AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **55** (2013), Pages 263–272

On mirror nodes in graphs without long induced paths

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Abstract

The notion of mirror nodes, first introduced by Fomin, Grandoni and Kratsch in 2006, turned out to be a useful item for their design of an algorithm for the maximum independent set problem (MIS). Given two nodes u, v of a graph $G = (V, E)$ with $\mathrm{dist}_G(u, v) = 2$, u is called a mirror of v if $N(v) \setminus N(u)$ induces a (possibly empty) clique. In order to have a well defined term, we add that every complete graph has a mirror node.

In general there might be no mirror nodes in arbitrary graphs, e.g. there exist graphs with no induced paths of length six without any mirror node. But in contrast every P_4-free graph contains a mirror node. Therefore we are interested in the existence of mirror nodes in P_5-free graphs. This class is well-studied in the context of MIS. For several subfamilies of P_5-free graphs with an additional forbidden induced subgraph, we demonstrate that mirror nodes always occur.

Whether there always exists a mirror node in P_5-free graphs is still open, even in (P_5, C_5)-free and $(P_5, \overline{P}_5)$-free graphs.

1 Introduction

The notion of mirror nodes was first introduced by Fomin, Grandoni and Kratsch, in their algorithm for the maximum independent set problem (MIS) in 2006 [3]. They used the *Measure and Conquer* approach and demonstrated, that a good choice of the measure can have a huge effect on the running time. They took advantage of the property that at each branching step the following holds: when you discard a node v, meaning that you do not take it into your maximum independent set, you

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can also discard its mirrors without modifying the maximum independent set size. So the existence of mirror nodes decreases the size of the problem. Therefore we are interested to know if all graphs do have mirror nodes.

The answer is positive for $P_k(k \leq 4)$-free graphs but negative for $P_k(k \geq 6)$-free graphs. This leads to the class of P_5-free graphs. The class of P_5-free graphs is of special interest since it is the only family of graphs defined by a single connected forbidden induced subgraph where the complexity status of MIS is still unknown. So far the best known running time for MIS for P_5-free graphs is a subexponential time algorithm by Randerath and Schiermeyer [8]. We will be able to verify the existence of mirror nodes in several subfamilies of P_5-free graphs with an additional forbidden induced subgraph. The question of mirror nodes in P_5-free graphs is still open, even for subclasses like (P_5, C_5)-free or $(P_5, \overline{P}_5)$-free graphs.

2 Preliminaries

Let $G = (V, E)$ be a simple, undirected, finite and connected graph with V the set of nodes ($|V| = n$) and E the set of edges ($|E| = m$). We call $N^d(v)$ the set of nodes having distance d to node v. $N^1(v) = N(v)$ induces the *neighborhood* of v and $N[v] = N(v) \cup \{v\}$ the *closed neighborhood*. We indicate the *degree* of v by $d(v) = |N(v)|$. Let P_k denote a path on k nodes with length $k-1$. A graph is called P_k-free if it does not have a P_k as an induced subgraph.

For a given graph G a node u is called a *mirror node* of v, if $dist_G(u, v) = 2$ and $N(v) \setminus N(u)$ induces a (possibly empty) clique. In terms of the well-definition, we say that there exists a mirror node in G, if G is a clique. By $N(v) \setminus N(u)$ we denote the *private neighborhood* of v with respect to u.

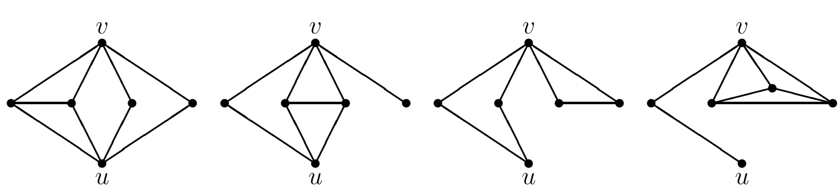

Figure 1: Examples of u being a mirror node of v.

Suppose we are looking at a graph G without any mirror nodes. Given two nodes $1, 2 \in V(G)$ of $dist_G = 2$, the sets $N(1) \setminus N(2)$ and $N(2) \setminus N(1)$ are not allowed to induce a (possibly empty) clique. This means nodes 1 and 2 each need at least two private, mutually non adjacent neighbors. Let the nodes 11, 12 be the private, mutually non adjacent neighbors of node 1, and 21, 22 the ones of node 2. This basic graph will be called G^*, see Figure 2, and will be important in our forthcoming proofs. Note that graph G might contain edges between some vertices of G^*.

