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A class of trees with equal broadcast and domination numbers

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Abstract

A *broadcast* on a graph G is a function $f : V \to \{0, 1, 2, \dots\}$. The *broadcast number* of G is the minimum value of $\sum_{v \in V} f(v)$ among all broadcasts f for which each vertex of G is within distance $f(v)$ from some vertex v with $f(v) \geq 1$. The broadcast number is bounded above by the radius and the domination number of G.

We consider a class of trees that contains the caterpillars and characterize the trees in this class that have equal domination and broadcast numbers, thus generalizing the results in: [S. M. Seager, Dominating broadcasts of caterpillars, *Ars Combin.* 88 (2008), 307–319].

1 Introduction

We place broadcast towers on some of the vertices of a graph and broadcast from each tower to all vertices within its range. The cost of the broadcast is proportional to the strength of the broadcast, and our goal is to broadcast to the entire graph with minimum cost. We need a few definitions to formalize this description.

A *broadcast* on a graph G is a function $f : V(G) \to \{0, 1, 2, \dots\}$. A *broadcast vertex* is a vertex v for which $f(v) \geq 1$. The set of all broadcast vertices is denoted $V_f^+(G)$, or V_f^+ when the graph under consideration is clear. A vertex u *hears* a broadcast from $v \in V_f^+$, and v *broadcasts to* u, if the distance between u and v is at most $f(v)$ (possibly $u = v$).

A broadcast f is a *dominating broadcast* if every vertex hears at least one broadcast. The *cost* of a broadcast f is defined as $\text{cost}(f) = \sum_{v \in V(G)} f(v)$, and the

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broadcast number of G is $\gamma_b(G) = \min\{\text{cost}(f) : f$ is a dominating broadcast of $G\}$. If f is a dominating broadcast such that $f(v) = 1$ for each $v \in V_f^+$, then V_f^+ is a *dominating set* of G, and the minimum cost of such a broadcast is the usual *domination number* $\gamma(G)$.

The *eccentricity* of a vertex v of a graph G is $e(v) = \max\{d(u, v) : u \in V(G)\}$. The *radius* and *diameter* of G are defined as $\operatorname{rad} G = \min\{e(v) : v \in V(G)\}$ and $\operatorname{diam} G = \max\{e(v) : v \in V(G)\}$, respectively.

Erwin [7, 8] was the first to consider the broadcast domination problem, and to observe the trivial bound $\gamma_b(G) \leq \min\{\operatorname{rad} G, \gamma(G)\}$ for any graph G. This bound immediately suggests the following questions:

For which graphs G is $\gamma_b(G) = \operatorname{rad} G$? For which graphs is $\gamma_b(G) = \gamma(G)$?

Graphs for which $\gamma_b(G) = \operatorname{rad} G$ are called *radial graphs.* The problem of characterizing radial trees was first addressed by Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi in [5] and also studied in [6, 13]. It was solved by Herke and Mynhardt [12] (see Theorem 2.1), who also showed that a tree T can be split into radial subtrees by deleting edges on a diametrical path of T.

Here we consider the second question for trees. A graph (tree) G such that $\gamma_b(G) = \gamma(G)$ is called a 1-*cap graph* (*tree*) – there exists a minimum cost broadcast where each tower broadcasts with a capacity equal to one. Heggernes and Lokshtanov [10] showed that minimum broadcast domination is solvable in polynomial time for any graph, while computing the domination number is NP-hard in general. Both the domination and broadcast numbers of a tree can be determined in linear time (see [2] and [4], respectively), but knowing that $\gamma(T) = \gamma_b(T)$ for some tree T (or for finitely many given trees) does not adequately reveal the properties of 1-cap trees, which merits investigation in its own right.

Seager [13] initiated this investigation and characterized 1-cap caterpillars. Cockayne, Herke and Mynhardt [3] showed that a tree is 1-cap if and only if it can be split into radial subtrees, each of which is 1-cap. However, their result does not show how such a split can be accomplished. There could be several ways of splitting a tree into radial subtrees, and while one split may yield 1-cap subtrees, another split may not. An example of such a 1-cap tree is given in [3, Figure 2]. In addition, the characterization of even **radial 1-cap trees** appears to be a difficult problem. We investigate this problem for a large class $\mathcal{H}^*$ of trees that contains the caterpillars.

We denote the class of all 1-cap trees T by $\mathcal{T}$ and let $\mathcal{T}_k = \{T \in \mathcal{T} : \gamma(T) = \gamma_b(T) = k\}$. We apply results from [3] and characterize the trees in $\mathcal{H}^*$ that are in $\mathcal{T}$, thus generalizing the results in [13].

After giving a few more definitions and earlier results in Section 2, we discuss the use of a special class of trees, called shadow trees, and isosceles right triangles in Section 3. Cockayne et al. [3] showed that one only needs to consider shadow trees when studying the class $\mathcal{T}$. A shadow tree consists of a longest path P with other paths, called boughs, attached to distinct vertices of P. In Section 4 we consider the subclass $\mathcal{H}$ of shadow trees where the boughs have length congruent to 1 (mod 3),

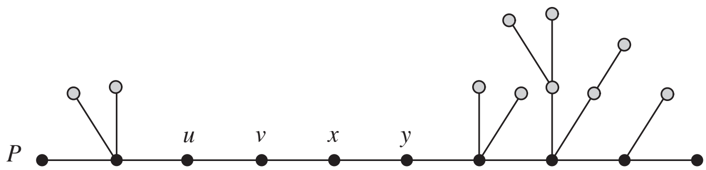

Figure 1: A tree with split-sets $\{uv\}$ and $\{xy\}$

which contains the shadow trees of caterpillars. We state some properties of trees in $\mathcal{H} \cap \mathcal{T}$ as lemmas. We also state and prove our main theorem, the characterization of the class $\mathcal{H} \cap \mathcal{T}$. Section 5 concerns the application of the characterization to caterpillars and to general trees in $\mathcal{H}^*$. Finally, the proofs of the lemmas in Section 4 are given in Section 6.

2 Definitions and background

For undefined concepts see [1, 9]. A dominating broadcast f of a graph G for which $\operatorname{cost}(f) = \gamma_b(G)$ is called a *γ_b-broadcast*, and a dominating set D such that $|D| = \gamma(G)$ is called a *γ-set*. The *open neighbourhood* $N(v)$ of $v \in V(G)$ is the set of vertices adjacent to v and the *closed neighbourhood* of v is $N[v] = N(v) \cup \{v\}$. For $v \in D$, the *private neighbourhood of v relative to D*, denoted by $\operatorname{pn}(v, D)$, is the set $N[v] - N[D - \{v\}]$. Define the subset D_{spn} of a dominating set D by $D_{\mathrm{spn}} = \{v \in D : \operatorname{pn}(v, D) = \{v\}\}$.

A *diametrical path* (abbreviated *d-path*) of a tree T is a path of length $\operatorname{diam} T$. A path is *even* or *odd*, corresponding to the parity of its length. A *central* vertex of a graph G is a vertex v such that $e(v) = \operatorname{rad} G$. A tree is either *central* or *bicentral*, depending on whether it has one or two (adjacent) central vertices; any d-path of a tree contains its *centre*, the set of all central vertices.

A set M of edges of a d-path P is a *split-P set* if, for each component T' of $T - M$, the path $P \cap T'$ is a d-path of T' of even positive length. A *split-set* of T is a split-P set for some d-path P of T, and a *maximum split-set* of T is a split-set of maximum cardinality. For example, the sets $\{uv\}$ and $\{xy\}$ are maximum split-P sets of the tree in Fig. 1, where P is the path of black vertices. Radial trees are characterized as follows.

Theorem 2.1 [11, 12] *A tree T is radial if and only if it has no nonempty split-set.*

Split-sets are used to determine the broadcast number of a tree.

Theorem 2.2 [11, 12] *If M is a split-set of maximum cardinality m of the tree T, and $T_1, \ldots, T_{m+1}$ are the components of $T - M$, then*

$$\gamma_b(T) = \left\lceil \frac{\operatorname{diam}(T) - m}{2} \right\rceil = \operatorname{rad} T - \left\lceil \frac{m}{2} \right\rceil = \sum_{i=1}^{m+1} \gamma_b(T_i).$$

Theorem 2.1 was used in [3] to prove the following result.

Theorem 2.3 [3] *A tree $T \in \mathcal{T}$ if and only if it has a maximum split-set M such that $T_i \in \mathcal{T}$ for each component T_i of $T - M$.*

3 Shadow trees and isosceles right triangles

Cockayne et al. [3] showed that one only needs to consider certain types of trees, called shadow trees, when studying the class $\mathcal{T}$. They used isosceles right triangles to describe the positions of the boughs on P and showed that the actual lengths of the boughs are not important, only their congruence classes modulo 3 and the number of edges by which two consecutive triangles overlap.

3.1 Shadow trees

Let $P = v_1, \ldots, v_n$ be a d-path of the tree T. For each i, let A_i be the set of all vertices of T that are connected to v_i by a (possibly trivial) path that is internally disjoint from P. Let B_i be a longest path in $T[A_i]$ that has initial vertex v_i. The *shadow tree of T with respect to P*, denoted $S_{T,P}$, is the subtree of T induced by $\bigcup_{i=1}^{n} V(B_i)$.

