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Existence of 4-factors in star-free graphs with high connectivity

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Abstract

Let $t \geq 6$ be an integer. We show that if G is a $\lceil (3t-3)/2 \rceil$-connected $K_{1,t}$-free graph, then G has a 4-factor.

1 Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges.

Let $G = (V(G), E(G))$ be a graph. For $x \in V(G)$, $\deg_G(x)$ denotes the degree of x in G. We let $\delta(G)$ denote the minimum of $\deg_G(x)$ as x ranges over $V(G)$. For an integer $r \geq 1$, a subgraph F of G such that $V(F) = V(G)$ and $\deg_F(x) = r$ for all $x \in V(F)$ is called an r-factor of G. The complete bipartite graph $K_{1,t}$ with partite sets of cardinalities 1 and t is called the t-star. We say that G is $K_{1,t}$-free or t-star-free if G does not contain $K_{1,t}$ as an induced subgraph.

The following two results concerning 2-factors and 4-factors were proved in [1] and [3].

Theorem 1. *Let $t \geq 3$ be an integer, and let G be a 2-connected $K_{1,t}$-free graph such that $\delta(G) \geq t$. Then G has a 2-factor.*

Theorem 2. *Let $t \geq 3$ be an integer, and let G be a 2-connected $K_{1,t}$-free graph such that $\delta(G) \geq \lceil (3t+1)/2 \rceil$. Then G has a 4-factor.*

In Theorems 1 and 2, the condition on $\delta(G)$ is best possible. However, it is natural to expect that we can weaken the condition on $\delta(G)$ if we replace the assumption that G is 2-connected by a stronger assumption. Along this line, the following theorem on 2-factors was proved by Aldred et al., in [1].

Theorem 3. *Let $t \geq 4$ be an integer, and let G be a $(t-1)$-connected $K_{1,t}$-free graph. Then G has a 2-factor.*

In [1], it was also shown that Theorem 3 is best possible in the sense that for each $t \geq 4$, there exist infinitely many $(t-2)$-connected $K_{1,t}$-free graphs G with $\delta(G) \geq t-1$ such that G has no 2-factor. The purpose of this paper is to prove a result about 4-factors which corresponds to Theorem 3. Our main theorem is as follows.

Theorem 4. *Let $t \geq 6$ be an integer, and let G be a $\lceil (3t-3)/2 \rceil$-connected $K_{1,t}$-free graph. Then G has a 4-factor.*

Theorem 4 is best possible in the sense that for each $t \geq 6$, there exist infinitely many $\lceil (3t-5)/2 \rceil$-connected $K_{1,t}$-free graphs G with $\delta(G) \geq \lceil (3t-1)/2 \rceil$ such that G has no 4-factor. To construct such graphs, fix $t \geq 6$ and set $r = \lceil (3t-5)/2 \rceil$. Let $m \geq t$ be an arbitrary integer relatively prime to $t-1$. Let $I_1, I_2, \ldots, I_m, J_1, J_2, \ldots, J_m, H_1, H_2, \ldots, H_{m(t-1)}$ be disjoint graphs such that I_k is isomorphic to the complete graph of order $\lceil r/2 \rceil$ for each $1 \leq k \leq m$, J_k is isomorphic to the complete graph of order $\lfloor r/2 \rfloor$ for each $1 \leq k \leq m$, and H_i is isomorphic to the complete graph of order 3 for each $1 \leq i \leq m(t-1)$. For each $1 \leq k \leq m$, set

$$T_k = \bigcup_{1 \leq j \leq t-1} V(H_{(k-1)(t-1)+j}),$$
$$T'_k = \bigcup_{1 \leq j \leq t-1} V(H_{(j-1)m+k}).$$

Now define a graph G by

$$V(G) = \Bigg(\bigcup_{1 \leq k \leq m} \Big(V(I_k) \cup V(J_k) \Big) \Bigg) \cup \Big(\bigcup_{1 \leq i \leq m(t-1)} V(H_i) \Big),$$

$$E(G) = \Bigg(\bigcup_{1 \leq k \leq m} \Big(E(I_k) \cup E(J_k) \Big) \cup \{xy \,|\, x \in V(I_k), y \in T_k\} \cup \{xy \,|\, x \in V(J_k), y \in T'_k\} \Bigg) \cup \Big(\bigcup_{1 \leq i \leq m(t-1)} E(H_i)) \Big).$$

Then G is $\lceil (3t-5)/2 \rceil$-connected and $K_{1,t}$-free, and satisfies $\delta(G) = \lceil (3t-1)/2 \rceil$. However, we easily see that G does not have a 4-factor (for example, if we apply

Lemma 2.1 in Section 2 with $S = \bigcup_{1\le k\le m}(V(I_k)\cup V(J_k))$ and $T = \bigcup_{1\le i\le m(t-1)} V(H_i)$, then we get $\theta(S,T) = -2m$ or $-4m$ depending on whether t is even or odd). In passing, we mention that Theorem 5 does not hold for $t \le 5$ (we plan to discuss in detail the case where $t \le 5$ in a subsequent paper). For related results, we refer the reader to Tokuda and Ota [4] and Yashima [6].

Our notation is standard, and is mostly taken from Diestel [2]. Possible exceptions are as follows. Let G be a graph. For $x \in V(G)$, $N(x) = N_G(x)$ denotes the set of vertices adjacent to x in G; thus $\deg_G(x) = |N_G(x)|$. For $A \subseteq V(G)$, we let $N(A)$ denote the union of $N(x)$ as x ranges over A. For A, $B \subseteq V(G)$ with $A \cap B = \emptyset$, $E(A,B)$ denotes the set of those edges of G which join a vertex in A and a vertex in B. For $A \subseteq V(G)$, the graph obtained from G by deleting all vertices in A together with the edges incident with them is denoted by $G - A$. For $A \subseteq V(G)$, we let $G[A]$ denote the subgraph induced by A in G. We often identify a vertex x of G with the set $\{x\}$; for example, when B is a subset of $V(G)$ with $x \notin B$, we write $E(x,B)$ for $E(\{x\},B)$.