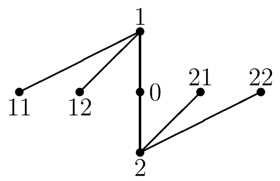

Figure 2: Basic graph G^*. For two nodes $1, 2$ of $dist = 2$ to not be mirror nodes the private neighborhood is not allowed to induce a clique.

3 $P_k(k \leq 4)$-free graphs

In this section we prove that P_k-free graphs with $k \leq 4$ do have mirror nodes. Note that every connected P_3-free graph is complete and therefore has a mirror node by definition.

Observation 3.1. *Every $P_k(k \leq 3)$-free graph G has at least one mirror node.*

So all we need to look at is the following theorem:

Theorem 3.2. *Every P_4-free graph G has at least one mirror node.*

Proof. We prove the result inductively on the order n of a connected P_4-free graph. The result is obviously true for P_4-free graphs of order at most 4. Let $n \geq 5$. The proof heavily relies on a structural characterisation of P_4-free graphs due to Seinesche [9]: Let G be P_4-free, then either G or its complement $\overline{G}$ is not connected.

Let $G = (V, E)$ be a connected P_4-free graph, thus $\overline{G}$ is not connected. Hence there must exist a partition of G into G_1, G_2, such that G contains all possible edges between G_1, G_2 and $|G_i| < n$, $i = 1, 2$. By induction over n we can assume that there already exist mirror nodes in partitions G_1 and G_2. Say G_1 contains at most as many components as G_2. We have to distinguish two cases:

- G_1 **has no nodes of** $dist_{G_1} = 2$. Then G_1 consists of disjoint cliques. Adding G_2 and all possible edges between G_1 and G_2 causes two nodes of two different cliques in G_1 to be mirror nodes in G. If there do not exist two different cliques in G_1, then G_1 is a clique and G contains a mirror node.

- G_1 **has nodes of** $dist_{G_1} = 2$. According to our assumption there exists a subgraph with at least one mirror. Analogously, adding G_2 and all possible edges between G_1 and G_2 does not change a mirror node's properties since all nodes of $dist_{G_1} = 2$ gain the same shared new neighbors in G.

□

4 $P_k (k \geq 6)$-free graphs

In contrast to the P_4-free graphs which always have a mirror node, there are graphs without long induced paths which do not have any mirror nodes.

Suppose a graph G does not have any mirror nodes. Hence for each two nodes u, v of $dist(u, v) = 2$ neither $N(v) \setminus N(u)$ nor $N(u) \setminus N(v)$ is allowed to induce a clique. The P_6-free Petersen graph is an example of a graph without any mirror nodes.

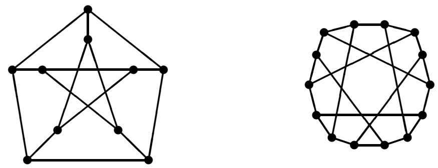

Figure 3: Examples of graphs without any mirror nodes, Petersen graph (left), Heawood graph (right).

The generalized Petersen graph $GP(n, k)$ has nodes $u_1, \ldots, u_n, v_1, \ldots, v_n$ and edges (u_i, u_{i+1}), (u_i, v_i) and (v_i, v_{i+k}) with subscripts modulo n, $n \geq 3$, $1 \leq k \leq \lfloor (n-1)/2 \rfloor$. The graph induced by the u_i's is called outer regular polygon and that induced by the v_i's is called inner star polygon.

Observation 4.1. *The generalized Petersen graph $GP(n, k)$, with $GP(n, k)$ being triangle- and C_4-free, does not have any mirror nodes.*

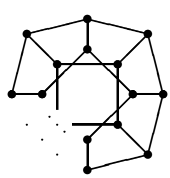

Figure 4: Generalized Petersen graph $GP(n, k)$, for $n \geq 9, k = 2$.

For $k = 2$, each node of the inner polygon is adjacent to the following second node in the inner polygon. In this notation the Petersen graph is $G(5, 2)$. In triangle-free GPs the private neighborhood cannot induce a clique in the inner polygon and in C_4-free GPs no two nodes of $dist = 2$ have two shared neighbors. Thus no mirror nodes are possible.