A tree T with d-path P is depicted in Fig. 2, which illustrates the construction of the shadow tree $S_{T,P}$. The path B_i is called a *bough* of $S_{T,P}$ at v_i. If $T = S_{T,P}$, we also call T a *shadow tree*; any shadow tree is the shadow tree of infinitely many trees. Note that if P and P' are different d-paths of T, then it is possible that $S_{T,P} \ncong S_{T,P'}$. If the d-path P is understood or irrelevant, we abbreviate $S_{T,P}$ to S_T. Herke and Mynhardt [12] demonstrated the relevance of shadow trees to the study of broadcast domination.

Theorem 3.1 [12] *For any shadow tree S_T of T, $\gamma_b(S_T) = \gamma_b(T)$.*

The following results show that shadow trees are of interest in the study of the class $\mathcal{T}$.

Corollary 3.2 [3] *(i) If $T \in \mathcal{T}_k$, then $\gamma(T) = \gamma(S_T)$.*

(ii) If $T \in \mathcal{T}_k$, then $S_T \in \mathcal{T}_k$.

(iii) If $S_T \in \mathcal{T}_k$ and $\gamma(T) = k$, then $T \in \mathcal{T}_k$.

The relatively simple structure of shadow trees suggests the following approach to the study of the sets $\mathcal{T}_k$.

Step 1 Find subsets of $\mathcal{T}_k$ containing only shadow trees.

Step 2 If T is a shadow tree in $\mathcal{T}_k$, use Corollary 3.2(*iii*) to find all trees in $\mathcal{T}_k$ that have T as shadow tree.

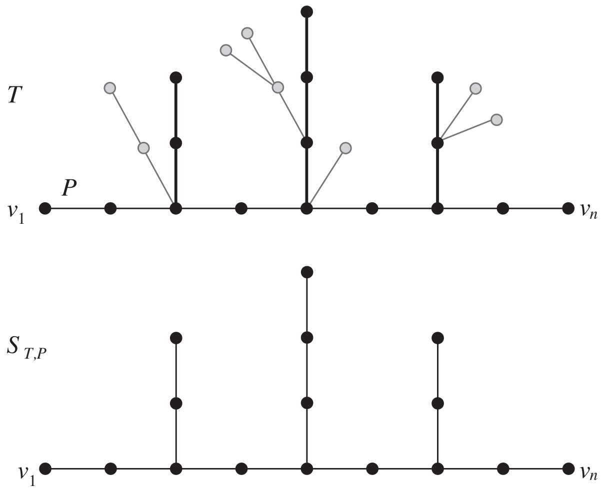

Figure 2: Shadow-tree construction

Necessary and sufficient conditions for a tree T and a subtree T' to have equal domination numbers were given in [3]. Let $W_1, \ldots, W_t$ be the nontrivial components of $T - E(T')$. For $i = 1, \ldots, t$, let u_i be the unique vertex of $V(T') \cap V(W_i)$. We call u_i the *hinge* of W_i and also say that W_i *is hinged at* u_i. Let U_1 (respectively U_2) be the set of hinges of subtrees W_i that are stars hinged at their centres, or at one of their leaves if $W_i = K_2$ (respectively at one of their leaves, where $W_i \neq K_2$). Note that $U_1 \cap U_2 = \varnothing$.

Proposition 3.3 [3] *Let T' be a subtree of the tree T. Then $\gamma(T) = \gamma(T')$ if and only if*

- (i) *each subtree W_i is either a star hinged at its centre or a star hinged one of its leaves, and*
- (ii) *T' has a γ-set D with $U_1 \subseteq D$ and $U_2 \subseteq D_{\mathrm{spn}}$.*

3.2 Isosceles right triangles

Let T be a shadow tree with d-path $P = v_1, \ldots, v_n$. Draw T in the positive $X - Y$ plane with P on the X-axis, v_1 at the origin, each edge of unit length, and each edge not on P parallel to the Y-axis. We henceforth assume that all shadow trees are drawn as described above. We may thus describe a vertex v_i as being *to the left* of v_j, or v_j as being *to the right* of v_i, if $i < j$. Further, v_i is the *leftmost* vertex of a sequence σ of vertices if it is to the left of all other vertices in σ; the *rightmost* vertex in a sequence is defined similarly.

Let $H(t)$ be the tree obtained from $K_{1,3}$ by subdividing each edge $t - 1$ times. If $H(t)$ is a subtree of T, then the leaves of $H(t)$ lie at the (geometric) vertices of an

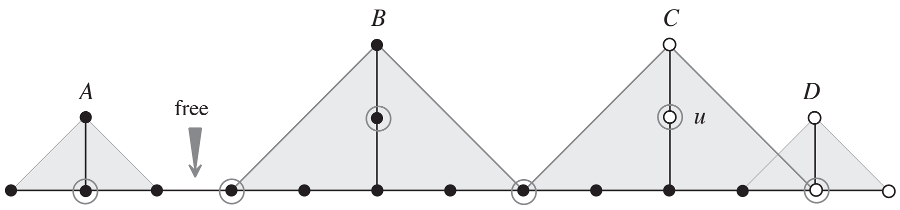

Figure 3: The triangles of a shadow tree

isosceles right triangle Δ whose hypotenuse lies on P and has length $2t$; we say that Δ has *radius* t. We use this observation below to better describe the positions of the boughs of T.

The vertices of the bough B_i of length t that begins at the vertex v_i are labelled $v_i, u_{i,1}, \ldots, u_{i,t}$. If $t \geq 1$, we place an isosceles right triangle Δ of radius t with its hypotenuse on P, centred at v_i, with B_i on the median and $u_{i,t}$ at the apex of Δ (see Fig. 3). We say that the vertices $v_{i-t}, \ldots, v_{i+t}, u_{i,1}, \ldots, u_{i,t}$ are *vertices of* Δ, and that Δ is a *triangle of* T. Thus we consider Δ to be a subtree of T isomorphic to $H(t)$.

An edge $v_i v_{i+1}$ of P is *free* if it does not lie on a triangle of T; in this case $\deg v_i, \deg v_{i+1} \leq 2$. Note that all split-edges of T are free, but not all free edges are split-edges. Also, $v_i v_{i+1}$ is free if and only if $v_1, \ldots, v_i$ and $v_{i+1}, \ldots, v_n$ are d-paths of the two subtrees of $T - v_i v_{i+1}$.

3.3 Properties of shadow trees

We now consider a shadow tree T with d-path $P = v_1, \ldots, v_n$. A triangle Δ of T is a *nested* triangle if it is contained in another triangle of T. Suppose Δ is a nested triangle of T of radius r and let T' be the tree obtained by deleting the vertices on the bough of Δ. An edge is a split-edge of T if and only if it is a split-edge of T', hence $\gamma_b(T') = \gamma_b(T)$ by Theorem 2.2. However, $\gamma(T') = \gamma(T)$ if and only if T and T' satisfy Proposition 3.3. Thus we assume henceforth that T does not contain nested triangles and deal with them later, when considering general trees (Section 5).

Let $v_{c_1}, \ldots, v_{c_k}$ be the branch vertices on P, let B_i be the branch of length (say) x_i of T at v_{c_i} and let Δ_i be the triangle of T with centre v_{c_i} and radius x_i associated with B_i. The sequence $\underline{x} = x_1, \ldots, x_k$ is called the *branch length sequence* of T. Let v_{ℓ_i} (v_{r_i}, respectively) be the vertex on P at distance x_i to the left (right) of v_{c_i}; that is, v_{ℓ_i} is the first and v_{r_i} is the last vertex of Δ_i on P. Further, let η_1 (η_{k+1}, respectively) be the number of edges on P preceding Δ_1 (succeeding Δ_k, respectively), and define $h_1 = -\eta_1$, $h_{k+1} = -\eta_{k+1}$. Then $h_1 = -(\ell_1 - 1)$ and $h_{k+1} = -(n - r_k)$. Also define $h_i = r_{i-1} - \ell_i$ for $i = 2, \ldots, k$; in this instance h_i is called the *overlap* of Δ_{i-1} and Δ_i. See Fig. 4.

Note that $h_1, h_{k+1} \leq 0$, but for $i = 2, \ldots, k$, h_i may be positive, zero or negative. Thus Δ_{i-1} and Δ_i may have a negative overlap, which indicates that there are free

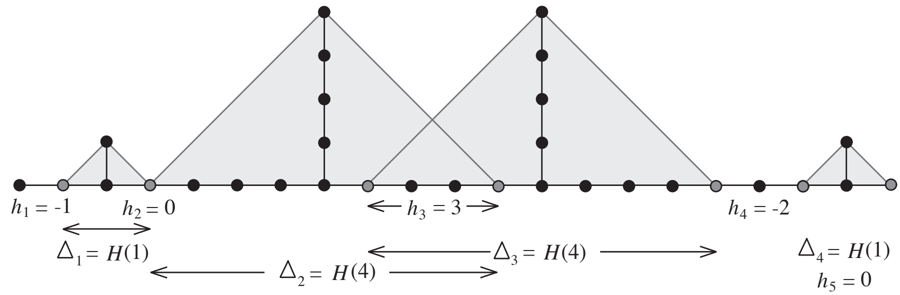

Figure 4: A tree with overlap sequence $\underline{h} = -1, 0, 3, -2, 0$

edges on the $v_{r_{i-1}} - v_{\ell_i}$ path in T (edges of neither Δ_{i-1} nor Δ_i). Similarly, if $h_1 < 0$ (or $h_{k+1} < 0$), then Δ_1 is preceded by free edges (or Δ_{k+1} is succeeded by free edges). The sequence $\underline{h} = h_1, \ldots, h_{k+1}$ is called the *overlap sequence* of T. Note that T is uniquely determined by its branch length sequence $\underline{x}$ and overlap sequence $\underline{h}$, and we also write $T = T(\underline{x}, \underline{h})$.