2 Preliminary results

In this section, we state preliminary lemmas, which we use in the proof of Theorem 4.

Let G be a graph. For S, $T \subseteq V(G)$ with $S \cap T = \emptyset$, define $\theta(S,T)$ by

$$\theta(S,T) = 4|S| + \Sigma_{y\in T}(\deg_{G-S}(y) - 4) - h(S,T),$$

where $h(S,T)$ denotes the number of those components C of $G - S - T$ such that $|E(T,V(C))|$ is odd. The following lemma is a special case of the f-Factor Theorem of Tutte [5].

Lemma 2.1. (i) *The graph G has a 4-factor if and only if $\theta(S,T) \ge 0$ for all S, $T \subseteq V(G)$ with $S \cap T = \emptyset$.*

(ii) *Whether G has a 4-factor or not, $\theta(S,T)$ is even for all S, $T \subseteq V(G)$ with $S \cap T = \emptyset$.* □

The following lemma seems to be well-known, but we include its proof for the convenience of the reader.

Lemma 2.2. *Let S, $T \subseteq V(G)$ be subsets of $V(G)$ with $S \cap T = \emptyset$ for which, $\theta(S,T)$ becomes smallest. Then the following hold.*

(i) *Let C be a component of $G - S - T$ such that $|E(T,V(C))| = 1$. Then $|V(C)| \ge 2$.*

(ii) *Suppose that S and T are chosen with $|T|$ is as small as possible, subject to the condition that $\theta(S,T)$ is smallest. Then $\deg_{G[T]}(y) \le 2$ for every $y \in T$.*

Proof. Suppose that there exists a component C of $G-S-T$ such that $|E(T,V(C))| = 1$ and $|V(C)| = 1$. Let $V(C) = \{v\}$, and set $T' = T \cup \{v\}$. Then $\deg_{G-S}(v) = 1$, and hence

$$\begin{aligned}\theta(S,T') &= 4|S| + \Sigma_{y\in T'}(\deg_{G-S}(y)-4) - h(S,T')\\ &= 4|S| + \Sigma_{y\in T}(\deg_{G-S}(y)-4) + \deg_{G-S}(v) - 4 - h(S,T) + 1\\ &= \theta(S,T) - 2,\end{aligned}$$

which contradicts the minimality of $\theta(S,T)$. Thus (i) is proved.

Now let $y \in T$, and set $T' = T - \{y\}$. Then

$$h(S,T') \geq h(S,T) - E(y, V(G) - S - T),$$

and hence

$$\begin{aligned}\theta(S,T') &\leq \theta(S,T) - \deg_{G-S}(y) + 4 + |E(y, V(G)-S-T)|\\ &= \theta(S,T) - \deg_{G[T]}(y) + 4.\end{aligned}$$

On the other hand, by the minimality of $|T|$ and Lemma 2.1 (ii), $\theta(S,T') \geq \theta(S,T) + 2$. Hence $\theta(S,T) + 2 \leq \theta(S,T) - \deg_{G[T]}(y) + 4$, which implies (ii). □

We also need the following technical result.

Lemma 2.3. *Suppose that $\delta(G) \geq 6$ and $|V(G)| \leq \delta(G)+3$. Then G has a 4-factor.*

Proof. Let S, $T \subseteq V(G)$ with $S \cap T = \emptyset$. In view of Lemma 2.1, it suffices to show that $\theta(S,T) \geq -1$. If $T = \emptyset$, then $h(S,T) = 0$ by definition, and hence $\theta(S,T) = 4|S| \geq 0$. Thus we may assume $T \neq \emptyset$. If $|S| \geq |V(G)|/2$, then since $\sum_{y\in T}(\deg_{G-S}(y) - 4) \geq -4|T|$ and $h(S,T) \leq |V(G)| - |S| - |T|$, we get

$$\begin{aligned}\theta(S,T) &\geq 4|S| - 4|T| - (|V(G)| - |S| - |T|)\\ &\geq 4|S| - 4|T| - 4(|V(G)| - |S| - |T|)\\ &\geq 0.\end{aligned}$$

Thus we may assume $|S| < |V(G)|/2$. Since $\delta(G) \geq 6$ and $\delta(G) \geq |V(G)| - 3$, we have $\deg_{G-S}(y) \geq \max\{6 - |S|, |V(G)| - |S| - 3\}$ for every $y \in T$. Hence

$$\theta(S,T) \geq 4|S| + |T| \cdot \max\{2 - |S|, |V(G)| - |S| - 7\} - h(S,T). \tag{2.1}$$

Note that we have $h(S,T) \leq 3$ because $\delta(G) \geq |V(G)| - 3$. Thus $\theta(S,T) \geq 4|S| + |T| \cdot \max\{2 - |S|, |V(G)| - |S| - 7\} - 3$. If $S = \emptyset$, then $\theta(S,T) \geq |T| \cdot 2 - 3 \geq -1$. Thus we may assume $S \neq \emptyset$. If $|S| \leq \max\{2, |V(G)| - 7\}$, then $\theta(S,T) \geq 4|S| - 3 > 0$. Thus we may assume $|S| \geq \max\{3, |V(G)| - 6\}$. Since$|S| < |V(G)|/2$, this implies $|V(G)| \leq 11$. Since $|S| \geq 3$, it now follows from (2.1) that

$$\begin{aligned}\theta(S,T) &\geq 4|S| + |T|(2 - |S|) - (|V(G)| - |S| - |T|)\\ &\geq 4|S| + |T|(2 - |S|) - (|V(G) - |S| - |T|)(|S| - 2)\\ &= |S|(|S| + 2) - |V(G)|(|S| - 2)\\ &\geq |S|(|S| + 2) - 11(|S| - 2)\\ &> 0,\end{aligned}$$

as desired. □

3 Notation

Let t, G be as in Theorem 4; thus $t \geq 6$, and G is a $\lceil(3t-3)/2\rceil$-connected $K_{1,t}$-free graph. In this section, we fix notation for the proof of Theorem 4.

Since G is $\lceil(3t-3)/2\rceil$-connected, we have $\delta(G) \geq \lceil(3t-3)/2\rceil \geq 8$. In view of Lemma 2.3, we may assume $|V(G)| \geq \delta(G)+4 \geq \lceil(3t+5)/2\rceil$.

