AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **55** (2013), Pages 205–233

Group developed weighing matrices*

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Abstract

A weighing matrix is a square matrix whose entries are $1, 0$ or -1, such that the matrix times its transpose is some integer multiple of the identity matrix. We examine the case where these matrices are said to be developed by an abelian group. Through a combination of extending previous results and by giving explicit constructions we will answer the question of existence for 318 such matrices of order and weight both below 100. At the end, we are left with 98 open cases out of a possible 1,022. Further, some of the new results provide insight into the existence of matrices with larger weights and orders.

1 Introduction

1.1 Group Developed Weighing Matrices

A weighing matrix $W = W(n, k)$ is a square matrix, of order n, whose entries are in the set $w_{i,j} \in \{-1, 0, +1\}$. This matrix satisfies $WW^t = kI_n$, where t denotes the matrix transpose, k is a positive integer known as the weight, and I_n is the identity matrix of size n.

Definition 1.1. Let G be a group of order n. An $n \times n$ matrix $A = (a_{gh})$ indexed by the elements of the group G (such that g and h belong to G) is said to be G-developed if it satisfies the condition

$$a_{gh} = a_{g+k,h+k} \quad \text{for all} \quad g, h, k \in G.$$

* Research partially supported by grants from the AFSOR and NSF.

The matrix A is said to be circulant if the underlying group G is cyclic. Clearly, a group developed matrix A is completely determined by its first row. Throughout this paper, we will only deal with abelian groups G.

To see how a matrix W is developed, we label the first row of the matrix by the elements in G. If $G = \{g_0, g_1, g_2 \ldots, g_{n-1}\}$, then we can define

$$P = \{g_i \mid w(1,i) = +1\}\,;\; N = \{g_i \mid w(1,i) = -1\} \tag{1}$$

From this definition we can see that $|P| + |N| = k$ and given only the first row of W, this allows for the remaining elements to be developed by

$$w_{i,j} = \begin{cases} +1 & \text{if} \quad g_i g_j^{-1} \in P \\ -1 & \text{if} \quad g_i g_j^{-1} \in N \\ 0 & \quad\text{otherwise} \end{cases}.$$

For any G-developed $W(n,k)$ the following are true:

1. $k = s^2$ for some positive integer s,
2. $\{|P|\,, |N|\} = \left\{\frac{s^2+s}{2}, \frac{s^2-s}{2}\right\}$.

Remark. Interchanging $+1$s and -1s in a weighing matrix does not change any of the properties of the matrix. Thus, it is only by convention that $|P|$ and $|N|$ are chosen so that $|P| > |N|$ (that is, the orders of P and N can be switched). For further information on these facts, see [22].

Definition 1.2. The support of a G-developed $W(n,k)$ is the set of elements which are non-zero, and is denoted by $\mathrm{supp}(W)$. It is clear that $\mathrm{supp}(W) = P \cup N$.

From the definition of support comes the natural idea of disjoint matrices.

Definition 1.3. Let W and X be two G-developed $W(n,k)$ matrices. Also, let

$$\begin{aligned} P_W &= \{g_i \mid W(1,i) = +1\} \\ N_W &= \{g_i \mid W(1,i) = -1\} \end{aligned}$$

and

$$\begin{aligned} P_X &= \{g_i \mid X(1,i) = +1\} \\ N_X &= \{g_i \mid X(1,i) = -1\}. \end{aligned}$$

Then we have $\mathrm{supp}(W) = P_W \cup N_W$ and $\mathrm{supp}(X) = P_X \cup N_X$. The two matrices are said to be disjoint if $\mathrm{supp}(W) \cap \mathrm{supp}(X)$ is the empty set.

A very useful notation for G-developed weighing matrices is the use of the group ring $\mathbb{Z}[G]$.

Definition 1.4. Let G be a finite group and R a ring where $G = \{g_0, g_1, g_2, \ldots, g_{n-1}\}$. Then the group ring of G over R is the set denoted by $R[G]$ defined as

$$R[G] = \left\{ \sum_{g \in G} a_g g \mid a_g \in R \right\},$$

with multiplication and addition defined similar to that of regular polynomials (i.e. addition is done term-wise and multiplication extends that of the group G using the distributive property).

Let $R[G]$ denote the group ring of a given group G over a ring R. We identify each subset S of G with the group ring element $\sum_{x \in S} x$. For $A = \sum_{g \in G} a_g g \in R[G]$ and any integer t, we define $A^{(t)} = \sum_{g \in G} a_g g^t$. Then the set of G-developed matrices with entries from R is isomorphic to the group ring $R[G]$. This isomorphism sends A^t, the transpose of A, to $A^{(-1)}$.

Any G-developed weighing matrix can be written in the group ring by using the sets P and N defined in Equation (1) and using the following formula:

$$W = \sum_{i=0}^{n-1} a_i g_i, \quad \text{where} \quad a_i = \begin{cases} 1 & \text{if } g_i \in P \\ -1 & \text{if } g_i \in N \\ 0 & \text{otherwise} \end{cases}.$$

The advantage of using group ring notation to investigate group developed matrices stems from character theory, as we shall see in Section 2.