Cockayne et al. [3] showed that whether $T \in \mathcal{T}$ does not depend on the size of the radii of the triangles of T, but only on their least residues modulo 3 and on the number of common edges of two consecutive triangles.

Theorem 3.4 [3] *If $T(\underline{x}, \underline{h}) \in \mathcal{T}$, then any shadow tree $T'(\underline{x'}, \underline{h})$, where $\underline{x'} = x'_1, \ldots, x'_k$ such that $x'_i \equiv x_i \pmod 3$ for each $i = 1, \ldots, k$, is also in $\mathcal{T}$.*

By Theorem 3.4 we may assume that, for each $i \geq 1$, $\ell_{i+1} \geq c_i$, for otherwise we may replace Δ_i by a triangle Δ'_i with radius $x_i + 3t$ for some suitable integer $t \geq 1$, thus replacing T by the tree T' with branch length sequence $\underline{x'} = x_1, \ldots, x_i + 3t, \ldots, x_k$ and the same overlap sequence as T, where now $\ell'_{i+1} \geq c'_i$. The exact procedure is described fully in [3]. We may similarly assume that $r_i \leq c_{i+1}$.

4 Branches of length congruent to 1 (mod 3)

Assume henceforth that the length of each branch is congruent to 1 (mod 3). Let $\mathcal{H}$ be the class of shadow trees with this property and without nested triangles. Let $\sigma = \Delta_i, \ldots, \Delta_j$, $j \geq i$, be a sequence of consecutive triangles of T, with branch vertices $v_{c_i}, \ldots, v_{c_j}$, such that $h_{i+1}, \ldots, h_j \geq 0$. We call σ a *nonnegative overlap sequence*. A nonnegative overlap sequence σ is a *maximal nonnegative overlap sequence (MNOS)* if it is not contained in a larger nonnegative overlap sequence. Let T_σ be the subtree of T induced by σ. We call T_σ the *subtree of T associated with σ*. Since T_σ has no free edges, it is radial. We now state a number of properties of trees in $\mathcal{H} \cap \mathcal{T}$, deferring their proofs to Section 6.

Lemma 4.1 *If σ is an MNOS of $T \in \mathcal{T}$, then $T_\sigma \in \mathcal{T}$.*

Lemma 4.2 *If σ is an MNOS, then $T_\sigma \in \mathcal{T}$ if and only if σ contains only overlaps of cardinality* $0,\ 1,\ 2,\ 3$ *or* 5*, and at most one overlap has odd cardinality.*

If σ is an MNOS containing only overlaps of size 0 or 2, then σ has even diameter and is called an *even MNOS*, otherwise, by Lemma 4.2, σ has odd diameter and is called an *odd MNOS*.

Now let $\sigma_i, \ldots, \sigma_j$, $j \geq i$, be a sequence of consecutive MNOS's of T, with h'_s, $s = i+1, \ldots, j$, the length of the negative overlap joining σ_{s-1} and σ_s, and assume that $h'_s = -1$ for each s. Such a sequence $\sigma_i, \ldots, \sigma_j$ is called a *tight sequence.* Let $S_{i,j}$ be the subtree of T associated with $\sigma_i, \ldots, \sigma_j$. For each $s = i, \ldots, j$ we simplify the notation to denote the subtree T_{σ_s} of T associated with σ_s by T_s.

Lemma 4.3 *If $\sigma_i, \ldots, \sigma_j$ is a tight sequence of T and $T_s \in \mathcal{T}$ for each $s = i, \ldots, j$, then $S_{i,j} \in \mathcal{T}$.*

The next lemma is clear from the proof of Lemma 4.3 (see Section 6).

Lemma 4.4 *If S is the subtree of T associated with the tight sequence $\sigma_1, \ldots, \sigma_t$, then S is radial if and only if at most one of the sequences $\sigma_1, \ldots, \sigma_t$ is even.*

A tight sequence is a *maximal tight sequence (MTS)* if it is not contained in a larger tight sequence. Let $S_1, \ldots, S_r$ be the MTS's of T. For simplicity we also consider the S_i to be subtrees of T, i.e., S_i is the subtree of T associated with the MTS S_i. (Hence $S_i = S_{i',j}$ for some i', j.) We also call S_i *even* or *odd* depending on the parity of its diameter.

Let Q_1 (Q_{r+1}, respectively) be the subpath of P induced by the free edges preceding S_1 (following S_r, respectively), and for $i = 1, \ldots, r$, let Q_i be the subpath of P induced by the free edges that join S_{i-1} to S_i. Say Q_i contains q_i vertices that do not lie on S_{i-1} or S_i. By the maximality of the S_i, each Q_i, $i = 2, \ldots, r$, has at least two edges and thus $q_i \geq 1$, while Q_1 and Q_{r+1} may have any nonnegative number of edges, and so $q_1, q_{r+1} \geq 0$.

Lemma 4.5 *Let $S_1, \ldots, S_r$ be the MTS's of the shadow tree T. Then $T \in \mathcal{T}$ if and only if $S_1, \ldots, S_r \in \mathcal{T}$ and the following conditions hold.*

(*i*) *If S_k is odd and radial, then $q_k \not\equiv 1 \pmod 3$ and $q_{k+1} \not\equiv 1 \pmod 3$.*

(*ii*) *If S_k is even and radial, then $q_k \not\equiv 1 \pmod 3$ or $q_{k+1} \not\equiv 1 \pmod 3$.*

(*iii*) *Suppose $j \geq 1$ and $S_k, \ldots, S_{k+j}$ are radial. If S_{k+s} is even for each integer s such that $0 < s < j$, and*

  (a) *S_k and S_{k+j} are odd, or*

  (b) *(without loss of generality) S_k is odd, S_{k+j} is even and $q_{k+j+1} \equiv 1 \pmod 3$, or*

  (c) *S_k and S_{k+j} are even and $q_k \equiv q_{k+j+1} \equiv 1 \pmod 3$,* *then* $q_{k+s} \equiv 0 \pmod 3$ *for at least one* $s \in \{1, \ldots, j\}$.

We are now ready to state and prove the characterization of trees in $\mathcal{H} \cap \mathcal{T}$. Note that each condition in the characterization concerns only the size of the overlaps of the triangles.

Theorem 4.6 *Let* T *be a shadow tree in* $\mathcal{H}$ *with MTS's* $S_1, \ldots, S_r$ *and define* $q_1, \ldots, q_{r+1}$ *as above. For each* $k \in \{1, \ldots, r\}$, *let* $\sigma_{k,1}, \ldots, \sigma_{k,t_k}$ *be the MNOS's of* S_k. *Then* $T \in \mathcal{T}$ *if and only if the following conditions hold.*

1. *Each* $\sigma_{k,i}$ *contains only overlaps of size* $0, 1, 2, 3, 5$, *and at most one odd overlap.*
2. *If* $\sigma_{k,1}, \ldots, \sigma_{k,t_k}$ *are all odd, then* $q_k \not\equiv 1 \pmod 3$ *and* $q_{k+1} \not\equiv 1 \pmod 3$.
3. *If exactly one of* $\sigma_{k,1}, \ldots, \sigma_{k,t_k}$ *is even, then* $q_k \not\equiv 1 \pmod 3$ *or* $q_{k+1} \not\equiv 1 \pmod 3$.
4. *Suppose* $k' \geq k + 1$ *and consider the MTS's* $S_k, S_{k+1}, \ldots, S_{k'}$. *If exactly one of* $\sigma_{i,1}, \ldots, \sigma_{i,t_i}$ *is even for each* i *such that* $k < i < k'$, *and*
   (a) $\sigma_{k,1}, \ldots, \sigma_{k,t_k}, \sigma_{k',1}, \ldots, \sigma_{k',t_{k'}}$ *are all odd, or*
   (b) *(without loss of generality)* $\sigma_{k,1}, \ldots, \sigma_{k,t_k}$ *are all odd, exactly one of* $\sigma_{k',1}, \ldots, \sigma_{k',t_{k'}}$ *is even and* $q_{k'+1} \equiv 1 \pmod 3$, *or*
   (c) *exactly one of* $\sigma_{k,1}, \ldots, \sigma_{k,t_k}$ *and exactly one of* $\sigma_{k',1}, \ldots, \sigma_{k',t_{k'}}$ *are even, and* $q_k \equiv q_{k'+1} \equiv 1 \pmod 3$,

   then $q_i \equiv 0 \pmod 3$ *for at least one* i *such that* $k < i \leq k'$.

Proof. Suppose $T \in \mathcal{T}$. By Lemma 4.1, each $T_{k,i} \in \mathcal{T}$ and (1) holds by Lemma 4.2. The other conditions hold by Lemmas 4.4 and 4.5.

