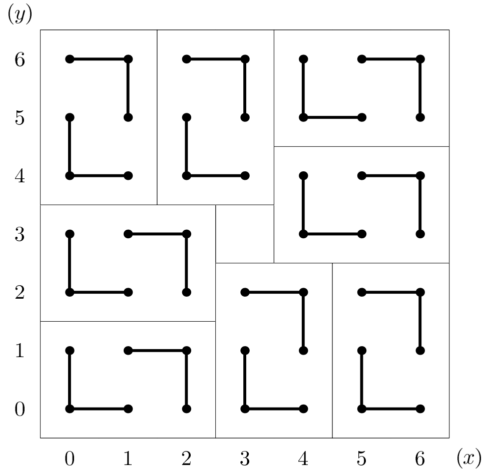

Figure 5: A tiling of $(\mathbb{Z}_7)^2 - \{(3,3)\}$.

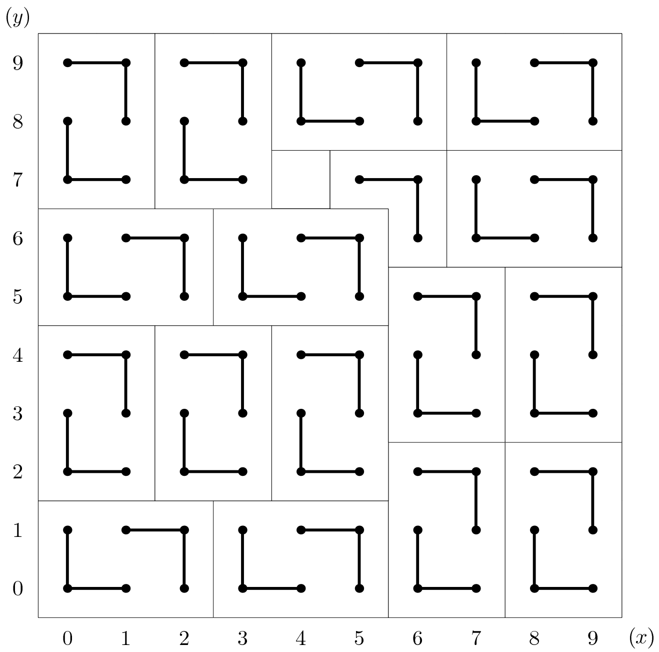

Figure 6: A tiling of $(\mathbb{Z}_{10})^2 - \{(4,7)\}$.

Observation 3.2.

(a) For each case in Example 3.1, the Hamilton cycles of the form $h^+_{x,y}$ are such that $x \neq m-1$ and $y \neq m-1$. Similarly, each Hamilton cycle $h^-_{x,y}$ is such that

$x \neq 0$ and $y \neq 0$.

(b) If X contains all edges from a 3×2 block of difference types $\{x, x+1, x+2\} \times \{y, y+1\} \subseteq \mathbb{Z}_m^2$, then X has a decomposition $\{h^+_{x,y}, h^-_{x+2,y+1}\}$, which we denote as $\mathcal{B}_{x,y}$.

If X contains all edges from a 2×3 block of difference types $\{x, x+1\} \times \{y, y+1, y+2\} \subseteq \mathbb{Z}_m^2$, then X has a decomposition $\{h^+_{x,y}, h^-_{x+1,y+2}\}$, denoted as $\mathcal{B}^{\mathrm{T}}_{x,y}$.

For example, the graph in Example 3.1(c) has the decomposition

$$\mathcal{H} = \mathcal{B}_{0,0} \cup \mathcal{B}^{\mathrm{T}}_{3,0} \cup \mathcal{B}^{\mathrm{T}}_{0,2} \cup \mathcal{B}_{2,3}.$$

In the previous examples, all Hamilton cycles came from sets of this type, with a single exception in the $m = 2$ or 10 case.

(c) For any two 1-factors T and T' of $K^{(3)}_{3\times m}$, there is an automorphism which sends T to T'.

Using this notation, we prove a simple, yet useful lemma.