1.2 Examples of G-developed Weighing Matrices

Definition 1.5. A circulant weighing matrix of order n and weight k, denoted by $CW(n, k)$, is a G-developed $W(n, k)$ weighing matrix where the group G is cyclic.

It should be clear that the identity matrix I_n is a $(\mathbb{Z}_n)$-developed $W(n, 1)$ and that this should be considered the trivial example of such a matrix. Here are some non-trivial examples of group weighing matrices:

Example 1. *A* $(\mathbb{Z}_7)$*-developed* $W(7, 4)$

Let W be the matrix given by

$$\begin{pmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

It can easily be checked that W is a weighing matrix with $WW^t = 4I_7$. Further, by labeling the first row with $\{1, x, x^2, x^3, x^4, x^5, x^6\}$, where $G = \mathbb{Z}_7 = \langle x \rangle$, the remaining rows may be developed. This means that $P = \{x, x^2, x^4\}$ and $N = \{1\}$. Here W may also be represented in $\mathbb{Z}[\mathbb{Z}_7]$ by $W = -1 + x + x^2 + x^4$.

Example 2. *A* $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$*-developed* $W(8, 4)$

Such a W is given by the matrix

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

Direct computation shows that $WW^t = 4I_8$. The matrix W may also be developed by labeling the first row with the elements $\{1, z, y, yz, x, xz, xy, xyz\}$ in $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle x\rangle \times \langle y\rangle \times \langle z\rangle$, then filling in the remaining rows by developing them. From this labeling, W may also be written as an element of $\mathbb{Z}[\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2]$ by $W = -1 + y + xz + xyz$. In this case, $P = \{y, xz, xyz\}$ and $N = \{1\}$.

Remark. We alert the reader that there are so-called 2-circulant weighing matrices (not 'group-developed') in the literature (e.g. see [4]).

2 Preliminaries

The study of G-developed weighing matrices is closely related to the study of difference sets. Since these are similar topics, we will give a brief explanation of the tools commonly used to study both. The tools needed here are group rings and character theory.

2.1 Group Rings and Character Theory

By definition of a G-developed $W(n, k)$, we know that $WW^t = kI_n$. So, if the matrix W can also be represented as an element of a group ring, then we must define how its transpose is to be handled. The following proposition is easy to prove.

Proposition 2.1. *Let W be a G-developed $W(n, k)$. Further, let W be written as an element in $\mathbb{Z}[G]$, as defined above in Definition 1.4. Then $WW^{(-1)} = k$, where*

$$\begin{aligned} W^{(-1)} &= \left(\sum_{i=0}^{n-1} a_i g_i\right)^{(-1)} \\ &= \sum_{i=0}^{n-1} a_i g_i^{-1} \end{aligned}$$

and g_i^{-1} denotes the usual group element inverse in G.

For more information on group rings in this context one may refer to [7, 23].

Example 3. Take the matrix W from Example 1. Written in $\mathbb{Z}[\mathbb{Z}_7]$, $W = -1 + x + x^2 + x^4$ and we see that

$$\begin{aligned} W^{(-1)} &= -1^{-1} + x^{-1} + x^{-2} + x^{-4} \\ &= -1 + x^6 + x^5 + x^3. \end{aligned}$$

Then, by direct computation,

$$\begin{aligned} WW^{(-1)} &= (-1 + x + x^2 + x^4)(-1 + x^6 + x^5 + x^3) \\ &= 1 - x - x^2 - x^3 + 3x^7 + x^8 + x^9 + x^{10} \\ &= 4, \end{aligned}$$

where $x^7 = 1$.

Definition 2.1. A homomorphism from an abelian group G to the field $\mathbb{C}$ of complex numbers is called a character of G and is denoted by χ.

Remark. The set of all characters of the abelian group G is represented by $\widehat{G}$ and it can be shown that $\widehat{G}$ is isomorphic to G. The principal character of G is defined to be the homomorphism that maps each element g of G to 1. This character is denoted χ_0. The character homomorphism can be extended linearly to the group rings defined above.

While we are restricting our topic to weighing matrices here, these tools are also used to study difference sets.

Definition 2.2. Let D be an element of $\mathbb{Z}[G]$ where D has coefficients from $\{0,1\}$. D is said to be a (v, k, λ)-difference set in G if

$$DD^{(-1)} = k - \lambda + \lambda G \ \text{ in } \ \mathbb{Z}[G].$$

There is much information in the literature about difference sets and the use of character theory and group rings. See [17, 24] for more information on such topics.

Definition 2.3. A prime p is called self-conjugate modulo an integer w if there exists a non-negative integer j with $p^j \equiv -1 (\text{mod } w')$, where w' denotes the p-free part of w.

2.2 Integer-G-developed Weighing Matrices

An interesting fact with G-developed $W(n, k)$ matrices is that the existence of one implies the existence of an integer-G-developed $W(m, k)$ matrix.

Definition 2.4. An integer-G-developed weighing matrix is a weighing matrix whose entries are not limited to $\{-1, 0, 1\}$. The entries may be any integer. Clearly, any G-developed $W(n, k)$ is also an integer-G-developed $W(n, k)$.

By taking an existing G-developed $W(n,k)$ we can create an integer-($^G/_H$)-developed $W(m,k)$ by folding.