Another example would be the P_8-free Heawood graph. So far all mentioned graphs without any mirror nodes are *3-regular*, have an *independent private neighborhood* and have *no induced path* with length at least five.

5 P_5-free graphs

The $P_k(k \leq 4)$-free graphs provide a family of graphs which always have mirror nodes. In contrast there are $P_k(k \geq 6)$-free graphs which do not always have a mirror node. So the class of P_5-free graphs is still missing. We prove that several subfamilies of P_5-free graphs with an additional forbidden induced subgraph have mirror nodes.

Figure 5: The forbidden induced subgraphs B of Theorem 5.2.

Theorem 5.1. *$(P_5, claw)$-free graphs have at least one mirror node.*

Proof. Suppose there exists a $(P_5, claw)$-free graph G without any mirror nodes. In order not to have any mirrors G must contain the basic subgraph G^*, see Figure 2. By assumption nodes $11, 12$ are not adjacent. Without loss of generality we enforce the edge $(0, 12)$ in order to avoid a claw with center node 1. The same argument can be applied to enforce the edge $(0, 21)$ with center node 2. The *claw*-freeness also guarantees the anti-edges $(0, 11)$ and $(0, 22)$ with center node 0 because no neighborhood contains a stable set of size three. Therefore we are facing an induced P_5: $11 - 1 - 0 - 2 - 22$, a contradiction. Note that $(11, 22)$ is an anti-edge as well since a further P_5 respectively a claw would occur. ☐

Theorem 5.2. *Let $B \in \{claw, paw, diamond, C_4\}$. Every (P_5, B)-free graph has at least one mirror node.*

Proof. Let G be a (P_5, B)-free graph. If $B = claw$, then we are done by Theorem 5.1.

CASE: $B = paw$

P_5-free, bipartite graphs have at least one mirror node.

A connected graph G is P_5-free and bipartite, if and only if G is $2K_2$-free and bipartite. Additionally, a graph G is called a *chain graph*, if it is $2K_2$-free and bipartite. Let $G = (V, E)$ with the partitions X and Y be a chain graph. Then G has the property that the neighborhood of the nodes in partition X form a chain, i.e., there is an ordering of the nodes of X say $[x_1, \ldots, x_p], p \geq 2$, such that $N(x_1) \supseteq N(x_2) \supseteq \ldots \supseteq N(x_p)$. All nodes in partition X have $dist_G = 2$, so the node x_1 is a mirror node by definition. If $p = 1$ and $|Y| \geq 2$ every node in partition Y is a mirror node. In case of $|X| = |Y| = 1$ graph G is a clique and therefore has a mirror node.

(P_5, K_3)-free graphs have at least one mirror node.

In [7] it was shown, that a connected triangle-free, P_5-free graph is bipartite, or it is maximal triangle-free and homomorphic with C_5, where a graph is maximal triangle-free, if joining two non-adjacent nodes creates a triangle.

We are looking at non-bipartite (P_5, K_3)-free graphs at this time so they have to be homomorphic with C_5. If the graph G in consideration is a C_5, then G obviously contains a mirror node. Otherwise we obtain the graph G by replacing the C_5 nodes' with independent sets. Every two nodes in those sets have $dist_G = 2$ and the same neighborhood. Thus they are mirror nodes to each other.

(P_5, paw)-free graphs have at least one mirror node.

By a result of Olariu [6] a graph G is paw-free if and only if G is triangle-free or complete multipartite. We have already proved the existence of mirror nodes in (P_5, K_3)-free graphs, so the complete multipartite graphs remain. Those obviously contain mirror nodes since every two nodes in a partition have $dist_G = 2$ and the same neighborhood or the graph in consideration forms a clique.

CASE: $B = diamond$

The proof relies on a structural result by Arbib and Mosca who presented a construction of $(P_5, diamond)$-free graphs with an induced C_5 by methods of multiplying and substituting nodes. They used the following expressions. Let $\{u_0, u_1, u_2, u_3, u_4\}$ be the induced C_5 and let D_l denote the set of nodes at distance l from this C_5. In addition let $v \in V(G)$ and k be a positive integer: G' is obtained by *multiplying v by k* if $V(G') = V(G) \cup \{v_1, \ldots, v_{k-1}\}$, where the set $\{v, v_1, \ldots, v_{k-1}\}$ is independent, and $N(v_i) = N(v)$ for $1 \leq i < k$. Let F be a graph: G' is obtained by *substituting F for v* if $V(G') = V(G) \cup V(F) - \{v\}$, and all nodes of $N(v)$ are adjacent to all nodes of F.