Let S, T be subsets of $V(G)$ with $S \cap T = \emptyset$ for which $\theta(S,T)$ becomes smallest. We choose S, $T \subseteq V(G)$ so that $|T|$ is as small as possible, subject to the condition that $\theta(S,T)$ is smallest. We show that $\theta(S,T) \geq 0$. By Lemma 2.1, this will imply that Theorem 4 holds.

If $T = \emptyset$, then $h(S,T) = 0$, and hence $\theta(S,T) = 4|S| \geq 0$. Thus we may assume that $T \neq \emptyset$.

We call a component C of $G-S-T$ an odd component or an even component according as $|E(T,V(C))|$ is odd or even.

Let $C_1,\dots,C_k$ be the components of $G-S-T$. We may assume that there exists a with $0 \leq a \leq k$ such that $|E(T,V(C_i))| = 1$ for each $1 \leq i \leq a$ and $|E(T,V(C_i))| \neq 1$ for each $a+1 \leq i \leq k$. Then the components $C_1,\dots,C_a$ are odd components. We may further assume that there exists b with $0 \leq b \leq k-a$ such that C_i is an odd component for each $a+1 \leq i \leq a+b$ and C_i is an even component for each $a+b+1 \leq i \leq k$. Then $h(S,T) = a+b$. Set $U = \cup_{i=1}^{a} V(C_i)$ and $U' = V(G)-S-T-U$. For each $y \in T$, set $\alpha(y) = |N(y)\cap U|$ and $\beta(y) = |N(y)\cap U'|$. Then

$$\alpha(y)+\beta(y) = |N(y) \cap (V(G)-S-T)|; \tag{3.1}$$

in particular,

$$\alpha(y)+\beta(y) \leq \deg_{G-S}(y). \tag{3.2}$$

We also set $\beta(A) = \sum_{y\in A}\beta(y)$ for a subset A of T.

Claim 3.1. (i) $a = \sum_{y\in T}\alpha(y)$.

(ii) *For each i with $a+1 \leq i \leq a+b$, $\sum_{y\in T}|N(y)\cap V(C_i)| \geq 3$.*

(iii) $b \leq (\sum_{y\in T}\beta(y))/3$.

Proof. We have

$$a = \sum_{1\leq i\leq a}|E(T,V(C_i))| = |E(T,U)| = \sum_{y\in T}|E(y,U)| = \sum_{y\in T}|N(y)\cap U| = \sum_{y\in T}\alpha(y),$$

which proves (i). Let $a+1 \leq i \leq a+b$. Then $|E(T,V(C_i))| \neq 1$ and $|E(T,V(C_i))|$ is odd. Hence $\sum_{y\in T}|N(y)\cap V(C_i)| = \sum_{y\in T}|E(y,V(C_i))| = |E(T,V(C_i))| \geq 3$. Thus

(ii) is proved. By (ii),

$$\begin{aligned} b &\le \left(\sum_{a+1 \le i \le a+b} \sum_{y \in T} |N(y) \cap V(C_i)| \right) /3 \\ &\le \left(\sum_{a+1 \le i \le k} \sum_{y \in T} |N(y) \cap V(C_i)| \right) /3 \\ &= \left(\sum_{y \in T} |N(y) \cap U'| \right) /3 \\ &= \left(\sum_{y \in T} \beta(y) \right) /3, \end{aligned}$$

which proves (iii). □

Let $1 \le i \le a$. Since $t \ge 6$ and G is $\lceil (3t-3)/2 \rceil$-connected, G is 2-connected. Since $|V(C_i)| \ge 2$ by Lemma 2.2(i), this implies that there exists an edge joining S and $V(C_i) - N(T)$. Let $x_i u_i$ be such an edge ($x_i \in S$, $u_i \in V(C_i) - N(T)$). Set

$$L = \{u_i \,|\, 1 \le i \le a\}.$$

Then

$$|L| = a. \tag{3.3}$$

For each $x \in S$, let $L(x) = \{u_i \,|\, 1 \le i \le a, x_i = x\}$. Clearly

$$L(x) \subseteq N(x) \quad \text{and} \quad L(x) \text{ is independent.} \tag{3.4}$$

Also $L = \bigcup_{x \in S} L(x)$ (disjoint union), and hence

$$\sum_{x \in S} |L(x)| = a \tag{3.5}$$

by (3.3).

We now look at components of $G[T]$. Recall that $T \ne \emptyset$. Let $H_1, H_2, \ldots, H_m$ be the components of $G[T]$. Then

$$T = \bigcup_{1 \le i \le m} V(H_i) \text{ (disjoint union).} \tag{3.6}$$

By Lemma 2.2 (ii), H_i is a path or a cycle for each $1 \le i \le m$. In the remainder of this section, we assign a real number θ_i to each H_i, and show that $\theta(S,T) \ge \sum_{1 \le i \le m} \theta_i$. We first prove several claims concerning H_i.

Claim 3.2. *Let $1 \le i \le m$, and suppose that H_i is a cycle of order 5. Then there exist vertices z, $z' \in V(H_i)$ with $zz' \in E(G)$ such that $|N(\{z, z'\}) \cap (S \cup U \cup U')| \ge \lceil (3t-5)/2 \rceil$.*

Proof. Note that we have $|N(y)\cap(S\cup U\cup U')| = \deg_G(y)-\deg_{H_i}(y) \geq \lceil(3t-3)/2\rceil - 2 = \lceil(3t-7)/2\rceil$ for every $y \in V(H_i)$. Suppose that $|N(\{z,z'\})\cap(S\cup U\cup U')| \leq \lceil(3t-7)/2\rceil$ for any z, $z' \in V(H_i)$ with $zz' \in E(G)$. Then since H_i is connected, it follows that $|N(H_i)\cap(S\cup U\cup U')| = \lceil(3t-7)/2\rceil$. Since G is $\lceil(3t-3)/2\rceil$-connected, this implies $V(G) = V(H_i)\cup(N(V(H_i)\cap(S\cap U\cap U'))$. But then $|V(G)| = 5+\lceil(3t-7)/2\rceil$, which contradicts the assumption that $|V(G)| \geq \lceil(3t+5)/2\rceil$ (see the second paragraph of this section). □

Claim 3.3. *Let* $1 \leq i \leq m$, *and suppose that* H_i *is a cycle of order* 3. *Let* z, $z' \in V(H_i)$ *with* $z \neq z'$. *Then* $|N(\{z,z'\})\cap(S\cup U\cup U')| \geq \lceil(3t-5)/2\rceil$.