Definition 2.5. Let $\sigma : G \longrightarrow H$ be some homomorphism between the groups G and H. Then σ can be extended linearly as a ring homomorphism from $\mathbb{Z}[G] \longrightarrow \mathbb{Z}[H]$. In particular, if $\sigma : G \longrightarrow {}^G/_H$ is the canonical homomorphism between groups, then we refer to the linearly extended form of σ between the rings $\mathbb{Z}[G] \longrightarrow \mathbb{Z}[{}^G/_H]$ as folding.

The following theorem follows along the same lines of its 'cyclic group' analogue, as given in [7] and [23].

Theorem 2.1. *Let W be a G-developed $W(n,k)$ and let H be a subgroup of G. Define $\sigma : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[{}^G/_H]$ as the canonical homomorphism between group rings (i.e. folding). Then W^σ is a (${}^G/_H$)-developed $W(|{}^G/_H|, k)$. Further, the coefficients of W^σ are confined by $[-|H|, |H|]$. Thus, this means that W^σ is an integer-(${}^G/_H$)-developed $W(|{}^G/_H|, k)$.*

2.3 Hadamard Matrices

There is much known about a special type of weighing matrix known as a Hadamard matrix. These are weighing matrices in which the weight and order are the same. That is to say, there are only non-zero entries in the matrix.

Definition 2.6. A Hadamard matrix is a weighing matrix which has all non-zero entries, i.e., a $W(n,k)$, where $n = k$.

The following theorem is well-known; (e.g., see [16]).

Theorem 2.2. *If a Hadamard matrix exists, then the order of the matrix must be* $1, 2$*, or a multiple of* 4.

Remark. This fact eliminates all G-developed $W(k,k)$ where $|G| = k$ and k is not a multiple of 4 (i.e. all groups of order 9, 25, 49, or 81 may not contain Hadamard matrices).

Conjecture 1. *(Hadamard) A Hadamard matrix exists if and only if n is a multiple of* 4.

Remark. This conjecture has been verified for orders up to $n = 668$, as mentioned in [14]. There are only 13 integers for which a Hadamard matrix is not known for orders $4m$ and $m < 500$. These values are $m = 167, 179, 223, 251, 283, 311, 347, 359, 419, 443, 479, 487$, and 491.

An interesting fact about Hadamard matrices is that there is only one non-trivial circulant Hadamard matrix known to exist. This matrix is the ($\mathbb{Z}_4$)-developed $W(4,4)$ and is given by

$$\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix},$$

or equivalently written as $W = -1 + x + x^2 + x^3$ in $\mathbb{Z}[\mathbb{Z}_4]$.

Conjecture 2. *(Hadamard) A circulant, non-trivial, Hadamard matrix exists if and only if $n = k = 4$.*

Remark. While this conjecture is so far unproven, it has been verified for orders up to 4×10^{26} with fewer than 1,600 exceptions [21].

Theorem 2.3. *(McFarland [20]) If a Hadamard matrix of order $4p^2$ exists, then p is prime and $p = 2$ or $p = 3$.*

3 Some Previously Known Results

3.1 Characterizations of cyclic-G-developed $W(n, k)$ of weights 4, 9 and 16

For weights 4, 9 and 16, the set of G-developed weighing matrices is fully classified if the group is cyclic. The following theorems illustrate this fact.

Theorem 3.1. *A $CW(n, 4)$ exists if and only if $n \geq 4$ is even or a multiple of 7.*

See [15] for more information.

Theorem 3.2. *A $CW(n, 9)$ exists if and only if n is a multiple of 13 or 24.*

See [2] for more information.

Theorem 3.3. *A $CW(n, 16)$ exists if and only if $n \geq 21$ is a multiple of 14, 21 or 31.*

See [6] for details.

3.2 Embedding and Construction Theorems

This next result comes from work by Arasu, Dillon, Jungnickel and Pott [3]. This produces an infinite family of circulant weighing matrices.

Theorem 3.4. *For each prime power q and positive integer d, there exists a $CW(\frac{q^{2d+1}-1}{q-1}, q^{2d})$.*

The following are other useful constructions for group weighing matrices.

Theorem 3.5. *If there exists both a G_1-developed $W(n_1, k_1)$ and a G_2-developed $W(n_2, k_2)$, then there exists a $(G_1 \times G_2)$-developed $W(n_1 n_2, k_1 k_2)$.*

Theorem 3.6. *If there exists a G-developed $W(n, k)$, then there exists an H-developed $W(m, k)$ for all groups H containing a subgroup isomorphic to G.*

This can also be seen in [8].

Theorem 3.7. *(Arasu and Dillon [8]) Let H be an abelian group and let $D_i \in \mathbb{Z}[H]$ for $i = 0, 1, 2, \ldots, (n-1)$. Assume that these three conditions are satisfied:*

1. *The coefficients of each* D_i *are in* $\{-1, 0, 1\}$,
2. $\sum_{i=0}^{n-1} D_i D_i^{(-1)} = n|H|$,
3. $D_i D_j^{(-1)} = 0$ *for all* $i \neq j$.

Further, let G *be an abelian group containing* H *as a subgroup of index* $l > n$. *Then there exists a* G*-developed* $W(|G|, n|H|)$.