Characterisation of Arbib and Mosca [1]*: Any connected $(P_5, diamond)$-free graph that properly contains an induced C_5 can be obtained from C_5 or from a graph among a set M of nine exceptional graphs of order at most 9 by multiplying some $v \in D_1 \cup C_5$ and/or substituting a P_3-free graph for some $v \in D_2$.*

Using Arbib and Mosca's characterisation, we are able to prove the existence of mirror nodes in (P_5, *diamond*)-free graphs with an induced C_5. By inspection, all members of M or C_5 contain a mirror node, see [1]. Note that this mirror node can be found among the nodes of C_5. So let the considered graph G not be a member of M or C_5.

Multiplying a node $v \in D_1 \cup C_5$ causes mirror nodes since all nodes in $\{v, v_1, \ldots, v_{k-1}\}$ have distance two and the same neighborhood.

Substituting a node $v \in D_2$ does not affect already existing mirror nodes in C_5 since it takes place in the set D_2.

Multiplying a node $v \in D_1 \cup C_5$ and substituting another node $v' \in D_2$ afterwards causes mirror nodes. Due to the multiplication mirror nodes occur in either the set D_1 or C_5. Those in C_5 will remain mirror nodes after a substitution in D_2. Those in the set D_1 will not be destroyed either since all nodes receive the same new neighbors caused by the substitution.

Still remaining is the case of (P_5, *diamond*)-free graphs without induced C_5. Assume G is a (P_5, C_5, *diamond*)-free graph without any mirror nodes. Recalling our basic subgraph G^*, see Figure 2, one can see that nodes 11 and 12 cannot be adjacent to the center node 0 at the same time since G is *diamond*-free. The same holds for the nodes 21,22. Thus the (C_5, *diamond*)-freeness gives reasons for the anti-edges (11,0),(22,0),(11,22) and an induced P_5:$11-1-0-2-22$ is achieved, a contradiction.

CASE: $B = C_4$

The following proof is based on a structural theorem by Fouquet et al. [4] which provides complete, homogeneous buoys and chordal subgraphs. It uses the fact that mirror nodes exist in chordal graphs due to the existence of a simplicial node.

First of all we are in need of further definitions. A graph G is called *chordal* or *triangulated* if every cycle of length at least 4 contains a chord. A node v is *simplicial* whenever its neighborhood is complete. A subset $S \subset V(G)$ satisfying $2 \leq |S| \leq |V(G)| - 1$ is *homogeneous* in G if every node of $V(G) - S$ is adjacent to either all nodes in S or none of them. In other words: for each two nodes $x, y \in S : N(x) \setminus S = N(y) \setminus S$. An induced subgraph of a $(P_5, \overline{P}_5)$-free graph G is a *buoy* whenever you can find a partition of its nodes into 5 subsets $A_i, i = 0, \dots, 4$, (i to be taken modulo 5), such that A_i and A_{i+1} are joined by every possibly edge, while no edges are allowed between A_i and A_j when $|j - i| \neq 1, 4$, and such that the $A_i's$ are maximal for these properties. A buoy is *complete*, if each $A_i, 0 \leq i \leq 4$, is complete. Since the class of (P_5, C_4)-free graphs is a subset of $(P_5, \overline{P}_5)$-free graphs it also applies to (P_5, C_4)-free graphs.

Hence the existence of a complete, homogeneous buoy in a graph G causes at least two mirror nodes, namely $a_i \in A_i$ and $a_{i+2} \in A_{i+2}$. These nodes have the same neighborhood outside the buoy and their private neighborhood induces a clique since the buoy is complete.

In addition we are able to prove mirror nodes in chordal graphs with the help of Dirac's conclusion saying that every chordal graph G has a simplicial node [2]. Thus G can either be complete and have a mirror node by definition, or G has a simplicial node p and a node u of $dist(p, u) = 2$ being a mirror node.