Conversely, suppose (1) – (4) hold. By Lemma 4.2, each $T_{k,i} \in \mathcal{T}$, and so each $S_k \in \mathcal{T}$ by Lemma 4.3. Now Lemmas 4.4 and 4.5 imply that $T \in \mathcal{T}$. ∎

5 Conclusions

We first apply Theorem 4.6 to caterpillars. Let C be any caterpillar, i.e. C consists of a d-path $P = v_1, \ldots, v_n$ together with any positive number of leaves attached to the branch vertices $v_{c_1}, \ldots, v_{c_k}$ of P, where $1 < c_1 < \cdots < c_k < n$. Since the number of leaves attached to each v_{c_i} is unimportant, we may assume without loss of generality that C is a shadow tree, and we thus continue to use the notation of Section 4. The only possible positive overlap is 1 and C contains no nested triangles. The MNOS's of C are maximal sequences of triangles just touching or overlapping in a single edge. If two triangles overlap in an edge, then the corresponding branch vertices are adjacent; we call these two vertices a *branching pair*. A *pairfree* MNOS is one without a branching pair. The following result is an immediate corollary of Theorem 4.6.

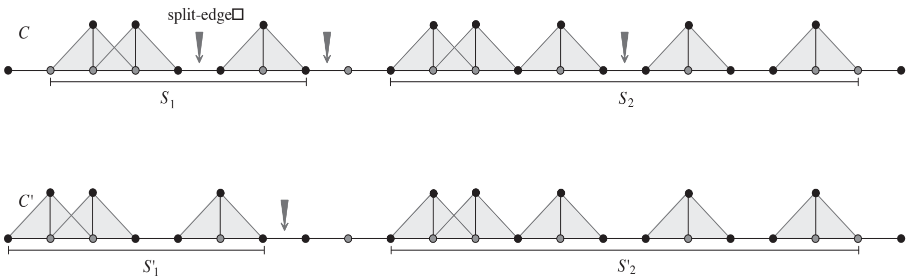

Figure 5: Caterpillars C and C' with $\gamma_b(C) = 9 < \gamma(C) = 11$ and $\gamma(C') = \gamma_b(C') = 10$

Corollary 5.1 *Let C be a caterpillar with MTS's $S_1, \ldots, S_r$ and define $q_1, \ldots, q_{r+1}$ as before. For each $k \in \{1, \ldots, r\}$, let $\sigma_{k,1}, \ldots, \sigma_{k,t_k}$ be the MNOS's of S_k. Then $\gamma(C) = \gamma_b(C)$ if and only if the following conditions hold.*

1. *Each $\sigma_{k,i}$ contains at most one branching pair.*
2. *If each $\sigma_{k,1}, \ldots, \sigma_{k,t_k}$ has a branching pair, then $q_k \not\equiv 1 \pmod 3$ and $q_{k+1} \not\equiv 1 \pmod 3$.*
3. *If exactly one of $\sigma_{k,1}, \ldots, \sigma_{k,t_k}$ is pairfree, then $q_k \not\equiv 1 \pmod 3$ or $q_{k+1} \not\equiv 1 \pmod 3$.*
4. *Suppose $k' \geq k+1$ and consider the MTS's $S_k, S_{k+1}, \ldots, S_{k'}$. If exactly one of $\sigma_{i,1}, \ldots, \sigma_{i,t_i}$ is pairfree for each i such that $k < i < k'$, and*
   (a) *each $\sigma_{k,1}, \ldots, \sigma_{k,t_k}, \sigma_{k',1}, \ldots, \sigma_{k',t_{k'}}$ contains a branching pair, or*
   (b) *(without loss of generality) each $\sigma_{k,1}, \ldots, \sigma_{k,t_k}$ contains a branching pair, exactly one of $\sigma_{k',1}, \ldots, \sigma_{k',t_{k'}}$ is pairfree and $q_{k'+1} \equiv 1 \pmod 3$, or*
   (c) *exactly one of $\sigma_{k,1}, \ldots, \sigma_{k,t_k}$ and exactly one of $\sigma_{k',1}, \ldots, \sigma_{k',t_{k'}}$ is pairfree, and $q_k \equiv q_{k'+1} \equiv 1 \pmod 3$,*

   then $q_i \equiv 0 \pmod 3$ for at least one i such that $k < i \leq k'$.

Fig. 5 shows two caterpillars C and C' with $C \notin \mathcal{T}$ and $C' \in \mathcal{T}$. The MTS S_1 of C has exactly one pairfree MNOS, yet $q_1 \equiv q_2 \equiv 1 \pmod 3$, thus violating Corollary 5.1(3). For S'_1, $q_1 \equiv 0 \pmod 3$, hence Corollary 5.1(3) is satisfied. None of the conditions (2) – (4) of Corollary 5.1 applies to S_2 or S'_2.

Now let T be an arbitrary tree with shadow tree S'_T, and let S_T be the shadow tree obtained by deleting all nested triangles of S'_T. Then $\gamma_b(T) = \gamma_b(S'_T) = \gamma_b(S_T)$. Let $W_1, \ldots, W_t$ be the nontrivial components of $T - E(S_T)$. If W_i is not a star for some i, then by Theorem 3.1 and Proposition 3.3(i), $\gamma(T) > \gamma(S_T) \geq \gamma_b(S_T) = \gamma_b(T)$ and thus $T \notin \mathcal{T}$. Assume that each W_i is a star, where (for some r) W_i, $i = 1, \ldots, r$, is hinged at its centre u_i or at a leaf if $W_i = K_2$, and for $i = r+1, \ldots, t$, $W_i \neq K_2$ is hinged at a leaf l_i. If S_T has no γ-set D such that $\{u_i : 1 \leq i \leq r\} \subseteq D$ and $\{l_i : r+1 \leq i \leq t\} \subseteq D_{\text{spn}}$, then by Theorem 3.1 and Proposition 3.3(ii), $T \notin \mathcal{T}$. On the other hand, if S_T does have a γ-set that satisfies Proposition 3.3(ii), then $T \in \mathcal{T}$ if and only if $S_T \in \mathcal{T}$, as determined by Theorem 4.6.

6 Proofs of Lemmas

Assume the bough $B_i = v_{c_i}, u_{i,1}, \ldots, u_{i,x_i}$ of T has length $x_i = 3m_i + 1$, $i = 1, \ldots, k$. If D is a γ-set of T, we may assume without loss of generality that D contains the vertex $u_{i,3m_i}$ of B_i, and then precisely every third vertex along the bough; thus $v_{c_i} \in D$. We may also assume that if $\sigma = \Delta_i, \ldots, \Delta_j$ is an MNOS of T, then D contains every third vertex to the left of v_{c_i}, so that D contains v_{ℓ_i+1}, and every third vertex to the right of v_{c_j}, so that D contains v_{r_j-1}. A γ-set with this property is called a *natural* γ-set of T.

Before proceeding with the proofs of the lemmas stated in Section 4, we determine an expression for $\gamma(T)$, $T \in \mathcal{H}$.

Lemma 6.1 *If D is a natural γ-set of $T \in \mathcal{H}$, then*

$$|D| = 3\sum_{i=1}^{k} m_i + k - \sum_{i=2}^{k} \left\lfloor \frac{h_i + 1}{3} \right\rfloor - \left\lfloor \frac{h_1}{3} \right\rfloor - \left\lfloor \frac{h_{k+1}}{3} \right\rfloor.$$

Proof. Define $P_1 = v_1, \ldots, v_{c_1}$, $P_{k+1} = v_{c_k}, \ldots, v_n$ and $P_i = v_{c_{i-1}}, \ldots, v_{c_i}$ for $i = 2, \ldots, k$. By the choice of D, $v_{c_i} \in D$ and $|D \cap V(B_i)| = m_i + 1$. Note that $d(v_{c_{i-1}}, v_{c_i}) = c_i - c_{i-1} = x_{i-1} + x_i - h_i$, and the number of vertices on the $v_{c_{i-1}} - v_{c_i}$ path P_i is $x_{i-1}+x_i-h_i+1$. Of these vertices, $v_{c_{i-1}}$ and its successor $v_{c_{i-1}+1}$, and v_{c_i} and its predecessor v_{c_i-1}, are dominated by $\{v_{c_{i-1}}, v_{c_i}\} \subseteq D$. Let $D^* = D - \bigcup_{i=1}^{k} V(B_i)$. Thus $x_{i-1} + x_i - h_i - 3$ vertices on P_i are dominated by D^*, $i = 2, \ldots, k$.

- If $d(v_{c_{i-1}}, v_{c_i}) \geq 3$, then $x_{i-1} + x_i - h_i - 3 \geq 0$. Hence at least
$$\lceil (x_{i-1} + x_i - h_i - 3)/3 \rceil$$
vertices in D^* are needed to dominate these remaining vertices on P_i. By the minimality of D,
$$|D^* \cap V(P_i)| = \lceil (x_{i-1} + x_i - h_i - 3)/3 \rceil . \tag{1}$$
- If $1 \leq d(v_{c_{i-1}}, v_{c_i}) \leq 2$, then $x_{i-1}+x_i-h_i-3 = -2$ or $x_{i-1}+x_i-h_i-3 = -1$ and $\lceil (x_{i-1} + x_i - h_i - 3)/3 \rceil = \lceil -1/3 \rceil = 0$. Obviously in this case no vertices on P_i need to be dominated by D^*, and by the minimality of D, $|D^* \cap V(P_i)| = 0$.