Theorem 3.8. *(Patching [4]) Let* M *and* N *be two disjoint* G*-developed* $W(n,k)$ *matrices as in Definition 1.3. Then* $(1+t)M + (1-t)N$ *is a* $(\mathbb{Z}_2 \times G)$*-developed* $W(2n, 4k)$, *where* $\mathbb{Z}_2 = \langle t \rangle$.

Proof. Let $W = (1+t)M + (1-t)N$ in $\mathbb{Z}[\mathbb{Z}_2 \times G]$. Then by direct calculation and noting that $(1+t)(1-t) = 1 - t^2 = 0$, we see that

$$\begin{aligned} WW^{(-1)} &= [(1+t)M + (1-t)N)][(1+t)M + (1-t)N]^{(-1)} \\ &= [(1+t)M + (1-t)N][(1+t)M^{(-1)} + (1-t)N^{(-1)}] \\ &= (1+t)^2 MM^{(-1)} + (1+t)(1-t)[MN^{(-1)} + M^{(-1)}N] + (1-t)^2 NN^{(-1)} \\ &= (1+t)^2 MM^{(-1)} + (1-t)^2 NN^{(-1)} \\ &= (1+t)^2 k + (1-t)^2 k \\ &= 4k. \end{aligned}$$

Remark. This patching scheme is a well-known result and, among many different sources, can be found in [4].

Theorem 3.9. *(Arasu and Linthicum [8]) A* $(\mathbb{Z}_2 \times \mathbb{Z}_p \times \mathbb{Z}_p)$*-developed* $W(2p^2, p^2)$ *exists for all primes* p *and can be constructed as such:*

$$W = \langle x, y \rangle - \langle x \rangle - \langle y \rangle - \sum_{n=1}^{p-3} x^n \left\langle x^{-n} y^{-n-2} \right\rangle + z[\langle xy \rangle - \left\langle xy^{-1} \right\rangle],$$

where $x^p = y^p = z^2 = 1$.

Remark. Take this W, from Theorem 3.9, and write it as $W = A + zB$, where

$$\begin{aligned} A &= \langle x, y \rangle - \langle x \rangle - \langle y \rangle - \sum_{n=1}^{p-3} x^n \left\langle x^{-n} y^{-n-2} \right\rangle, \\ B &= \langle xy \rangle - \left\langle xy^{-1} \right\rangle. \end{aligned}$$

Then A and B can be shown to have the following properties:

1. $AA^{(-1)} + BB^{(-1)} = k$;
2. $AB^{(-1)} = 0$;
3. the coefficients of A and B are all in $\{-1, 0, 1\}$.

A and B are said to form an orthogonal pair. Further, note that this is a well-known idea and has been used often by Craigen and others as in [13]. Sure enough, A and B can be used as the 'pieces' in Theorem 3.7.

3.3 Other Key Theorems

The following few theorems are critical to the development of the new results found in this paper.

Theorem 3.10. *(McFarland [8]) For every positive integer m, there exists an integer $M(m)$ such that if G is a finite abelian group whose order v is relatively prime to $M(m)$, then the only solutions of $A \in \mathbb{Z}[G]$ satisfying $AA^{(-1)} = m^2$ in $\mathbb{Z}[G]$ are $A = \pm mg$ for some $g \in G$. The value $M(m)$ is defined as follows: $M(1) = 1$, $M(2) = 2 \times 7$, $M(3) = 2 \times 3 \times 11 \times 13$, $M(4) = 2 \times 3 \times 7 \times 31$, and for any $m > 4$ let $M(m)$ be the product of distinct prime factors of m and*

$$M\left(\frac{m^2}{p^{2e}}\right),\ p-1,\ p^2-1,\ldots,\ p^{u(m)-1},$$

where p is a prime dividing m such that $p^e|m$ but $p^{e+1} \nmid m$, and we have $u(m) = \frac{m^2-m}{2}$.

Theorem 3.11. *(Inversion Formula [11]) Let G be a finite abelian group and $\hat{G}$ be the group of all characters of G. If $A = \sum_{g\in G} a_g g \in \mathbb{C}[G]$, then*

$$a_g = \frac{1}{|G|}\sum_{\chi\in\hat{G}}\chi(A)\chi(g^{-1}).$$

The following are key theorems by Turyn and Ma that are required to generate the main result of this paper (The mentioned result is Theorem 4.2).

Theorem 3.12. *(Turyn [24]) Let p be a prime and $G = H \times P$ an abelian group, where P is the Sylow p-subgroup of G. Assume that there exists an integer f such that $p^f \equiv 1 \pmod{exp(H)}$. Let χ be a non-principal character of G and let α be a positive integer. Suppose $A \in \mathbb{Z}[G]$ satisfies $\chi(A)\overline{\chi(A)} \equiv 0 \pmod{p^{2\alpha}}$. Then $\chi(A) \equiv 0 \pmod{p^\alpha}$.*

The congruences involving character sums in Theorem 3.12 occur in the ring of algebraic integers $\mathbb{Z}[\xi_v]$, where ξ_v is a primitive v^{th} root of unity, where $v = |G|$.