Lemma 3.3. *Let $t > 1$. If X is a set of all edges from a $6 \times t$ block of difference types, then X has a decomposition. Similarly, if X is a set of all edges from a $t \times 6$ block of difference types, the X has a decomposition.*

Proof. Suppose X contains all edges of difference types from the block $\{x, \dots, x+5\} \times \{y, \dots, y+t-1\}$, and first assume that t is even. Then

$$\mathcal{H} = \{\mathcal{B}_{x,y+2i}, \mathcal{B}_{x+3,y+2i} : \ 0 \leq i < \tfrac{t}{2}\}$$

is a decomposition of X. If t is odd, then $t \geq 3$ and

$$\begin{aligned} \mathcal{H} \;=\; & \{\mathcal{B}^{\mathrm{T}}_{x,y}, \mathcal{B}^{\mathrm{T}}_{x+2,y}, \mathcal{B}^{\mathrm{T}}_{x+4,y}\} \quad \cup \\ & \{\mathcal{B}_{x,y+3+2i}, \mathcal{B}_{x+3,y+3+2i} : \ 0 \leq i < \tfrac{t-3}{2}\} \end{aligned}$$

is a decomposition of X. Note that by swapping $\mathcal{B}$ with $\mathcal{B}^{\mathrm{T}}$ in this argument, we make the same conclusion if X is the set of all edges from a $t \times 6$ block of difference types. $\square$

We use this lemma and the previous observations to prove the following classification.

Theorem 3.4. *Let $m > 1$ and $3 \nmid m$. Let T be a 1-factor of $K^{(3)}_{3\times m}$, and let $G = K^{(3)}_{3\times m} - T$. Then G has a decomposition if and only if $m \neq 4$.*

Proof. If G has a decomposition, then $m \neq 4$ by Example 3.1(b).

Suppose that $m \neq 4$ and $3 \nmid m$. We will proceed by induction on m. Assume that $m > 7$ and $K^{(3)}_{3\times(m-6)} - T''$ has a decomposition $\mathcal{H}$ for some 1-factor T'' consisting of edges of the same difference type. Furthermore, assume each Hamilton cycle in this decomposition is cyclic of the form $h^+_{x,y}$ or $h^-_{x,y}$, and each satisfies the conditions in Observation 3.2(a). Let X^+ and X^- be subsets of $\mathbb{Z}_m^2$ defined as

$$\begin{aligned} X^+ &= \{(x,y) : \ h^+_{x,y} \in \mathcal{H}\}, \text{ and} \\ X^- &= \{(x,y) : \ h^-_{x,y} \in \mathcal{H}\}. \end{aligned}$$

Furthermore, let H^+_X and H^-_X be Hamilton cycles of $K^{(3)}_{3\times m}$ defined as

$$\begin{aligned} H^+_X &= \{h^+_{x,y} : \ (x,y) \in X^+\}, \text{ and} \\ H^-_X &= \{h^-_{x,y} : \ (x,y) \in X^-\}. \end{aligned}$$

By Lemma 3.3, the edges from the block of difference types $\{m-6,\ldots,m-1\} \times \{0,\ldots,m-7\}$ has a decomposition $\mathcal{H}'$ and the edges from the block of difference types $\{0,\ldots,m-1\} \times \{m-6,\ldots,m-1\}$ has a decomposition $\mathcal{H}''$. Thus, the collection of Hamilton cycles

$$H^+_X \cup H^-_X \cup \mathcal{H}' \cup \mathcal{H}''$$

is a decomposition of $K^{(3)}_{3\times m} - T'$, where T' is a 1-factor of edges in $K^{(3)}_{3\times m}$ which have the same difference type as those in T''. By Observation 3.2(c), the theorem follows. ☐

Example 3.5. Let $m = 8$. Following the construction in Theorem 3.4, we find a decomposition of $K^{(3)}_{3\times 8} - T$, with T being the edges of difference type $(0,0)$. First use the decomposition if $K^{(3)}_{3\times 2} - T'$, and then "pad" with blocks $\mathcal{B}$ and $\mathcal{B}^{\mathrm{T}}$. See Figure 7.

4 Decompositions of $K^{(k)}_{k\times m}$, $k \mid m$

Let $G = K^{(k)}_{k\times m}$. For each $i \in \mathbb{Z}_k$, let $V^i = \{0^i, 1^i, \ldots, (m-1)^i\}$. Denote the vertex set of G as $V = V^0 \cup V^1 \cup \cdots \cup V^{k-1}$. Let $(x_0, x_1, \ldots, x_{k-1}) \in \mathbb{Z}_m^k$ denote the edge $\{x^0_0, x^1_1, \ldots, x^{k-1}_{k-1}\} \in E(G)$. As with $k = 3$, a Hamilton cycle H of G is necessarily of the form

$$\begin{aligned} H = (\ & a^0_{0,0}, a^1_{1,0}, \ldots, a^{k-1}_{k-1,0}, \\ & a^0_{0,1}, a^1_{1,1}, \ldots, a^{k-1}_{k-1,1}, \\ & \vdots \\ & a^0_{0,m-1}, a^1_{1,m-1}, \ldots, a^{k-1}_{k-1,m-1}, a^0_{0,0} \ ), \end{aligned} \tag{2}$$

where $\mathbb{Z}_m = \{a_{i,j} : j \in \mathbb{Z}_m\}$ for each $i \in \mathbb{Z}_k$, using the convention that $a^i_{i,j} = (a_{i,j})^i \in V^i$.