Thus (1) holds for each $i = 2, \ldots, k$. Since $d(v_1, v_{c_1}) = x_1 - h_1$, a similar argument shows that $|D^* \cap V(P_1)| = \lceil (x_1 - h_1 - 1)/3 \rceil$ and, correspondingly, $|D^* \cap V(P_{k+1})| = \lceil (x_k - h_{k+1} - 1)/3 \rceil$. Hence

$$\begin{aligned}
|D^*| &= \sum_{i=2}^{k} \left\lceil \frac{x_{i-1} + x_i - h_i - 3}{3} \right\rceil + \left\lceil \frac{x_1 - h_1 - 1}{3} \right\rceil + \left\lceil \frac{x_k - h_{k+1} - 1}{3} \right\rceil \\
&= \sum_{i=2}^{k} \left\lceil \frac{3(m_{i-1} + m_i) - (h_i + 1)}{3} \right\rceil + \left\lceil \frac{3m_1 - h_1}{3} \right\rceil + \left\lceil \frac{3m_k - h_{k+1}}{3} \right\rceil \\
&= 2\sum_{i=1}^{k} m_i - \sum_{i=2}^{k} \left\lfloor \frac{h_i + 1}{3} \right\rfloor - \left\lfloor \frac{h_1}{3} \right\rfloor - \left\lfloor \frac{h_{k+1}}{3} \right\rfloor,
\end{aligned}$$

so that

$$\begin{aligned}
|D| &= |D^*| + \sum_{i=1}^{k} |D \cap V(B_i)| \\
&= 2\sum_{i=1}^{k} m_i - \sum_{i=2}^{k} \left\lfloor \frac{h_i+1}{3} \right\rfloor - \left\lfloor \frac{h_1}{3} \right\rfloor - \left\lfloor \frac{h_{k+1}}{3} \right\rfloor + \sum_{i=1}^{k} m_i + k \\
&= 3\sum_{i=1}^{k} m_i + k - \sum_{i=2}^{k} \left\lfloor \frac{h_i+1}{3} \right\rfloor - \left\lfloor \frac{h_1}{3} \right\rfloor - \left\lfloor \frac{h_{k+1}}{3} \right\rfloor
\end{aligned}$$

as required. ■

Proof of Lemma 4.1. Assume that $T \in \mathcal{T}$ and that D is a natural γ-set of T. Let $\sigma = \Delta_i, \ldots, \Delta_j$ and $D_\sigma = D \cap V(T_\sigma) - \{v_{\ell_i}, v_{r_j}\}$. By the choice of D, D_σ dominates T_σ but no vertices of $T - T_\sigma$. Since D is a γ-set of T, D_σ is a γ-set of T_σ. By Lemma 6.1,

$$\gamma(T_\sigma) = |D_\sigma| = 3\sum_{s=i}^{j} m_s + j - i + 1 - \sum_{s=i+1}^{j} \left\lfloor \frac{h_s+1}{3} \right\rfloor. \tag{2}$$

Since T_σ contains no nested triangles,

$$\operatorname{rad} T_\sigma = \sum_{s=i}^{j} x_s - \left\lfloor \sum_{s=i}^{j} \frac{h_s}{2} \right\rfloor = 3\sum_{s=i}^{j} m_s + j - i + 1 - \left\lfloor \frac{1}{2} \sum_{s=i+1}^{j} h_s \right\rfloor. \tag{3}$$

Let

$$\alpha = \left\lfloor \frac{1}{2} \sum_{s=i+1}^{j} h_s \right\rfloor \quad \text{and} \quad \beta = \sum_{s=i+1}^{j} \left\lfloor \frac{h_s+1}{3} \right\rfloor \tag{4}$$

and note that $\alpha \geq \beta$ because $h_{i+1}, \ldots, h_j \geq 0$. Suppose $\alpha > \beta$. Then $\operatorname{rad} T_\sigma < |D_\sigma|$. Let v be a central vertex of T_σ and note that v lies on the path $v_{\ell_i}, \ldots, v_{r_j}$. Define the broadcast f on T by

$$f(u) = \begin{cases} 1 & \text{if } u \in D - D_\sigma \\ \operatorname{rad} T_\sigma & \text{if } u = v \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, v broadcasts to all of T_σ, while $D - D_\sigma$ dominates $T - T_\sigma$, hence f is a dominating broadcast of T. But $\operatorname{cost}(f) = |D| - |D_\sigma| + \operatorname{rad} T_\sigma < |D| = \gamma(T)$, contradicting $T \in \mathcal{T}$. Therefore $\alpha = \beta$, i.e. $\operatorname{rad} T_\sigma = \gamma(T_\sigma)$, and since T_σ is radial it follows that $\gamma_b(T_\sigma) = \gamma(T_\sigma)$, i.e., $T_\sigma \in \mathcal{T}$. ■

Proof of Lemma 4.2. Assume that $T_\sigma \in \mathcal{T}$ and suppose $h_{i'} = 4$ for some $i' \in \{i+1, \ldots, k\}$. With α and β as defined in (4), this implies that

$$\alpha = \left\lfloor \frac{1}{2} \sum_{s=i+1,\ s\neq i'}^{j} h_s + \frac{4}{2} \right\rfloor = \left\lfloor \frac{1}{2} \sum_{s=i+1,\ s\neq i'}^{j} h_s \right\rfloor + 2$$

and

$$\beta = \sum_{s=i+1,\ s\neq i'}^{j} \left\lfloor \frac{h_s+1}{3} \right\rfloor + \left\lfloor \frac{5}{3} \right\rfloor = \sum_{s=i+1,\ s\neq i'}^{j} \left\lfloor \frac{h_s+1}{3} \right\rfloor + 1 < \alpha.$$

By (2) and (3), $\gamma(T_\sigma) > \operatorname{rad} T_\sigma = \gamma_b(T_\sigma)$, contradicting $T_\sigma \in \mathcal{T}$. Therefore $h_{i'} \neq 4$ for all $i' \in \{i+1, \ldots, k\}$. Similarly, $h_{i'} \notin \{6, 7, 8, \ldots\}$ for all $i' \in \{i+1, \ldots, k\}$.

Suppose next that σ contains two odd overlaps; say $h_z, h_{z'} \in \{1, 3, 5\}$ for some z, z', while $h_t \in \{0, 2\}$ otherwise. Assume there are r values of t such that $h_t = 2$, and that $h_z = 2w+1$, $h_{z'} = 2w'+1$. Then $\alpha = r + w + w' + 1$. But $w, w' \in \{0, 1, 2\}$, and for these values, $\left\lfloor \frac{2w+2}{3} \right\rfloor = w$, so that $\beta = w + r + w' < \alpha$. Now (2) and (3) imply that $\gamma_b(T_\sigma) < \gamma(T_\sigma)$, which is a contradiction as above.

Conversely, if σ contains only the stated overlaps, then it is easy to verify that $\alpha = \beta$ and the result follows from (2) and (3). ■

Proof of Lemma 4.3. Abbreviate the notation D_{σ_s} (defined as in the proof of Lemma 4.1) to D_s; as before D_s is a γ-set of T_s. Since $h'_s = -1$ for each $s = i+1, \ldots, j$, Lemma 6.1 applied to $S_{i,j}$ shows that

$$\gamma(S_{i,j}) = \sum_{s=i}^{j} |D_s| - \sum_{s=i+1}^{j} \left\lfloor \frac{h'_s+1}{3} \right\rfloor = \sum_{s=i}^{j} \gamma(T_s). \tag{5}$$

Further, since each $T_s \in \mathcal{T}$ and each T_s is radial, $\operatorname{rad} T_s = \gamma_b(T_s) = \gamma(T_s)$ for each $s = i, \ldots, j$. Substitution in (5) gives

$$\gamma(S_{i,j}) = \sum_{s=i}^{j} \operatorname{rad} T_s. \tag{6}$$

Let $\Sigma = \sigma_i, \ldots, \sigma_j$ and say δ of the sequences in Σ are even and $j+1-i-\delta$ are odd. Then

$$\begin{aligned} \operatorname{diam} S_{i,j} &= \sum_{\sigma_s \text{ even}} \operatorname{diam} T_s + \sum_{\sigma_s \text{ odd}} \operatorname{diam} T_s - \sum_{s=i+1}^{j} h'_s \\ &= 2 \sum_{\sigma_s \text{ even}} \operatorname{rad} T_s + 2 \sum_{\sigma_s \text{ odd}} \operatorname{rad} T_s - (j+1-i-\delta) + j - i \\ &= 2 \sum_{s=i}^{j} \operatorname{rad} T_s + \delta - 1, \end{aligned}$$

so that

$$\operatorname{rad} S_{i,j} = \sum_{s=i}^{j} \operatorname{rad} T_s + \left\lceil \frac{\delta-1}{2} \right\rceil. \tag{7}$$

For each $s = i, \ldots, j-1$, let e_s be the edge joining σ_s to σ_{s+1}. Let m be the number of edges in a maximum split-set of $S_{i,j}$. Note that any split-set is contained in $\{e_i, \ldots, e_{j-1}\}$. We prove that either $m = 0$ and $\delta \in \{0, 1\}$, or $m = \delta - 1$.