In order to complete the proof we use the following theorem.

Characterisation of Fouquet et al. *[4]: A connected graph G is a (P_5, C_4)-free graph if and only if there exists a partition $V_1 \cup V_2$ with the following property*

- *V_1 induces a triangulated P_5-free subgraph of G.*
- *If V_2 is not empty then it can be partitioned into distinct complete buoys, each of them being a homogeneous part of G whose neighborhood is a complete subgraph of G, contained in V_1. Moreover there is a complete subgraph of G, contained in V_1, whose open neighborhood contains each buoy of G.*

So we either have a chordal graph G or a homogenous, complete buoy in G and therefore, as seen before, there exist mirror nodes in (P_5, C_4)-free graphs. This completes the proof of Theorem 5.2. □

Proposition 5.3. *$(P_5, \overline{P}_5, C_5)$-free graphs have at least one mirror node.*

We prove an even stronger result, by saying that Hole-, House- and Domino-free graphs, (H, H, D)-free graphs, have at least one mirror node. An (H, H, D)-free graph does not contain holes (cycles of length at least five), a house or a domino as induced subgraphs. Since the class of (H, H, D)-free graphs is a superclass of $(P_5, \overline{P}_5, C_5)$-free graphs, the proposition holds as well. The proof relies on a structural result of Hoang and Khouzam [5], who guarantee a (H, H, D)-free graph G to

- be a clique, or
- have at least two non-adjacent simplicial nodes, or
- have a homogeneous set S.

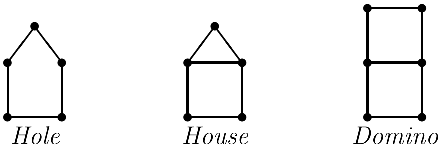

Figure 6: The graphs *Hole*, *House* and *Domino*.

Mirror nodes have already been proved in the first two cases. So the case of a homogeneous set is the only one being left over. Let $G = (V, E)$ be a (H, H, D)-free graph with $|V| \leq 4$, then G has at least one mirror node. Assume all (H, H, D)-free graphs G with $|V| < n$ and having a homogeneous SET S contain a mirror node. Induction over n will bring the desired result.

Let $S \subset V$ be the homogeous set with $|S| \geq 2$. Let G_S be the subgraph induced by S. G_S has to have a mirror node since $|S| < n$. If a mirror node exists in G_S, it likewise exists in G because all nodes in S have the same neighborhood in $V \setminus S$ and the private neighborhood is not touched.

If a trivial mirror node exists, G_S can either be a clique or consists of several disjoint cliques. In the latter case adding $V \setminus S$ causes two nodes of two cliques to be mirrors. In contrast suppose G_S is a complete graph. All nodes in G_S have the same neighbors in $V \setminus S$ since S is a homogeneous set. The private neighborhood properties will not be changed in G by melting S into one single node s'. The occurring graph $G' = (V', E')$ fulfills $|V'| < n$ and must therefore contain a mirror node. Remelting of s' cannot destroy existing mirror nodes since only a clique is added.
So we are able to formulate our theorem:

Theorem 5.4. *(H, H, D)-free graphs have at least one mirror node.*

6 Final Remarks

In this paper we proved the existence of mirror nodes in several graph classes. With the class of $P_k(k \leq 4)$-free graphs we found a class of graphs in which there always exists a mirror node. On the other hand the class of $P_k(k \geq 6)$-free graphs provides examples of graphs which do not have a mirror node. We do assume that graphs without any mirror nodes are at least P_6-free and have minimum degree three in order to avoid an induced clique in the private neighborhood.

Our aim was to fill the gap with the P_5-free graphs. Several subfamilies of P_5-free graphs with one or two additional, forbidden, induced subgraphs do have mirror nodes. Nevertheless the question is still open even for (P_5, C_5)-free, $(P_5, \overline{P}_5)$-free graphs or (P_5, K_4)-free graphs. So far our proofs have been based on structural results. That is why we assume that one might be able to answer our big conjecture that *every P_5-free graph has a mirror node* by means of an ingenious decomposition theorem for the class of P_5-free graphs.

Acknowledgment

We thank the unknown referees for their careful reading and very helpful comments and suggestions.

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(Received 29 Feb 2012; revised 9 Oct 2012)