Proof. Suppose that $|N(\{z,z'\})\cap(S\cup U\cup U')| \leq \lceil(3t-7)/2\rceil$. Then $|N(\{z,z'\})\cap(V(G)-V(H))| = |(N(\{z,z'\})\cap(S\cup U\cup U'))\cup(V(H_i)-\{z,z'\})| \leq \lceil(3t-5)/2\rceil$. Since G is $\lceil(3t-3)/2\rceil$-connected, this implies $|V(G)| = |V(H_i)\cup(N(\{z,z'\})\cap(S\cup U\cup U'))| \leq \lceil(3t-1)/2\rceil$, which contradicts the assumption that $|V(G)| \geq \lceil(3t+5)/2\rceil$. □

Claim 3.4. *Let* $1 \leq i \leq m$, *and suppose that* H_i *is a path of order* 1 *or* 2, *or a cycle of order* 3. *Then* $|N(V(H_i))\cap(S\cup U\cup U')| \geq \lceil(3t-3)/2\rceil$.

Proof. If $|N(V(H_i))\cap(S\cup U\cup U')| \leq \lceil(3t-5)/2\rceil$, then since G is $\lceil(3t-3)/2\rceil$-connected, it follows that $|V(G)| = |V(H_i)\cap(N(V(H_i))\cap(S\cup U\cup U'))| \leq \lceil(3t+1)/2\rceil$, a contradiction. □

Before defining the numbers θ_i ($1 \leq i \leq m$), we choose, for each $1 \leq i \leq m$, disjoint subsets $T_{i,1}$, $T_{i,2}\ldots,T_{i,j_i}$ of $V(H_i)$ such that $E(T_{i,j},T_{i,j'}) = \emptyset$ for any j, j' with $j \neq j'$, where j_i is the independence number of H_i. Our choice of the $T_{i,j}$ depend on whether $t \geq 7$ or $t = 6$.

(1) Let $t \geq 7$.

- **(D1-1)** Assume that H_i is a path of order l, where $l \geq 3$, or a cycle of order l, where $l = 4$ or $l \geq 6$. Let $z_1,\ldots,z_{j_i}$ be j_i independent vertices of H_i. Under this notation, we set $T_{i,j} = \{z_j\}$ for each $1 \leq j \leq j_i$.
- **(D1-2)** Assume that H_i is a path of order 1 or 2, or a cycle of order 3. In this case, $j_i = 1$. We set $T_{i,1} = V(H_i)$.
- **(D1-3)** Assume that H_i is a cycle of order 5. In this case, $j_i = 2$. By Claim 3.2, there exist vertices z, $z' \in V(H_i)$ such that $zz' \in E(G)$ and $|N(\{z,z'\})\cap(S\cup U\cup U')| \geq \lceil(3t-5)/2\rceil$. Let z'' be the unique vertex in $V(H_i)-\{z,z'\}$ such that $E(z'',\{z,z'\}) = \emptyset$. We set $T_{i,1} = \{z,z'\}$ and $T_{i,2} = \{z''\}$.

(2) Let $t = 6$.

- **(D2-1)** Assume that H_i is a path of order l, where l is odd. Let $z_1,\ldots,z_{j_i}$ be j_i independent vertices of H_i. Under this notation, we set $T_{i,j} = \{z_j\}$ for each $1 \leq j \leq j_i$.

(D2-2) Assume that H_i is a path of order l, where l is even and $l \geq 4$, or H_i is a cycle of order l, where $l = 4$ or $l \geq 6$. Note that we can take two independent sets $Y, Y' \subseteq V(H_i)$ having cardinality j_i so that $Y \cap Y' = \emptyset$. We may assume $\beta(Y) \leq \beta(Y')$. Write $Y = \{z_1, \ldots, z_{j_i}\}$. Under this notation, we set $T_{i,j} = \{z_j\}$ for each $1 \leq j \leq j_i$. We have $\beta(V(H_i)) \geq \beta(Y) + \beta(Y') \geq 2\beta(Y) = 2\sum_{1\leq j\leq j_i} \beta(T_{i,j})$.

(D2-3) Assume that H_i is a path of order 2, or a cycle of order 3. In this case, $j_1 = 1$. We set $T_{i,1} = V(H_i)$.

(D2-4) Assume that H_i is a cycle of order 5. In this case, $j_i = 2$. First assume that $\beta(V(H_i)) = 0$. By Claim 3.2, there exist vertices z, $z' \in V(H_i)$ such that $zz' \in E(G)$ and $|N(\{z, z'\}) \cap (S \cup U \cup U')| \geq 7$. Let z'' be the unique vertex in $V(H_i) - \{z, z'\}$ such that $E(z'', \{z, z'\}) = \emptyset$. We set $T_{i,1} = \{z, z'\}$ and $T_{i,2} = \{z''\}$. Next assume that $\beta(V(H_i)) \geq 1$. Note that we can take two independent sets Y, Y' having cardinality 2 so that $Y \cap Y' = \emptyset$. We may assume $\beta(Y) \leq \beta(Y')$. Write $Y = \{z_1, z_2\}$. We set $T_{i,1} = \{z_1\}$ and $T_{i,2} = \{z_2\}$. We have $\beta(V(H_i) - (T_{i,1} \cup T_{i,2})) \geq \beta(Y') \geq \beta(T_{i,1}) + \beta(T_{i,2})$ and $\beta(V(H) - (T_{i,1} \cup T_{i,2})) \geq 1$.

For each $x \in S$, let $\mathcal{N}(x) = \{T_{i,j} \mid 1 \leq i \leq l, 1 \leq j \leq i_j, x \in N(T_{i,j})\}$.