Definition 4.1. A Hamilton cycle H is *cyclic* if there exists a *difference* $d \in \mathbb{Z}_m$ such that $a_{i,j} - a_{i,j+1} = d$ for each $i \in \mathbb{Z}_k$ and $j \in \mathbb{Z}_m$. Again, this is possible only when d is a unit of $\mathbb{Z}_m$. Also, along the lines of Observation 2.1, such a cyclic Hamilton

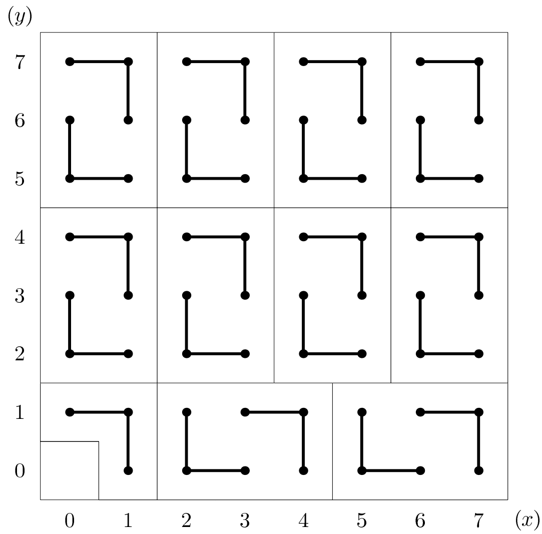

Figure 7: A tiling of $(\mathbb{Z}_8)^2 - \{(0,0)\}$ arising from the tiling in Figure 3.

cycle has the property that $a_{i,j} - a_{i+1,j} = a_{i,l} - a_{i+1,l}$ for each $i \in \mathbb{Z}_k \backslash \{k-1\}$ and $j, l \in \mathbb{Z}_m$.

Similarly, the *difference type* of an edge $(a_0, a_1, \ldots, a_{k-1})$ is the ordered $(k-1)$-tuple $(x_1, \ldots, x_{k-1})$, where $x_i = a_i - a_{i-1}$ for each $i \in \mathbb{Z}_k \backslash \{0\}$. Again, there are exactly m edges in G which have a specified difference type, which form the 1-factor

$$\{(a, a + x_1, a + x_1 + x_2, \ldots, a + x_1 + x_2 + \cdots + x_{k-1}) : \; a \in \mathbb{Z}_m\}. \tag{3}$$

From these definitions, we have the following generalization of Lemma 2.2:

Lemma 4.2. *Let H be a cyclic Hamilton cycle as given in* (2) *with difference d. Then there exists a difference type $(x_1, \ldots, x_{k-1}) \in \mathbb{Z}_m^{k-1}$ for which H is the union of all edges of the k difference types*

$$\begin{array}{l} (x_1, x_2, \ldots, x_{k-1}), \\ (x_1 + d, x_2, \ldots, x_{k-1}), \\ (x_1, x_2 + d, \ldots, x_{k-1}), \\ \quad\vdots \\ (x_1, x_2, \ldots, x_{k-1} + d). \end{array} \tag{4}$$

Proof. Let H be a cyclic Hamilton cycle as given in (2) with difference d. Define $x_i = a_{i-1,0} - a_{i,0}$ for each $i \in \mathbb{Z}_k \backslash \{0\}$.