If $\sigma_k, \dots, \sigma_{k'}$ are consecutive odd sequences for some $k, k' \in \{i, \dots, j\}$, then $S_{k,k'}$ has odd diameter, because each h'_s is odd. Now, if $M \neq \varnothing$ is a split-set of $S_{i,j}$, then each component of $S_{i,j} - M$ has even diameter and thus contains an even σ_k. Let $\sigma_{\ell_1}, \dots, \sigma_{\ell_\delta}$ be the even sequences in Σ. If $\delta \geq 2$, then $\{e_{\ell_1}, \dots, e_{\ell_{\delta-1}}\}$ is a maximum split-set and thus $m = \delta - 1$. If $\delta \in \{0, 1\}$, then $S_{i,j}$ has no split-edges and thus $m = 0$. By (7),

$$\operatorname{rad} S_{i,j} = \sum_{s=i}^{j} \operatorname{rad} T_s + \left\lceil \frac{m}{2} \right\rceil .$$

By Theorem 2.2, $\gamma_b(S_{i,j}) = \operatorname{rad} S_{i,j} - \lceil \frac{m}{2} \rceil$, from which it follows that $\gamma_b(S_{i,j}) = \sum_{s=i}^{j} \operatorname{rad} T_s$. Thus by (6), $\gamma_b(S_{i,j}) = \gamma(S_{i,j})$, as required. ■

Proof of Lemma 4.5. Suppose $T \in \mathcal{T}$ and let D be a natural γ-set of T. Let α_i and ω_i be the first and last vertices, respectively, of S_i on P, and let α_i^- and α_i^+ (ω_i^- and ω_i^+) be the vertices to the left and right of α_i (ω_i, respectively). Then D contains α_i^+ and ω_i^-. Let $X_i = (D \cap S_i) - \{\alpha_i, \omega_i\}$. Then X_i dominates S_i but no vertices of $T - S_i$ and is a γ-set of S_i. Therefore $|X_i| = \gamma(S_i)$. By Lemmas 4.1 and 4.3, $S_1, \dots, S_r \in \mathcal{T}$, hence $|X_i| = \gamma(S_i) = \gamma_b(S_i)$ and thus (similar to Lemma 6.1)

$$|D| = \sum_{i=1}^{r} |X_i| + \sum_{i=1}^{r+1} \left\lceil \frac{q_i}{3} \right\rceil = \sum_{i=1}^{r} \gamma_b(S_i) + \sum_{i=1}^{r+1} \left\lceil \frac{q_i}{3} \right\rceil . \tag{8}$$

We prove that (i), (ii) and (iii) hold. In each case we broadcast to T with a cost of one from each vertex in D, except where otherwise stated.

(i) Let S_k be odd and radial and suppose $q_k \equiv 1 \pmod 3$; say $q_k = 3a + 1$. By (8), D contains $\lceil \frac{3a+1}{3} \rceil = a + 1$ vertices of Q_k. Since S_k is odd, it is bicentral. Let c be the leftmost central vertex of S_k and broadcast from c with a cost of $\operatorname{rad} S_k = |X_k|$. Then the internal vertex α_k^- of Q_k hears this broadcast, and the remaining internal vertices of Q_k can be reached by broadcasting from $\lceil \frac{3a}{3} \rceil = a$ vertices on Q_k, with a cost of 1 in each case. Hence

$$\gamma_b(T) \leq \sum_{i=1}^{r} \gamma_b(S_i) + \sum_{\substack{i=1 \\ i \neq k}}^{r+1} \left\lceil \frac{q_i}{3} \right\rceil + \left\lceil \frac{q_k}{3} \right\rceil - 1 < |D| \text{ by (8),}$$

in contradiction to $T \in \mathcal{T}$. Hence $q_k \not\equiv 1 \pmod 3$. By symmetry, $q_{k+1} \not\equiv 1 \pmod 3$.

(ii) Let S_k be even and radial and suppose $q_k = 3a + 1$ and $q_{k+1} = 3b + 1$. As above D contains $a + 1$ vertices of Q_k and $b + 1$ vertices of Q_{k+1}. Let c be the central vertex of S_k and broadcast from c with a cost of $\operatorname{rad} S_k + 1 = |X_k| + 1$. Then the internal vertices α_k^- of Q_k and ω_k^+ of Q_{k+1} hear this broadcast. The remaining internal vertices of Q_k and Q_{k+1} can be reached by broadcasting from a vertices on Q_k and b vertices on Q_{k+1}, in each case with a cost of 1. Again,

$$\gamma_b(T) \leq \sum_{\substack{i=1 \\ i \neq k}}^{r} \gamma_b(S_i) + (\gamma_b(S_k) + 1) + \sum_{\substack{i=1 \\ i \neq k,k+1}}^{r+1} \left\lceil \frac{q_i}{3} \right\rceil + \left(\left\lceil \frac{q_k}{3} \right\rceil - 1 \right) + \left(\left\lceil \frac{q_{k+1}}{3} \right\rceil - 1 \right) < |D|$$

by (8), a contradiction.

(iii) Assume $j \geq 1$, $S_k, \ldots, S_{k+j}$ are radial and S_{k+s} is even for each $s \in \{1, \ldots, j-1\}$.

(a) Suppose S_k and S_{k+j} are odd, but $q_{k+s} \not\equiv 0 \pmod 3$ for each $s \in \{1, \ldots, j\}$. Let c_k be the rightmost central vertex of S_k, c_{k+j} be the leftmost central vertex of S_{k+j}, and for each s with $0 < s < j$, let c_{k+s} be the central vertex of S_{k+s}. Broadcast from c_k and c_{k+j} with a cost of $\operatorname{rad} S_k$ and $\operatorname{rad} S_{k+j}$, respectively, and from c_{i+s} with a cost of $\operatorname{rad} S_{k+s} + 1$, $0 < s < j$. If $q_{k+s} = 1$, then the unique internal vertex of Q_{k+s} hears a broadcast from c_{k+s-1} and from c_{k+s}, otherwise, at least two vertices on Q_{k+s} hear broadcasts from c_{k+s-1} or c_{k+s}. Since $q_{k+s} \equiv 1$ or $2 \pmod 3$, the remainder of the vertices of Q_{k+s} can be reached by a broadcast from at most $\lceil \frac{q_{k+s}}{3} \rceil - 1$ vertices on Q_{k+s}, each with a cost of one. Hence

$$\gamma_b(T) \leq \sum_{i=1}^{r} \gamma_b(S_i) + j - 1 + \sum_{i=1}^{r+1} \left\lceil \frac{q_i}{3} \right\rceil - j < |D| \text{ by (8)},$$

a contradiction as before.

(b) Suppose S_k is odd, S_{k+j} is even and $q_{k+j+1} \equiv 1 \pmod 3$, while $q_{k+s} \equiv 1$ or $2 \pmod 3$ for each $s \in \{1, \ldots, j\}$. We proceed as in (a), except that we also broadcast from the unique central vertex of S_{k+j} with a cost of $\operatorname{rad} S_{k+j} + 1$. Thus we increase the strength of the broadcast vertex of each of the j subtrees $S_{k+1}, \ldots, S_{k+j}$ by one, and decrease the number of broadcast vertices on each of the $j+1$ paths $Q_{k+1}, \ldots, Q_{k+j+1}$ by one, resulting in a contradiction as above.

(c) Similar to (a) and (b).

Before proving the converse of Lemma 4.5, we formulate and prove two more lemmas.

Lemma 6.2 *Let T be a radial shadow tree, R the subtree of T obtained by deleting all leading and trailing free edges on the d-path $P = v_1, \ldots, v_n$ of T, and ρ the cardinality of a maximum split-set of R. Then $\rho \leq 2$.*

Proof. Let M_R be a maximum split-set of R and suppose $|M_R| \geq 3$; say $\{v_i v_{i+1}, v_j v_{j+1}, v_k v_{k+1}\} \subseteq M_R$. Let $P_R = v_f, \ldots, v_t$ be the d-path of R that is a subpath of the d-path $P = v_1, \ldots, v_n$ of T. Then $v_f, \ldots, v_i$; $v_{i+1}, \ldots, v_j$; $v_{j+1}, \ldots, v_k$ and $v_{k+1}, \ldots, v_t$ are even. If $v_1, \ldots, v_f$ is even, then $\{v_i v_{i+1}\}$ or $\{v_i v_{i+1}, v_j v_{j+1}\}$ (depending on the parity of the diameter of T) is a split-set of T, which is not the case, so $v_1, \ldots, v_f$ is odd. Thus $v_1, \ldots, v_j$ is even. Now either $\{v_j v_{j+1}\}$ or $\{v_j v_{j+1}, v_k v_{k+1}\}$ is a split-set of T, depending on whether $v_{j+1}, \ldots, v_n$ is even or odd, contradicting the radiality of T ■

Lemma 6.3 *If T is radial, $S_1, \ldots, S_r \in \mathcal{T}$ and (i), (ii) and (iii) hold, then $T \in \mathcal{T}$.*

Proof. Define R, M_R, P_R and ρ as in the proof of Lemma 6.2; then $\rho \in \{0,1,2\}$. Since T is radial, $\gamma_b(T) = \operatorname{rad} T$. Since $S_1, \ldots, S_r \in \mathcal{T}$, $\gamma(T) = \sum_{i=1}^{r} \gamma_b(S_i) + \sum_{i=1}^{r+1} \left\lceil \frac{q_i}{3} \right\rceil$ by (8). We consider three cases, depending on the value of ρ, to prove that

$$\operatorname{rad} T = \sum_{i=1}^{r} \gamma_b(S_i) + \sum_{i=1}^{r+1} \left\lceil \frac{q_i}{3} \right\rceil . \tag{9}$$

Case 1 $\rho = 2$. By Theorem 2.2, $\gamma_b(R) = \operatorname{rad} R - 1$, and P_R is even. Say $M_R = \{v_j v_{j+1}, v_{j'} v_{j'+1}\}$. Then $v_f, \ldots, v_j$; $v_{j+1}, \ldots, v_{j'}$ and $v_{j'+1}, \ldots, v_t$ are even. Since neither edge is a split-edge of T, $v_1, \ldots, v_j$ and $v_{j'+1}, \ldots, v_n$ are odd, so q_1 and q_{r+1} are odd, and P is even. If $q_1 \geq 3$, then $\{v_3 v_4, v_j v_{j+1}\}$ is a split-set of T, which is impossible, hence $q_1 = 1$; similarly, $q_{r+1} = 1$. Hence $\operatorname{rad} T = \operatorname{rad} R + 1$.