Claim 3.5. (i) *If $(i,j) \neq (i',j')$, then $E(T_{i,j}, T_{i',j'}) = \emptyset$. In particular, for each $x \in S$, we have $E(T_{i,j}, T_{i',j'}) = \emptyset$ for any $T_{i,j}$, $T_{i'j'} \in \mathcal{N}(x)$ with $(i,j) \neq (i',j')$.*

(ii) *Let $x \in S$. Then $E(u, T_{i,j}) = \emptyset$ for any $u \in L(x)$ and for any $T_{i,j} \in \mathcal{N}(x)$.*

Proof. Statement (i) follows from the definition of H_i and $T_{i,j}$. Since $L(x) \cap N(T) = \emptyset$ by the definition of $L(x)$, (ii) also holds. ☐

Claim 3.6. $(t-1)|S| \geq \sum_{1\leq i\leq m} \sum_{1\leq j\leq j_i} \big|N(T_{i,j}) \cap (S \cup U)\big|$.

Proof. Since G is $K_{1,t}$-free, it follows from (3.4) and Claim 3.5(i), (ii) that $|\mathcal{N}(x)| + |L(x)| \leq t-1$ for every $x \in S$. Note that $\sum_{x\in S} |\mathcal{N}(x)| = \sum_{1\leq i\leq m} \sum_{1\leq j\leq j_i} |N(T_{i,j}) \cap S|$, and that $\sum_{x\in S} |L(x)| = a = \sum_{y\in T} \alpha(y) = \sum_{1\leq i\leq m} \sum_{y\in V(H_i)} \alpha(y)$ by (3.5), Claim 3.1(i) and (3.6). By the definition of $\alpha(y)$, we also have

$$\sum_{y\in V(H_i)} \alpha(y) \geq \sum_{1\leq j\leq j_i} \sum_{y\in T_{i,j}} \alpha(y) = \sum_{1\leq j\leq j_i} |N(T_{i,j}) \cap U|$$

for each $1 \leq i \leq m$. Consequently

$$\begin{aligned}(t-1)|S| &\geq \sum_{x\in S} \big(|\mathcal{N}(x)| + |L(x)|\big) \\ &= \sum_{1\leq i\leq m} \Big(\sum_{1\leq j\leq j_i} \big|N(T_{i,j}) \cap JS\big| + \sum_{y\in V(H_i)} \alpha(y)\Big) \\ &\geq \sum_{1\leq i\leq m} \Big(\sum_{1\leq j\leq j_i} (|N(T_{i,j}) \cap S| + |N(T_{i.j}) \cap U|)\Big),\end{aligned}$$

as desired. ☐

We now estimate $\theta(S,T)$ from below. For each $1 \le i \le m$, set

$$\theta_i = \frac{4}{t-1} \sum_{1\le j\le j_i} \left|N(T_{i,j}) \cap (S \cup U)\right| + \frac{2}{3}\beta(V(H_i)) - \sum_{y\in V(H_i)} (4 - \deg_{H_i}(y)).$$

Claim 3.7. $\theta(S,T) \ge \sum_{1\le i\le m} \theta_i$.

Proof. Note that $h(S,T) = a + b \le \sum_{1\le j\le m} \sum_{y\in V(H_i)} (\alpha(y) + \beta(y)/3)$ and

$$\sum_{y\in T} (\deg_{G-S}(y) - 4) = \sum_{1\le j\le m} \sum_{y\in V(H_i)} (\deg_{G-S}(y) - 4)$$

by Claim 3.1 (i), (iii) and (3.6). Also $\deg_{G-S}(y) = \deg_{H_i}(y) + \alpha(y) + \beta(y)$ for every $y \in T$. Therefore it follows from Claim 3.6 that

$$\begin{aligned}
\theta(S,T) &= 4|S| + \sum_{y\in T} \big(\deg_{G-S}(y) - 4\big) - h(S,T) \\
&\ge \frac{4}{t-1} \sum_{1\le i\le m} \sum_{1\le j\le j_i} \left|N(T_{i,j}) \cap (S\cup U)\right| \\
&\qquad + \sum_{1\le i\le m} \sum_{y\in V(H_i)} \left(\deg_{H_i}(y) + \alpha(y) + \beta(y) - 4 \right) \\
&\qquad - \sum_{1\le i\le m} \sum_{y\in V(H_i)} \left(\alpha(y) + \frac{1}{3}\beta(y) \right) \\
&= \sum_{1\le i\le m} \left\{ \frac{4}{t-1} \sum_{1\le j\le j_i} \left|N(T_{i,j}) \cap (S\cup U)\right| + \sum_{y\in V(H_i)} \left(\deg_{H_i}(y) - 4 \right) \right. \\
&\qquad \left. + \sum_{y\in V(H_i)} \frac{2}{3}\beta(y) \right\} \\
&= \sum_{1\le i\le m} \left\{ \frac{4}{t-1} \sum_{1\le j\le j_i} \left(\left|N(T_{i,j}) \cap (S\cup U)\right| \right) + \frac{2}{3}\beta(V(H_i)) \right. \\
&\qquad \left. - \sum_{y\in V(H_i)} (4 - \deg_{H_i}(y)) \right\},
\end{aligned}$$

as desired. ☐

4 Proof of Theorem 4

We continue with the notation of the proceeding section, and complete the proof of Theorem 4. We divide the proof into two cases.

Case 1. Suppose $t \geq 7$.
For each $1 \leq i \leq m$, set

$$p_i = \frac{4}{t-1}\sum_{1\leq j\leq j_i} \left|N(T_{i,j}) \cap (S \cup U \cup U')\right| - \sum_{y\in V(H_i)} (4 - \deg_{H_i}(y)).$$

Since $4/(t-1) \leq 2/3$ and $\beta(V(H_i)) \geq \sum_{1\leq j\leq j_i} \beta(T_{i,j}) \geq \sum_{1\leq j\leq j_i} |N(T_{i,j})\cap U'|$ by the definition of β, it follows that $\theta_i \geq (4/(t-1))\sum_{1\leq j\leq j_i} \big(|N(T_{i,j})\cup(S\cap U)| + |N(T_{i,j})\cap U'|\big) - \sum_{y\in V(H_i)}(4-\deg_{H_i}(y)) = p_i$. By Claim 3.7, we see that $\theta(S,T) \geq \sum_{1\leq i\leq m} p_i$. Hence in order to prove Theorem 4, it suffices to show that $p_i \geq 0$ for each $1 \leq i \leq m$. Thus fix i with $1 \leq i \leq m$. Recall that we have $\delta(G) \geq (3t-3)/2$ by the assumption that G is $\lceil (3t-3)/2\rceil$- connected.