Lemma 3.13. *(Ma [18]) Let p be a prime and G an abelian group with a cyclic Sylow p-subgroup. If $A \in \mathbb{Z}G$ satisfies $\chi(A) \equiv 0 \pmod{p^\alpha}$ for all non-principal characters of G, then there exist $x_1, x_2 \in \mathbb{Z}[G]$ such that*

$$A = p^\alpha x_1 + Qx_2,$$

where Q is the unique subgroup of order p.

Theorem 3.14. *(Arasu, Seberry [10]) Suppose that a $CW(n,k)$ exists. Let p be a prime such that $p^{2t}|k$ for some positive integer t. Assume that m is a divisor of n and write $m = m'p^u$ where $\gcd(p, m') = 1$. Also, assume there exists an integer f such that $p^f \equiv -1 \pmod{m'}$. Then*

1. *$\frac{2n}{m} \geq p^t$ if $p|m$,*
2. *$\frac{n}{m} \geq p^t$ if $p \nmid m$.*

4 New Results

In this section we answer the question of existence for G-developed $W(n,k)$ for a total of 318 matrices. Specifically, we exclude the existence of 211 by use of Theorem 4.2, and give affirmation of the existence of 107 by constructions (70 by Kronecker products, 27 by patching, and 10 by the constructions in Section 4.5). Recall that only orders and weights under 100 are being examined and thus some of these results can be extended directly to higher orders and weights. In total we are left with 98 open cases out of the 1022 cases investigated. This first item is an extension of the result by Eades and Haine (see Theorem 3.1 and [15]).

4.1 The Complete Characterization of G-developed $W(n,4)$

Theorem 4.1. *An abelian G-developed $W(n,4)$ exists if and only if $n \geq 4$ and $2|n$ or $7|n$.*

Proof. Suppose that a G-developed $W(n,4)$ exists for some abelian group G of order n. Then by Theorem 3.10 and using $p = 2$, we must have that $\gcd(n, M(2)) = \gcd(n, 14) \neq 1$. Thus, either $2|n$ or $7|n$ as otherwise only trivial examples exist and they have a coefficient of 2 (Note that clearly, $n \geq 4$ since n must be greater than $k = 4$).
Conversely, suppose that $2|n$ and $n \geq 4$. Then let $W = -1 + g + h + gh$ for $g, h \in G$ where $g^2 = 1$, $h \neq 1$ or g. Then

$$\begin{aligned} WW^{(-1)} &= (-1+g+h+gh)(-1+g+h+gh)^{(-1)} \\ &= (-1+g+h+gh)(-1+g+h^{-1}+gh^{-1}) \\ &= 1-g-h-gh^{-1}-g+1+gh^{-1}+h^{-1}-h+gh+1+g-gh+h+g+1 \\ &= 4. \end{aligned}$$

So W is a G-developed $W(n,4)$.
On the other hand, if $7|n$, then let $W = -1 + x + x^2 + x^4$. By Example 1, W is a $\mathbb{Z}_7$-developed $W(7,4)$. Further, since $7|n$, we know that there exists a subgroup of order 7 in G which is isomoprhic to $\mathbb{Z}_7$ and thus we can use Theorem 3.6 to see that a G-developed $(n,4)$ exists.

Conjecture 3. *If a G-developed $W(n,9)$ exists, where G is abelian, then $n \geq 13$ and one the following is true:*

1. 13 divides n;
2. 24 divides n;
3. G contains a subgroup isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Remark. Even though this remains unproven, the table of known matrices in this paper supports the conjecture. Note that the converse is not true as a $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4)$-developed $W(24,9)$ does not exist by Theorem 4.2. The characterization of all G-developed $W(n,9)$ should probably follow from the analogous result given by Ang, Arasu, Ma and Strassler in [2], for the case when G is a cyclic group of order n.

The following theorem is the main result of this paper. It is an extension of the result by Arasu and Seberry for cyclic groups. See Theorem 3.14 above. Later, this extension is used to exclude numerous examples from existence.

4.2 Main Result

Theorem 4.2. *Suppose an abelian G-developed $W(n,k)$ exists and let p be a prime such that $p^{2t}|k$ for some $t \in \mathbb{N}$. Further, let H be a subgroup of G, of order $|H| = \frac{n}{m}$, where ${}^{G}/_{H} = J \times P$ for P the cyclic-Sylow-p-subgroup of ${}^{G}/_{H}$ and $\gcd(p, |J|) = 1$. Assume also that there exists an $f \in \mathbb{Z}$ such that $p^f \equiv -1 \pmod{exp(J)}$. Then*

1. *If $p|m$ then $\frac{2n}{m} \geqslant p^t$.*
2. *If $p \nmid m$ then $\frac{n}{m} \geqslant p^t$.*

Proof: Since W is a G-developed weighing matrix, we know that $WW^t = kI_n$, or by Equation (1), that

$$(A-B)(A-B)^{(-1)} = k$$

in $\mathbb{Z}[G]$ for some abelian group G. Now, let $H \leq G$, where $|{}^{G}/_{H}| = m$. Define $\sigma : G \longrightarrow {}^{G}/_{H}$ to be the canonical homomorphism. Then we get

$$(A^\sigma - B^\sigma)(A^\sigma - B^\sigma)^{(-1)} = k$$

in $\mathbb{Z}[{}^{G}/_{H}]$. Now, for each non-principal character of ${}^{G}/_{H}$, as in Definition 2.1, we have

$$\chi(A^\sigma - B^\sigma)\overline{\chi(A^\sigma - B^\sigma)} = k \equiv 0 \pmod{p^{2t}}.$$