If $\{e_k, e_{k+1}\}$ and $\{e_\ell, e_{\ell+1}\}$ are two sets of consecutive free edges of R separated by at least one S_i, then one of $\{e_k, e_\ell\}$, $\{e_k, e_{\ell+1}\}$, $\{e_{k+1}, e_\ell\}$, $\{e_{k+1}, e_{\ell+1}\}$ is a split-set of T, which is not the case, so T contains at most one set Y of two or more consecutive free edges. A similar argument shows that $|Y| \leq 4$. In particular, R consists of at most two MTS's.

Case 1.1 R consists of one MTS S_1. Since $R = S_1 \in \mathcal{T}$, $\gamma(S_1) = \gamma_b(S_1) = \gamma_b(R) = \operatorname{rad} R - 1$. Since $q_1 = q_2 = 1$, $\gamma_b(T) = \gamma_b(S_1) + \sum_{i=1}^{2} \left\lceil \frac{q_i}{3} \right\rceil = \operatorname{rad} R + 1 = \operatorname{rad} T$, hence (9) holds.

Case 1.2 R has two MTS's S_1 and S_2. Since $2 \leq |Y| \leq 4$, $1 \leq q_2 \leq 3$.

- If $q_2 = 1$, then (i), (ii) and the fact that $q_1 = q_3 = 1$ imply that neither S_1 nor S_2 is radial. Since $\rho = 2$, each S_i therefore has a maximum split-set of cardinality 1, and so has odd diameter. Thus $\operatorname{rad} R = (\operatorname{rad} S_1 + \operatorname{rad} S_2 - 1) + \operatorname{rad} Q_2 = \operatorname{rad} S_1 + \operatorname{rad} S_2$. By Theorem 2.2, $\gamma_b(S_i) = \operatorname{rad} S_i - 1$, $i = 1, 2$. Hence $\sum_{i=1}^{2} \gamma_b(S_i) + \sum_{i=1}^{3} \left\lceil \frac{q_i}{3} \right\rceil = \operatorname{rad} S_1 - 1 + \operatorname{rad} S_2 - 1 + 3 = \operatorname{rad} R + 1 = \operatorname{rad} T$, as required in (9).
- If $q_2 = 2$, then (i) implies that if S_i is radial, then it is even. If S_1 and S_2 are both radial, then $(iii)(c)$ applies (with $k = j = 1$) and we obtain a contradiction because $q_2 \not\equiv 0 \pmod 3$. Hence at most one of S_1 and S_2 is radial. If neither S_1 nor S_2 is radial, then each S_i has a maximum split-set of cardinality 1 (since $\rho = 2$), and so has odd diameter. But Q_2 is odd, hence P_R is odd, a contradiction. Hence assume without loss of generality that S_1 is radial and S_2 is not. Then S_1 is even. Since Q_2 is odd and P_R is even, S_2 is odd and has a maximum split-set of cardinality one. Hence $\operatorname{rad} R = \operatorname{rad} S_1 + \operatorname{rad} S_2 + 1$ and, by Theorem 2.2, $\gamma_b(S_2) = \operatorname{rad} S_2 - 1$. Therefore $\sum_{i=1}^{2} \gamma_b(S_i) + \sum_{i=1}^{3} \left\lceil \frac{q_i}{3} \right\rceil = \operatorname{rad} S_1 + \operatorname{rad} S_2 - 1 + 3 = \operatorname{rad} R + 1 = \operatorname{rad} T$, hence (9) holds.
- Say $q_2 = 3$. Since P_R and Q_2 are even, S_1 and S_2 are either both even or both odd. Say $Q_2 = u_1, u_2, u_3, u_4, u_5$, where $u_1 \in V(S_1)$ and $u_5 \in V(S_2)$.

  If S_1 and S_2 are both even, then $\{u_1 u_2, u_4 u_5\}$ is a maximum split-set of R, hence S_1 and S_2 are radial (for otherwise R has a larger split-set). In this case

$\sum_{i=1}^{2} \gamma_b(S_i) + \sum_{i=1}^{3} \lceil \frac{q_i}{3} \rceil = \operatorname{rad} S_1 + \operatorname{rad} S_2 + 3 = \operatorname{rad} R + 1 = \operatorname{rad} T$, hence (9) holds.

Assume S_1 and S_2 are both odd. Then $\operatorname{rad} R = (\operatorname{rad} S_1 + \operatorname{rad} S_2 - 1) + \operatorname{rad} Q_2 = \operatorname{rad} S_1 + \operatorname{rad} S_2 + 1$. If both S_1 and S_2 are nonradial, let $v_j v_{j+1}$ be a split-edge of S_1 and $v_{j'} v_{j'+1}$ a split-edge of S_2. Then $v_{j+1}, \ldots, u_1$ is even, as is $u_5, \ldots, v_{j'}$. But then $\{v_j v_{j+1}, u_1 u_2, u_4 u_5, v_{j'} v_{j'+1}\}$ is a split-set of R, which is not the case. If both S_1 and S_2 are radial, then R has no nonempty split-set, which is also not the case. Hence S_1 (say) is radial and S_2 is nonradial. Therefore $\sum_{i=1}^{2} \gamma_b(S_i) + \sum_{i=1}^{3} \lceil \frac{q_i}{3} \rceil = \operatorname{rad} S_1 + \operatorname{rad} S_2 - 1 + 3 = \operatorname{rad} R + 1 = \operatorname{rad} T$, as required.

This completes the proof of Case 1.

Case 2 $\rho = 1$. By Theorem 2.2, $\gamma_b(R) = \operatorname{rad} R - 1$, and P_R is odd. Say $M_R = \{v_j v_{j+1}\}$. If q_1 and q_{r+1} are both even, then M_R is a split-set of T, which is not the case, so assume without loss of generality that q_1 is odd. If $q_1 \geq 3$, then either $\{v_3 v_4\}$ (if q_{r+1} is odd) or $\{v_3 v_4, v_j v_{j+1}\}$ (if q_{r+1} is even) is a split-set of T, which is impossible, hence $q_1 = 1$. Similarly, if q_{r+1} is odd, then $q_{r+1} = 1$.

Case 2.1 q_{r+1} is odd. Then P is odd and $q_{r+1} = 1$. If e_k, e_{k+1} are two consecutive free edges of R, then one of e_k and e_{k+1} is a split-edge of T; hence R has no consecutive free edges and consists of one MTS S_1, which is nonradial. Therefore $\gamma_b(S_1) + \sum_{i=1}^{2} \lceil \frac{q_i}{3} \rceil = \operatorname{rad} R - 1 + 2 = \operatorname{rad} T$ and (9) holds.

Case 2.2 q_{r+1} is even. Then P is even and $\operatorname{rad} T = \operatorname{rad} R + \frac{1}{2} q_{r+1}$. Since $\rho = 1$, R contains at most one set Y of two or more consecutive free edges; if there exists such a Y, then $|Y| \leq 4$. Thus R consists of at most two MTS's. If $q_{r+1} \geq 6$, then $\{v_{n-6} v_{n-5}, v_{n-3} v_{n-2}\}$ is a split-set of T, so $q_{r+1} \in \{0, 2, 4\}$.

If R consists of one MTS S_1, then $\gamma_b(S_1) + \sum_{i=1}^{2} \lceil \frac{q_i}{3} \rceil = (\operatorname{rad} R - 1) + 1 + \lceil \frac{q_2}{3} \rceil = \operatorname{rad} T$, as required. Hence assume R consists of two MTS's S_1 and S_2, together with Q_2 (where $E(Q_2) = Y$), and $1 \leq q_2 \leq 3$. Now if $q_3 = 4$ and $e_k, e_{k+1} \in Y$, then $\{e_k, v_{n-3} v_{n-2}\}$ or $\{e_{k+1}, v_{n-3} v_{n-2}\}$ is a split-set of T, a contradiction. Assume therefore that $q_3 \in \{0, 2\}$. We only consider the case $q_3 = 2$; the case $q_3 = 0$ is similar.

- Say $q_2 = 1$. Then Q_2 and exactly one of S_1 and S_2 are even. If S_1 is even and S_2 is odd, then by (i) and (ii), neither S_1 nor S_2 is radial. Since Q_2 is even, the union of a split-set of S_1 and a split-set of S_2 is a split-set of R containing more than one edge, a contradiction. Hence S_1 is odd and S_2 is even. By (i), S_1 is nonradial, and thus, following the same reasoning as above, S_2 is radial. Since P_R is odd, $\operatorname{rad} R = \operatorname{rad} S_1 + \operatorname{rad} S_2 + \operatorname{rad} Q_2 = \operatorname{rad} S_1 + \operatorname{rad} S_2 + 1$. Now $\sum_{i=1}^{2} \gamma_b(S_i) + \sum_{i=1}^{3} \lceil \frac{q_i}{3} \rceil = \operatorname{rad} S_1 - 1 + \operatorname{rad} S_2 + 3 = \mathrm{radS_1} + \operatorname{rad} S_2 + 2 = \operatorname{rad} R + 1$, as required.
- Say $q_2 = 2$. Then Q_2 is odd, so S_1 and S_2 are both odd or both even. Assume firstly that both are odd. Then by (iii) with $j = 1$, one of them is nonradial. If

S_2 is nonradial, let v_jv_{j+1} be a split-edge of S_2 and let u_1u_2 be the first edge of Q_2 (i.e., $u_1 \in V(S_1)$). Then $\{u_1u_2, v_jv_{j+1}\}$ is a split-set of T, a contradiction. Hence S_1 is nonradial while S_2 is radial, so that $\sum_{i=1}^{2} \gamma_b(S_i) + \sum_{i=1}^{3} \lceil \frac{q_i}{3} \rceil = \operatorname{rad} S_1 - 1 + \operatorname{rad} S_2 + 3 = \operatorname{rad} T$ as above.