Let $i \in \mathbb{Z}_m$. The edge $(a_{0,i}, a_{1,i}, \ldots, a_{k-1,i})$ has difference type

$$\begin{array}{rl} & (a_{0,i} - a_{1,i}, a_{1,i} - a_{2,i}, \ldots, a_{k-2,i} - a_{k-1,i}) \\ = & (a_{0,0} - a_{1,0}, a_{1,0} - a_{2,0}, \ldots, a_{k-2,0} - a_{k-1,0}) \\ = & (x_1, \ldots, x_{k-1}). \end{array}$$

Let $j \in \mathbb{Z}_k \backslash \{k-1\}$. The edge

$$(a_{0,i+1}, a_{1,i+1}, \ldots, a_{j,i+1}, a_{j+1,i}, \ldots, a_{k-1,i})$$

has difference type

$$\begin{aligned} & (\; a_{0,i+1} - a_{1,i+1}, a_{1,i+1} - a_{2,i+1}, \ldots, a_{j-1,i+1} - a_{j,i+1}, \\ & \quad a_{j,i+1} - a_{j+1,i}, a_{j+1,i} - a_{j+2,i}, \ldots, a_{k-2,i} - a_{k-1,i} \;) \\ = & (\; a_{0,0} - a_{1,0}, a_{1,0} - a_{2,0}, \ldots, a_{j-1,0} - a_{j,0}, \\ & \quad a_{j,1} - a_{j+1,0}, a_{j+1,0} - a_{j+2,0}, \ldots, a_{k-2,0} - a_{k-1,0} \;) \\ = & (\; x_1, x_2, \ldots, x_{j-1}, x_j + d, x_{j+1}, \ldots, x_{k-1} \;). \end{aligned}$$

Again, since there are m edges of each difference type and km edges in H, it follows that H is the union of all edges of these k difference types. □

Definition 4.3. The cyclic Hamilton cycle H from Lemma 4.2 with difference d is *centered* at $X = (x_0, x_1, \ldots, x_{k-1})$, and denoted as $H = h_X^d$. We abbreviate H as h_X^+ (or simply h_X) when $d = 1$, or h_X^- when $d = -1$. A set of k difference types which are of the form given in (4) is called a *permissible k-tuple*.

Example 4.4. Let $k = 4$ and $m = 2$. Let $V^i = \{0^i, 1^i\}$ for each $i \in \mathbb{Z}_4$. Then $K_{4\times 2}^{(4)}$ has 8 difference types from $\mathbb{Z}_2^3$, which can be partitioned into two permissible 4-tuples:

$$\begin{array}{l} \{(0,0,0),(0,0,1),(0,1,0),(1,0,0)\} \text{ and} \\ \{(1,1,1),(1,1,0),(1,0,1),(0,1,1)\}, \end{array}$$

which gives a decomposition of $K_{4\times 2}^{(4)}$:

$$\begin{aligned} h_{0,0,0}^+ &= (0^0, 0^1, 0^2, 0^3, 1^0, 1^1, 1^2, 1^3, 0^0), \text{ and} \\ h_{1,1,1}^- &= (0^0, 1^1, 0^2, 1^3, 1^0, 0^1, 1^2, 0^3, 0^0). \end{aligned}$$

The following is a generalization of the constructions in Section 2. We partition the m^{k-1} difference types of Z_m^{k-1} into $\frac{m^{k-1}}{k}$ permissible k-tuples, thereby giving a decomposition of $K_{k\times m}^{(k)}$ into cyclic Hamilton cycles. For each $i \in \mathbb{Z}_k$, define $\mathcal{A}_i$ as

$$\mathcal{A}_i = \left\{ (x_1, \ldots, x_{k-1}) \in \mathbb{Z}_m^{k-1} : \sum_{j=1}^{k-1} j \cdot x_j \equiv i \bmod k \right\}.$$

At this point, we assume that $k \mid m$ for these sets to be well-defined. Suppose that $(x_1, \ldots, x_{k-1}) \in \mathcal{A}_0$. Let $i \in \mathbb{Z}_k \backslash \{0\}$. Then

$$\sum_{j=1}^{i-1} (j \cdot x_j) + i \cdot (x_i + d) + \sum_{j=i+1}^{k-1} (j \cdot x_j) = i \cdot d + \sum_{j=1}^{k-1} j \cdot x_j \equiv i \cdot d \bmod k.$$

So, $(x_1, \ldots, x_{i-1}, x_i + d, x_{i+1}, \ldots, x_{k-1}) \in \mathcal{A}_{id}$. Thus, since d is a unit of $\mathbb{Z}_m$, each element in any given permissible k-tuple belongs to a distinct $\mathcal{A}_i$. So $|\mathcal{A}_i| = \frac{m^{k-1}}{k}$. Using this construction, we prove the following theorem.

Theorem 4.5. *The graph* $G = K^{(k)}_{k\times m}$ *has a decomposition if* $k \mid m$.