Next, since ${}^{G}/_{H}$ contains a Sylow-p-subgroup and we have self-conjugacy, we may now use Theorem 3.12 to yield

$$\chi(A^\sigma - B^\sigma) \equiv 0 \pmod{p^t}.$$

Further, since the Sylow-p-subgroup of ${}^{G}/_{H}$ is cyclic, we may invoke Lemma 3.13 to obtain

$$A^\sigma - B^\sigma = p^t x_1 + Q x_2$$

for some $x_1, x_2 \in \mathbb{Z}[{}^{G}/_{H}]$, and $Q = \langle h \rangle$ is the unique subgroup of ${}^{G}/_{H}$ with $|Q| = p$. Next, we show that:

$$(A^\sigma - B^\sigma)(1-h) \equiv 0 \pmod{p^t}. \tag{2}$$

This is true since:

$$\begin{aligned}
(A^\sigma - B^\sigma)(1-h) &= (p^t x_1 + Q x_2)(1-h) \\
&= p^t x_1(1-h) + Q x_2(1-h) \\
&= p^t x_1(1-h) + x_2(Q - Qh) \\
&= p^t x_1(1-h) + x_2(0) \\
&= p^t x_1(1-h) \\
&\equiv 0 \pmod{p^t}.
\end{aligned}$$

Now, notice that the coefficients of $A^\sigma - B^\sigma$ are in $[-\frac{n}{m}, \frac{n}{m}]$, as $A^\sigma - B^\sigma$ is an integer-G-developed $W(n,k)$ (see Theorem 1.0). So $(A^\sigma - B^\sigma)(1-h)$ must have coefficients which are bounded by $[-\frac{2n}{m}, \frac{2n}{m}]$. Finally, from Equation (2), and since $(A^\sigma - B^\sigma)(1-h)$ must contain at least one non-zero coefficient (as otherwise we have a violation of the weight of W being k), we have shown that $\frac{2n}{m} \geq p^t$.
For the second part, suppose that $p \nmid m$ (i.e. P is trivial) and let χ_0 be the principal character of $\mathbb{Z}[G/H]$; then we get $\chi_0(A^\sigma - B^\sigma) = k \equiv 0 \pmod{p^{2t}}$ and Theorem 3.12 again implies $\chi(A^\sigma - B^\sigma) \equiv 0 \pmod{p^t}$. Now applying Theorem 3.11 to $A^\sigma - B^\sigma$, we see that each coefficient of $A^\sigma - B^\sigma$ has a factor of p^t. As the coefficients of $A^\sigma - B^\sigma$ are in $[-\frac{n}{m}, \frac{n}{m}]$, it follows that $\frac{n}{m} \geq p^t$ as needed, since at least one coefficient of W must be non-zero.

4.3 Non-Existence by Theorem 4.2

Here are a few examples of how to use Theorem 4.2.

Example 4. A $(\mathbb{Z}_3 \times \mathbb{Z}_9)$-developed $W(27, 25)$ does not exist.
Proof Let $G = \mathbb{Z}_3 \times \mathbb{Z}_9$ and $p = 5$. Then we see first of all that $p^{2t} = 5^{2t}$ must divide 25 and so $t = 1$. Next, we must be able to write G/H in the form $J \times P$. Let $H = 1$ and then $G/H = \mathbb{Z}_3 \times \mathbb{Z}_9$, and so $J = \mathbb{Z}_3 \times \mathbb{Z}_9$, because we only have a trivial cyclic 5-Sylow subgroup. Now we can readily compute the value of $\exp(J) = \exp(\mathbb{Z}_3 \times \mathbb{Z}_9) = 9$. Further, doing a quick calculation shows that $p^f = 5^3 = 125 \equiv -1 \pmod 9$, meaning that we have self-conjugacy. So, we must have that

$$\begin{aligned} \frac{n}{m} &\geq p^t, \\ 1 &\geq 5, \end{aligned}$$

which is a contradiction. Therefore, W does not exist.

Example 5. A $(\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5)$-developed $W(75, 49)$ does not exist.

Proof. Let $G = \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ and $p = 7$. Then we see first of all that $p^{2t} = 7^{2t}$ must divide 49 and so $t = 1$. Next, we must be able to write G/H in the form $J \times P$. Let $H = \mathbb{Z}_3$ and then $G/H = \mathbb{Z}_5 \times \mathbb{Z}_5$, and so $J = \mathbb{Z}_5 \times \mathbb{Z}_5$, as we only have a trivial cyclic 7-Sylow subgroup. Now we can readily compute the value of $\exp(J) = \exp(\mathbb{Z}_5 \times \mathbb{Z}_5) = 5$. Further, doing a quick calculation shows that $p^f = 7^2 = 49 \equiv -1 \pmod 5$, meaning that we have self-conjugacy. So, we must have that

$$\begin{aligned} \frac{n}{m} &\geq p^t, \\ \tfrac{75}{25} &\geq 7, \\ 3 &\geq 7, \end{aligned}$$

which is a contradiction. Therefore, W does not exist.