Subcase 1.1. H_i is a path of order l, where $l \geq 3$, or H_i is a cycle of order l, where $l = 4$ or $l \geq 6$.

Let z_j be as in (D1-1). Thus $T_{i,j} = \{z_j\}$ for each $1 \leq j \leq j_i$. Since $\delta(G) \geq (3t-3)/2$, we get

$$\begin{aligned}\sum_{1\leq j\leq j_i} \left|N(T_{i,j}) \cap (S \cup U \cup U')\right| &= \sum_{1\leq j\leq j_i} \Big(\deg_G(z_j) - \deg_{H_i}(z_j)\Big)\\ &\geq \frac{(3t-3)j_i}{2} - \sum_{1\leq j\leq j_i} \deg_{H_i}(z_j).\end{aligned}$$

First assume that H_i is a path of order l, where l is odd and $l \geq 3$. Then $j_i = (l+1)/2$ and $\sum_{1\leq j\leq j_i} \deg_{H_i}(z_j) = 2j_i - 2$. Also $\sum_{y\in V(H_i)}(4-\deg_{H_i}(y)) = 2l+2$. Hence

$$\begin{aligned}p_i &\geq \frac{4}{t-1}\Big(\frac{(3t-3)j_i}{2} - (2j_i-2)\Big) - (2l+2)\\ &\geq 6j_i - \frac{2}{3}(2j_i-2) - (2l+2) = \frac{l}{3} + \frac{5}{3} > 0.\end{aligned}$$

Next assume that H_i is a path of order l, where l is even and $l \geq 4$. Then $j_i = l/2$, $\sum_{1\leq j\leq j_i} \deg_{H_i}(z_j) = 2j_i - 1$ or $2j_i - 2$, and $\sum_{y\in V(H_i)}(4-\deg_{H_i}(y)) = 2l+2$. Hence

$$\begin{aligned}p_i &\geq \frac{4}{t-1}\Big(\frac{(3t-3)j_i}{2} - (2j_i-1)\Big) - (2l+2)\\ &\geq 6j_i - \frac{2}{3}(2j_i-1) - (2l+2) = \frac{1}{3}l - \frac{4}{3} \geq 0.\end{aligned}$$

Now assume that H_i is a cycle of order l, where $l = 4$ or $l \geq 6$. Then $\sum_{1\leq j\leq j_i} \deg_{H_i}(z_j) = 2j_i$ and $\sum_{y\in V(H_i)}(4-\deg_{H_i}(z_j)) = 2l$. In the case where l is even, $j_i = l/2$, and hence

$$\begin{aligned}p_i &\geq \frac{4}{t-1}\Big(\frac{(3t-3)j_i}{2} - 2j_i\Big) - 2l\\ &\geq 6j_i - \frac{2}{3}\cdot 2j_i - 2l = \frac{1}{3}l > 0;\end{aligned}$$

in the case where l is odd, $j_i = (l-1)/2$ and $l \geq 7$, and hence

$$\begin{aligned} p_i &\geq \frac{4}{t-1}\Big(\frac{(3t-3)j_i}{2} - 2j_i\Big) - 2l \\ &> 6j_i - \frac{2}{3}\cdot 2j_i - 2l = \frac{1}{3}l - \frac{7}{3} \geq 0. \end{aligned}$$

This completes the discussion for Subcase 1.1.

Subcase 1.2. H_i is a path of order 1 or 2, or a cycle of order 3.

By (D1-2) and Claim 3.4, $j_i = 1$ and

$$|N(T_{i,1}) \cap (S \cup U \cup U')| = |N(V(H_i)) \cap (S \cup U \cup U')| \geq (3t-3)/2.$$

Also $\sum_{y \in V(H_i)}(4 - \deg_{H_i}(y)) \leq 6$. Hence

$$p_i \;\geq\; \frac{4}{t-1}\cdot\frac{(3t-3)}{2} - 6 \;=\; 0.$$

Subcase 1.3. H_i is a cycle of order 5.

By (D1-3), $j_i = 2$ and

$$\big|N(T_{i,1}) \cap (S \cup U \cup U')\big| \geq \frac{3t-5}{2}.$$

Also, letting $T_{i,2} = \{z''\}$, we get

$$\begin{aligned} \big|N(T_{i,2}) \cap (S \cup U \cup U')\big| &= \deg_G(z'') - \deg_{H_i}(z'') \\ &\geq \delta(G) - 2 \;\geq\; \frac{3t-3}{2} - 2 \\ &= \frac{3t-7}{2}. \end{aligned}$$

Further $\sum_{y \in V(H_i)}(4 - \deg_{H_i}(y)) = 10$. Hence

$$p_i \;\geq\; \frac{4}{t-1}\left(\frac{3t-5}{2} + \frac{3t-7}{2}\right) - 10 \;=\; \frac{12t-24}{t-1} - 10 \;\geq\; 0.$$

Thus the proof for the case where $t \geq 7$ is completed.

Case 2. Suppose $t = 6$.

In view of Claim 3.7, it suffices to show that $\theta_i \geq 0$ for each $1 \leq i \leq m$. Fix i with $1 \leq i \leq m$. Note that $\lceil (3t-3)/2 \rceil = 8$. Thus G is 8-connected, and $\delta(G) \geq 8$.

Subcase 2.1. H_i is a path of order l, where l is odd.