Now assume that S_1 and S_2 are even. Then both are radial, for otherwise we obtain a split-set of T. Then $\operatorname{rad} R = \operatorname{rad} S_1 + \operatorname{rad} S_2 + 2$, so that $\sum_{i=1}^{2} \gamma_b(S_i) + \sum_{i=1}^{3} \lceil \frac{q_i}{3} \rceil = \operatorname{rad} S_1 + \operatorname{rad} S_2 + 3 = \operatorname{rad} T$, and again (9) is satisfied.

- Say $q_2 = 3$ and let $Q_2 = u_1, u_2, u_3, u_4, u_5$, where $u_1 \in V(S_1)$ and $u_5 \in V(S_2)$. Since Q_2 is even, exactly one of S_1 and S_2 is even. If S_1 is odd, then $\{u_1u_2, u_4u_5\}$ is a split-set of T, contradicting the radiality of T. Thus S_1 is even and S_2 is odd. If e is a split-edge of S_2, then $\{u_4u_5, e\}$ is a split-set of T, which is not the case, hence S_2 is radial. Similarly, S_1 is radial. Hence $\sum_{i=1}^{2} \gamma_b(S_i) + \sum_{i=1}^{3} \lceil \frac{q_i}{3} \rceil = \operatorname{rad} S_1 + \operatorname{rad} S_2 + 3 = \operatorname{rad} R + 1 = \operatorname{rad} T$.

The proof of Case 2 is now complete.

Case 3 $\rho = 0$, i.e., R is radial. Now P_R may be even or odd.

Case 3.1 P_R is odd. Then R consists of a single MTS S_1, for if e_1 and e_2 are consecutive free edges, then one of them is a split-edge of R. Necessarily, S_1 is radial. By (i), $q_1, q_2 \not\equiv 1 \pmod 3$. By symmetry we may assume without loss of generality that $q_1 \le q_2$. If $\min\{q_1, q_2\} \ge 3$, then $\{v_3v_4\}$ or $\{v_3v_4, v_{n-3}v_{n-2}\}$ is a split-set of T, so we may assume that $q_1 \le 2$ and hence that $q_1 \in \{0, 2\}$. Similarly, $q_2 \le 5$. Now if $q_1 = 0$ and

$$q_2 = \left\{ \begin{array}{l} 2 \\ 3, \\ 5 \end{array} \right. \quad \text{then } \gamma_b(S_1) + \sum_{i=1}^{2} \left\lceil \frac{q_i}{3} \right\rceil = \left\{ \begin{array}{l} \operatorname{rad} S_1 + 1 \\ \operatorname{rad} S_1 + 1 \\ \operatorname{rad} S_1 + 2 \end{array} \right\} = \operatorname{rad} T.$$

A similar argument works if $q_1 = 2$.

Case 3.2 P_R is even. As in Case 1, R contains at most one set Y of two or more consecutive free edges, where $|Y| \le 4$, and R consists of at most two MTS's. As above we may assume that $q_1 \le q_{r+1} \le 5$ and $q_1 \le 2$. Moreover, if q_1 is even and $q_{r+1} \ge 3$ is odd, then P is odd and $v_{n-3}v_{n-2}$ is a split-edge of T, a contradiction. Thus if q_1 is even, then $q_{r+1} \in \{0, 1, 2, 4\}$.

- Suppose firstly that R consists of a single MTS S_1. Since R is radial, S_1 is radial. By (ii), $q_1 \not\equiv 1 \pmod 3$ or $q_2 \not\equiv 1 \pmod 3$. Using this and various parity arguments for the radii of R and T, we see that, for $a \in \{0, 1\}$ and $b \in \{0, 1, 2\}$,

$$\text{if } q_1 = \left\{ \begin{array}{l} 1 \\ 1 \\ 1 \\ 0 \\ 2a \end{array} \right. \text{ and } q_2 = \left\{ \begin{array}{l} 2 \\ 3 \\ 5, \\ 1 \\ 2b \end{array} \right. \text{then } \gamma_b(S_1) + \sum_{i=1}^{2} \left\lceil \frac{q_i}{3} \right\rceil = \left\{ \begin{array}{l} \operatorname{rad} S_1 + 2 \\ \operatorname{rad} S_1 + 2 \\ \operatorname{rad} S_1 + 3 \\ \operatorname{rad} S_1 + 1 \\ \operatorname{rad} S_1 + a + b \end{array} \right\} = \operatorname{rad} T.$$

- Now suppose that R consists of two MTS's S_1 and S_2, and the path Q_2. Let e_1 and e_2 be adjacent edges of Q_2. If $q_3 \geq 4$, then $\{v_{n-3}v_{n-2}\}$, $\{e_1, v_{n-3}v_{n-2}\}$ or $\{e_2, v_{n-3}v_{n-2}\}$ is a split-set of T, which is impossible. Thus $q_3 \leq 3$.

  ⋆ Say $q_2 = 1$. Since P_R and Q_2 are even, S_1 and S_2 are both even or both odd. If both are odd, then by (i) neither is radial. But then the union of split-sets of S_1 and S_2 is a split-set of R, which is impossible. Hence S_1 and S_2 are even. A split-set of either S_1 or S_2 is also a split-set of R. But R is radial, hence S_1 and S_2 are radial. By (ii), $q_1, q_3 \not\equiv 1 \pmod 3$. Thus $q_1 = 2a$, $a \in \{0,1\}$, and by the above analysis, $q_3 = 2b$, $b \in \{0,1\}$. Now $\sum_{i=1}^{2} \gamma_b(S_i) + \sum_{i=1}^{3} \lceil \frac{q_i}{3} \rceil = \operatorname{rad} S_1 + \operatorname{rad} S_2 + a + 1 + b = \operatorname{rad} R + a + b = \operatorname{rad} T$.

  ⋆ Say $q_2 = 2$. Then exactly one of S_1 and S_2 is even, hence $\operatorname{rad} R = \operatorname{rad} S_1 + \operatorname{rad} S_2 + 1$. If S_i is odd, then a split-edge e of S_i together with a suitably chosen edge of Q_2 is a split-set of R. If S_i is even, then any split-set of S_i is a split-set of R, a contradiction. Thus both S_1 and S_2 are radial. Now if S_1 is odd, then by (i), $q_1 = 2a$, $a \in \{0,1\}$, and as shown above, $q_3 \in \{0,1,2\}$. However, if $q_3 = 1$, then the middle edge of Q_2 is a split-edge of T, a contradiction. Hence $q_3 = 2b$, $b \in \{0,1\}$. Therefore $\sum_{i=1}^{2} \gamma_b(S_i) + \sum_{i=1}^{3} \lceil \frac{q_i}{3} \rceil = \operatorname{rad} R + a + b = \operatorname{rad} T$.

  On the other hand, if S_2 is odd, then by (i), $q_3 \in \{0,2,3\}$. It is routine to verify that (9) holds for all possible choices of (q_1, q_3).

  ⋆ Finally, say $q_2 = 3$ and let $Q_2 = u_1, u_2, u_3, u_4, u_5$, where $u_1 \in V(S_1)$ and $u_5 \in V(S_2)$. Then S_1 and S_2 have the same parity. But if S_1 and S_2 are even, then $\{u_1u_2, u_4u_5\}$ is a split-set of R, which is not the case. Hence S_1 and S_2 are odd, and both are radial, otherwise R has a nonempty split-set. Therefore $\operatorname{rad} R = (\operatorname{rad} S_1 + \operatorname{rad} S_2 - 1) + \operatorname{rad} Q_2 = \operatorname{rad} S_1 + \operatorname{rad} S_2 + 1$. By (i), $q_1, q_3 \not\equiv 1 \pmod 3$, and by the above restrictions on q_1 and q_3 we thus have $q_1, q_3 \in \{0, 2\}$. In all cases (9) is satisfied.

This completes the proof of Case 3 and also the proof of the lemma. ■

Proof of the converse of Lemma 4.5. Assume that $S_1, \ldots, S_r \in \mathcal{T}$ and that (i), (ii) and (iii) hold. Let M be a maximum split-set of T with $|M| = m$. Let $T_1, \ldots, T_{m+1}$ be the (radial) components of $T - M$. By Theorem 2.2, $\gamma_b(T) = \sum_{i=1}^{m+1} \gamma_b(T_i)$. By Lemma 6.3, $T_i \in \mathcal{T}$ for each i. Obviously, $\gamma(T) \leq \sum_{i=1}^{m+1} \gamma(T_i)$, and since $T_i \in \mathcal{T}$ for each i, it follows that $\gamma(T) \leq \sum_{i=1}^{m+1} \gamma_b(T_i) = \gamma_b(T)$. The result follows from the trivial bound $\gamma_b(T) \leq \gamma(T)$. ■

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