Proof. Let $X = (x_1, \dots, x_{k-1})$ and $X' = (x'_1, \dots, x'_{k-1})$ be distinct elements of $\mathcal{A}_0$ and $H = h_X$ and $H' = h_{X'}$ be their corresponding cyclic Hamilton cycles.

Assume H and H' are not disjoint. Since X and X' are distinct, there is a common difference type to both H and H' belonging to $\mathcal{A}_i$ for some $i \in \mathbb{Z}_k\backslash\{0\}$. Those difference types in H and H' belonging to $\mathcal{A}_i$ are $(x_1, \dots, x_i + 1, \dots, x_{k-1})$ and $(x'_1, \dots, x'_i + 1, \dots, x'_{k-1})$, respectively. Therefore, $x_j = x'_j$ for each $j \in \mathbb{Z}_k\backslash\{0\}$, which contradicts X and X' being distinct. So H and H' are edge-disjoint Hamilton cycles.

Define $\mathcal{H} = \{h_X : X \in \mathcal{A}_0\}$. This is a partition of the edges of G, and $\mathcal{H}$ consists of $\frac{m^{k-1}}{k}$ Hamilton cycles, giving a decomposition. ☐

Example 4.6. The results from Theorem 4.5 imply that if 4 divides m, then $G = K^{(4)}_{4\times m}$ has a decomposition. The graph in Example 4.4 has a decomposition, but $4 \nmid m$. Furthermore, if m is even, then $4 \mid m^3$, so the necessary condition is satisfied. Define $\mathcal{H}$ as

$$\mathcal{H} = \{h^+_X : X \in (2\mathbb{Z}_m)^3\} \cup \{h^-_X : X \in (1 + 2\mathbb{Z}_m)^3\}$$

This produces $\frac{m^3}{4}$ Hamilton cycles which are disjoint, so $\mathcal{H}$ is a decomposition of $K^{(4)}_{4\times m}$ for m even.

A necessary condition for the existence of a decomposition of $K^{(k)}_{k\times m}$ is that $k \mid m^{k-1}$. The previous example leads to the following conjecture.

Conjecture 4.7. *Let* $G = K^{(k)}_{k\times m}$. *Then* G *has a decomposition if and only if* $k \mid m^{k-1}$.

5 Summary

We summarize the results from the previous sections in the following theorem.

Theorem 5.1. *Let* $m > 1$ *and* $k \geq 2$ *be given. Then a hypergraph* G *has a decomposition if*

- $G = K^{(3)}_{3\times m} - T$, *where* $3 \nmid m$, $m \neq 4$ *and* T *is a 1-factor,*
- $G = K^{(k)}_{k\times m}$ *and* $k \mid m$, *or*
- $G = K^{(4)}_{4\times m}$ *and* m *is even.*

Let $k \geq 2$. Define k' as the largest square-free divisor of k. Then $k \mid (k')^{k-1}$ and k' is the smallest such integer for which such divisibility holds. Thus to prove Conjecture 4.7, the remaining cases one must consider are values of m for which $k' \mid m$, $k \nmid m$, and $k \neq 4$.

Symmetric Decompositions

The constructions from the previous sections have nice structure which relates to symmetry as defined in [8].

Definition 5.2. Let ϕ be an automorphism of a graph G. A Hamilton cycle decomposition is *ϕ-symmetric* if each Hamilton cycle in the decomposition is ϕ-invariant as an edge set.

In [8], [1] and [4], the existence of ϕ-symmetric Hamilton cycle decompositions is classified for K_n, $K_n - F$, $K_{n,n}$, $K_{n,n} - F$, and $K_{m\times t}$, where F is a 1-factor. This notion can be extended to hypergraphs.

Observation 5.3. Let $G = K^{(k)}_{k\times m}$ and let $X = (x_1, \dots, x_{k-1})$ be a difference type of G. Then the set of edges with difference type X is ϕ-invariant, where $\phi : V \to V$ is the map $\phi(a_i) = (a + 1)_i$ in $\mathbb{Z}_m$, as shown in (3). Furthermore, any cyclic Hamilton cycle is also ϕ-symmetric.

We conclude with the following theorem.

Theorem 5.4. *Let $m > 1$ and $k \geq 2$ be given. Then for each graph G given in Theorem 5.1, there exists a Hamilton cycle decomposition of G which is ϕ-symmetric for an appropriate order m automorphism.*

Proof. In each of the cases above, the constructions in the previous sections involve only cyclic Hamilton cycles of the type given in (2) and Lemma 4.2, and the result follows from Observation 5.3. ☐

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