Let z_j be as in (D2-1). Thus $T_{i,j} = \{z_j\}$ for each $1 \leq j \leq j_i$. We have $j_i = (l+1)/2$, $\sum_{1 \leq j \leq j_i} \deg_{H_i}(z_j) = 2j_i - 2$ and $\sum_{y \in V(H_i)}(4 - \deg_{H_i}(y)) = 2l + 2$ (note that these equalities hold for $l = 1$ as well). Further $4/(t-1) = 4/5 \geq 2/3$, and $\beta(V(H_i)) \geq \sum_{1\leq j\leq j_i} |N(z_j) \cap U'|$ by the definition of β. Since $\delta(G) \geq 8$, we obtain

$$\begin{aligned}
\theta_i &= \frac{4}{5} \sum_{1\leq j\leq j_i} |N(z_j) \cap (S \cup U)| + \frac{2}{3}\beta(V(H_i)) - (2l+2) \\
&\geq \frac{4}{5} \sum_{1\leq j\leq j_i} \big|N(z_j) \cap (S \cup U)\big| + \frac{2}{3} \sum_{1\leq j\leq j_i} |N(z_j) \cap U'| - (2l+2) \\
&\geq \frac{2}{3} \sum_{1\leq j\leq j_i} \big|N(z_j) \cap (S \cup U \cup U')\big| - (2l+2) \\
&= \frac{2}{3} \sum_{1\leq j\leq j_i} \Big(\deg_G(z_j) - \deg_{H_i}(z_j)\Big) - (2l+2) \\
&\geq \frac{2}{3}(8j_i - 2j_i + 2) - (2l+2) = \frac{4}{3} > 0.
\end{aligned}$$

Subcase 2.2. H_i is a path of order l, where l is even and $l \geq 4$, or H_i is a cycle of order l, where $l = 4$ or $l \geq 6$.

Let z_j be as in (D2-2). Set

$$p_i = \frac{32}{5} j_i - \frac{4}{5} \sum_{1\leq j\leq j_i} \deg_{H_i}(z_j) - \sum_{y\in V(H_i)} (4 - \deg_{H_i}(y)).$$

By (D2-2), $\beta(V(H_i)) \geq 2\sum_{1\leq j\leq j_i} \beta(z_j) \geq 2\sum_{1\leq j\leq j_i} |N(z_j) \cap U'|$. Hence

$$\begin{aligned}
\theta_i &= \frac{4}{5} \sum_{1\leq j\leq j_i} |N(z_j) \cap (S \cup U)| + \frac{2}{3}\beta(V(H_i)) - \sum_{y\in V(H_i)} (4 - \deg_{H_i}(y)) \\
&\geq \frac{4}{5} \sum_{1\leq j\leq j_i} \big|N(z_j) \cap (S \cup U)\big| + \frac{4}{3} \sum_{1\leq j\leq j_i} |N(z_j) \cap U'| - \sum_{y\in V(H_i)} (4 - \deg_{H_i}(y)) \\
&\geq \frac{4}{5} \sum_{1\leq j\leq j_i} \big|N(T_{i,j}) \cap (S \cup U \cup U')\big| - \sum_{y\in V(H_i)} (4 - \deg_{H_i}(y)) \\
&= \frac{4}{5} \sum_{1\leq j\leq j_i} (\deg_G(z_j) - \deg_{H_i}(z_j)) - \sum_{y\in V(H_i)} (4 - \deg_{H_i}(y)) \\
&\geq \frac{4}{5} \cdot 8 \cdot j_i - \frac{4}{5} \sum_{1\leq j\leq j_i} \deg_{H_i}(z_j) - \sum_{y\in V(H_i)} (4 - \deg_{H_i}(y)) \\
&= p_i.
\end{aligned}$$

Therefore it suffices to show that $p_i \geq 0$.

First assume that H_i is a path of order l, where l is even and $l \geq 4$. Then $j_i = l/2$, $\sum_{1\leq j\leq j_i} \deg_{H_i}(z_j) = 2j_i - 1$, and $\sum_{y\in V(H_i)}(4 - \deg_{H_i}(y)) = 2l + 2$. Hence

$$p_i = \frac{2}{5}l - \frac{6}{5} > 0.$$

Next assume that H_i is a cycle of order l, where l is even and $l \geq 4$. Then $j_i = l/2$, $\sum_{1 \leq j \leq j_i} \deg_{H_i}(z_j) = 2j_i$, and $\sum_{y \in V(H_i)}(4 - \deg_{H_i}(y)) = 2l$. Hence

$$p_i = \frac{2}{5}l > 0.$$

Now assume that H_i is a cycle of order l, where l is odd and $l \geq 7$. Then $j_i = (l-1)/2$, $\sum_{1 \leq j \leq j_i} \deg_{H_i}(z_j) = 2j_i$, and $\sum_{y \in V(H_i)}(4 - \deg_{H_i}(y)) = 2l$. Hence

$$p_i = \frac{2}{5}l - \frac{12}{5} > 0.$$

Subcase 2.3. H_i is a path of order 2.

Note that $\sum_{y \in V(H_i)}(4 - \deg_{H_i}(y)) = 6$. By (D2-3), $j_i = 1$ and $T_{i,1} = V(H_i)$. First assume that $\beta(V(H_i)) = 0$. Then by the definition of $\beta(V(H_i))$ and Claim 3.4, $|N(V(H_i)) \cap (S \cup U)| = |N(V(H_i)) \cap (S \cup U \cup U')| \geq 8$. Hence

$$\theta_i = \frac{4}{5}|N(V(H_i)) \cup (S \cap U)| - 6 \geq \frac{4}{5} \cdot 8 - 6 > 0.$$

Next assume that $\beta(V(H_i)) \geq 1$. Write $V(H_i) = \{z, z'\}$ with $\beta(z) \leq \beta(z')$. Then $\beta(z') \geq 1$. Hence

$$\begin{aligned}
\theta_i &= \frac{4}{5}|N(\{z,z'\}) \cap (S \cup U)| + \frac{2}{3}\beta(z) + \frac{2}{3}\beta(z') - 6 \\
&\geq \frac{4}{5}\big|N(z) \cap (S \cup U)\big| + \frac{2}{3}\beta(z) + \frac{2}{3}\beta(z') - 6 \\
&= \frac{4}{5}\big|N(z) \cap (S \cup U \cup U')\big| - \frac{2}{15}\beta(z) + \frac{2}{3}\beta(z') - 6 \\
&\geq \frac{4}{5}\big|N(z_1) \cap (S \cup U \cup U')\big| + \frac{8}{15}\beta(z') - 6 \\
&\geq \frac{4}{5}(\deg_G(z_1) - \deg_{H_i}(z_1)) + \frac{8}{15} - 6 \\
&\geq \frac{4}{5}(8-1) + \frac{8}{15} - 6 > 0.
\end{aligned}$$

Subcase 2.4. H_i is a cycle of order 3.