Example 6. A $(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9)$-developed $W(72, 64)$ does not exist.
Proof. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9$ and $p = 2$. Then we see first of all that $p^{2t} = 2^{2t}$ must divide 64 and so $t = 3$. Next, we must be able to write G/H in the form $J \times P$. Let

$H = \mathbb{Z}_2$ and then ${}^{G}/{}_{H} = \mathbb{Z}_4 \times \mathbb{Z}_9$, and so $J = \mathbb{Z}_9$ and $P = \mathbb{Z}_4$. Now we can readily compute the value of $\exp(J) = \exp(\mathbb{Z}_9) = 9$. Further, doing a quick calculation shows that $p^f = 2^3 = 8 \equiv -1 \pmod 9$, meaning that we have self-conjugacy. So, we must have that

$$\begin{aligned} \frac{2n}{m} &\geq p^t, \\ \tfrac{2\times 72}{36} &\geq 2^3, \\ 4 &\geq 8, \end{aligned}$$

which is a contradiction. Therefore, W does not exist.

The Tables 1–8 given in the Appendix, ordered by the weight of the matrices, show the G-developed $W(n,k)$ matrices found not to exist by Theorem 4.2 for the abelian groups G. The necessary parameters to exclude their existence are given as well.

Remark. Examples 4, 5 and 6 above, as well as several others discussed in this paper, only establish the 'group-developed' property of the weighing matrices in question. Existence status of such weighing matrices can be found in Table 2.85 of [12].

4.4 Existence by Patching or Kronecker

4.4.1 Patching

The following examples are constructed using a given group weighing matrix M and N written as group ring elements. Using the patching method from Theorem 3.8, one can compute the sought after G-developed $W(n,k)$.

A $(\mathbb{Z}_2 \times \mathbb{Z}_{2s})$-developed $W(4s,16)$ exists for $s \geq 4$.

Let $M = -1 + x + x^s + x^{s+1}$ and $N = x^2 M$ where $M, N \in (\mathbb{Z}_{2s})$-developed $W(2s,4)$. Then M and N have disjoint supports. Thus, by patching, $(1+t)M + (1-t)N \in (\mathbb{Z}_2 \times \mathbb{Z}_{2s})$-developed $W(4s,16)$.

A $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2s})$-developed $W(8s,64)$ exists for $s \geq 8$.

Let $A = -1 + x + x^s + x^{s+1}$, $B = x^2 A$, $C = x^2 B$, and $D = x^2 C$, where $A, B, C, D \in (\mathbb{Z}_{2s})$-developed $W(2s,4)$ and have pairwise disjoint supports by construction. Then $M = (1+t)\,A + (1-t)\,B$ and $N = (1+t)\,C + (1-t)\,D$ are both in $(\mathbb{Z}_2 \times \mathbb{Z}_{2s})$-developed $W(4s,16)$ and also are disjoint. Continue by patching M and N to get a $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2s})$-developed $W(8s,64)$.

A $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$-developed $W(36,36)$ exists.

Let $M = 1 + y + x + xy - x^2y^2 + z(1 - y^2 - x^2 + x^2y^2)$ and $N = y^2 + xy^2 - x^2 - x^2y + z(y + x - xy + xy^2 + x^2y)$ where both are $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$-developed $W(18,9)$ and disjoint. Then patch to obtain the sought after matrix.

A $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$-developed $W(54,36)$ exists.

Let $M = 1 + y + x + xy - x^2y^2 + z(1 - y^2 - x^2 + x^2y^2)$ and $N = y^2 + xy^2 - x^2 - x^2y + z(y + x - xy + xy^2 + x^2y)$ where both are $(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$-developed $W(27,9)$ and are disjoint. Then patch to obtain the sought after matrix.

4.4.2 Kronecker

Tables 9–13 given in Appendix list all G-developed $W(n,k)$ matrices which can be generated by use of the Kronecker product as in Theorem 3.6. The left columns give the matrix to be generated and the right columns give the products needed. The Kronecker product is denoted by $\otimes$ and the trivial weighing matrix $CW(n,1)$ is denoted by I_n. For formatting, the G-developed $W(n,k)$ matrices are being written in the form (G,n,k).

4.5 Other Constructions

The following constructions come from the G-developed $W(n,k)$ given in Theorem 3.9 by Arasu and Linthicum where $G = \mathbb{Z}_2 \times \mathbb{Z}_p \times \mathbb{Z}_p$ and $k = p^2$.

Theorem 4.3. *An* $(L \times \mathbb{Z}_p \times \mathbb{Z}_p)$*-developed* $W(rp^2, p^2)$ *exists for all* p *prime and* all *groups* L *of order* r.