Note that $\sum_{y \in V(H_i)}(4 - \deg_{H_i}(y)) = 6$. By (D2-3), $j_i = 1$ and $T_{i,1} = V(H_i)$. If $\beta(V(H_i)) = 0$, then arguing as in the first paragraph of Subcase 2.3, we get $\theta_i > 0$. Thus we may assume that $\beta(V(H_i)) \geq 1$. Write $V(H_i) = \{z, z', z''\}$ with $\beta(z) \leq \beta(z') \leq \beta(z'')$. First assume that $\beta(z') = 0$. Then $\beta(z) = 0$ and $\beta(z'') \geq 1$. By the definition of β and Claim 3.3, $|N(\{z,z'\}) \cap (S \cup U))| \geq |N(\{z,z'\}) \cap (S \cup U \cup U')| \geq 7$. Hence

$$\begin{aligned}
\theta_i &= \frac{4}{5}|N(\{z,z',z''\}) \cap (S \cup U)| + \frac{2}{3}\beta(z'') - 6 \\
&\geq \frac{4}{5}\big|N(\{z,z'\}) \cap (S \cup U)\big| + \frac{2}{3}\beta(z'') - 6 \\
&\geq \frac{4}{5} \cdot 7 + \frac{2}{3} - 6 > 0.
\end{aligned}$$

Next assume that $\beta(z') \geq 1$. Then $\beta(z'') \geq 1$. Hence

$$\begin{aligned}
\theta_i &= \frac{4}{5}|N(\{z, z', z''\}) \cap (S \cup U)| + \frac{2}{3}\beta(z) + \frac{2}{3}\beta(z') + \frac{2}{3}\beta(z'') - 6 \\
&\geq \frac{4}{5}\big|N(z) \cap (S \cup U)\big| + \frac{2}{3}\beta(z) + \frac{2}{3}\beta(z') + \frac{2}{3}\beta(z'') - 6 \\
&= \frac{4}{5}\big|N(z) \cap (S \cup U \cup U')\big| - \frac{2}{15}\beta(z) + \frac{2}{3}\beta(z') + \frac{2}{3}\beta(z'') - 6 \\
&\geq \frac{4}{5}\big|N(z) \cap (S \cup U \cup U')\big| + \frac{8}{15}\beta(z') + \frac{2}{3}\beta(z'') - 6 \\
&\geq \frac{4}{5}(\deg_G(z) - \deg_{H_i}(z)) + \frac{8}{15} + \frac{2}{3} - 6 \\
&\geq \frac{4}{5}(8-2) + \frac{8}{15} + \frac{2}{3} - 6 \\
&= 0.
\end{aligned}$$

Subcase 2.5. H_i is a cycle of order 5.

Note that $\sum_{y \in V(H_i)}(4 - \deg_{H_i}(y)) = 10$. By (D2-4), $j_i = 2$. First assume that $\beta(V(H_i)) = 0$. By (D2-4), $\big|N(T_{i,1}) \cap (S \cup U)\big| \geq 7$, and we also have $\big|N(T_{i,2}) \cap (S \cup U)\big| \geq \delta(G) - 2 \geq 8 - 2 = 6$. Hence

$$\begin{aligned}
\theta_i &= \frac{4}{5}\Big(\big|N(T_{i,1}) \cap (S \cup U)\big| + \big|N(T_{i,2}) \cap (S \cup U)\big|\Big) - 10 \\
&\geq \frac{4}{5}(7+6) - 10 \ > \ 0.
\end{aligned}$$

Next assume that $\beta(V(H_i)) \geq 1$. Let z_1, z_2 be as in (D2-4). By (D2-4), $\beta(V(H_i) - \{z_1, z_2\}) \geq \beta(z_1) + \beta(z_2)$ and $\beta(V(H_i) - \{z_1, z_2\}) \geq 1$. Therefore we obtain

$$\begin{aligned}
\theta_i &= \frac{4}{5}\big|N(z_1) \cap (S \cup U)\big| + \frac{4}{5}\big|N(z_2) \cap (S \cup U)\big| \\
&\qquad + \frac{2}{3}\beta(z_1) + \frac{2}{3}\beta(z_2) + \frac{2}{3}(\beta(V(H_i) - \{z_1, z_2\}) - 10 \\
&= \frac{4}{5}\big|N(z_1) \cap (S \cup U \cup U')\big| + \frac{4}{5}\big|N(z_2) \cap (S \cup U \cup U')\big| \\
&\qquad - \frac{2}{15}\beta(z_1) - \frac{2}{15}\beta(z_2) + \frac{2}{3}\beta(V(H_i) - \{z_1, z_2\}) - 10 \\
&\geq \frac{4}{5}\big|N(z_1) \cap (S \cup U \cup U')\big| + \frac{4}{5}\big|N(z_2) \cap (S \cup U \cup U')\big| \\
&\qquad + \frac{8}{15}\beta(V(H_i) - \{z_1, z_2\}) - 10 \\
&\geq \frac{4}{5}(\deg_G(z_1) - \deg_{H_i}(z_1)) + \frac{4}{5}(\deg_G(z_2) - \deg_{H_i}(z_2)) + \frac{8}{15} - 10 \\
&\geq \frac{4}{5} \cdot 6 + \frac{4}{5} \cdot 6 + \frac{8}{15} - 10 \ > \ 0.
\end{aligned}$$

Now the proof for the case where $t = 6$ is completed. This completes the proof of Theorem 4.

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