Proof. By the construction in Theorem 3.9, the matrix $W = \langle x, y\rangle - \langle x\rangle - \langle y\rangle - \sum_{n=1}^{p-3} x^n \langle x^{-n}y^{-n-2}\rangle + z[\langle xy\rangle - \langle xy^{-1}\rangle]$ can be written as $W = A + zB$, where $A = \langle x, y\rangle - \langle x\rangle - \langle y\rangle - \sum_{n=1}^{p-3} x^n \langle x^{-n}y^{-n-2}\rangle$ and $B = \langle xy\rangle - \langle xy^{-1}\rangle$. These two blocks form an orthogonal pair. Now, simply let $x^p = y^p = 1$ and z be some element in L. Then we see that by computing $WW^{(-1)}$, without restriction on the nature of z:

$$\begin{aligned} WW^{(-1)} &= (A+zB)(A+zB)^{(-1)} \\ &= (A+zB)(A^{(-1)} + z^{-1}B^{(-1)}) \\ &= AA^{(-1)} + z^{-1}AB^{(-1)} + zA^{(-1)}B + BB^{(-1)} \\ &= AA^{(-1)} + BB^{(-1)} \\ &= k. \end{aligned}$$

Here z is canceled out by the orthogonality of the two blocks A and B. Thus, z can be any element of any group and the lifted group weighing matrix exists.

Theorem 4.4. *A* $(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$*-developed* $W(p^3, p^2)$ *exists for all primes* p.

Proof. Take $W = A + zB$ as before. Note that $A, B \in \mathbb{Z}[\langle x, y\rangle]$ (i.e. they contain no z elements). Here, $x^p = y^p = 1$. Now, we need to create new blocks $\overline{A}, \overline{B} \in \mathbb{Z}[\langle x, t\rangle]$. This can be done by simply using the homomorphic mapping $y \longrightarrow t$, where $t = y^p$ in both A and B. In this new group we allow for $x^p = t^p = 1$, where $t = y^p$. Next, examine the new $(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$-developed matrix $\overline{W} = \overline{A} + y\overline{B}$.

$$\begin{aligned} \overline{WW}^{(-1)} &= (\overline{A} + y\overline{B})(\overline{A} + y\overline{B})^{(-1)} \\ &= (\overline{A} + y\overline{B})(\overline{A}^{(-1)} + y^{-1}\overline{B}^{(-1)}) \\ &= \overline{AA}^{(-1)} + \overline{BB}^{(-1)} + y\overline{BA}^{(-1)} + y^{-1}\overline{B}^{(-1)}\overline{A} \\ &= k \end{aligned}$$

by the orthogonality of $\overline{A}$ and $\overline{B}$.
Further, the coefficients of $\overline{W}$ lie in $\{-1, 0 ,1\}$ still, as A and B were already disjoint, and we now have $\overline{A}, \overline{B} \in \mathbb{Z}[\langle x, t\rangle]$, where t has the same order as y. This means that t is actually the same element as y in the original group. Therefore, $\overline{W}$ is a $(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$-developed $W(p^3, p^2)$.

5 Table of abelian-G-developed Weighing Matrices

Table 14 given in the Appendix is the up-to-date survey for group weighing matrices. It includes the updated Strassler Table from [9] plus the added material above. The labels are as follows: '.' denotes previously known non-existence; 'N' denotes non-existence given in this paper; '+' denotes previously known existence; 'Y' denotes existence given in this paper; 'K' denotes existence given by the Kronecker product; and blank spaces remain open cases. (Note the obviously omitted $k = 1$ set since this represents only trivial weighing matrices.)

6 Conclusion

The research reported here was partly due to the open questions in [9], where the authors looked at studying the case of cyclic-G-developed weighing matrices. Here we have studied the case for abelian-G-developed weighing matrices. Clearly, there is a very significant amount of work that has been done in this field — simply look at all of the other publications involving this topic. However, even with so much done there are still many unanswered questions that require work. From the above data alone it appears that there are up to 98 abelian-G-developed weighing matrices which need to be checked out. Also, remember that only groups of order less than 100 are considered. Effort should certainly be placed into answering the open cases above. There are many different approaches that could be attempted: for example, computer searches to construct examples or possibly extending theorems to help fill in the gaps. Also, attempting to generate similar data for non-abelian groups might be an interesting endeavor.

Appendix

Table 1: Non-Existence for $k = 9$, $p = 3$, $t = 1$

G	n	m	$\exp(J)$	f
$\mathbb{Z}_2 \times \mathbb{Z}_6$	12	12	2	0
$\mathbb{Z}_4 \times \mathbb{Z}_4$	16	16	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_8$	16	8	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	16	16	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	16	16	2	0
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$	20	20	10	2
$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	24	24	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	24	24	2	0
$\mathbb{Z}_5 \times \mathbb{Z}_5$	25	25	5	2
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$	28	28	14	3
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	32	32	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	32	32	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	32	32	2	0
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$	32	16	4	1
$\mathbb{Z}_4 \times \mathbb{Z}_8$	32	16	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$	36	36	2	0
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5$	40	20	10	2
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$	40	40	10	2
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	48	48	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	48	48	2	0
$\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	48	48	4	1
$\mathbb{Z}_7 \times \mathbb{Z}_7$	49	49	7	3
$\mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	50	50	10	2
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$	56	56	14	3
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_7$	56	56	28	3
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	60	60	10	2
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$	64	32	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	64	64	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	64	64	2	0
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	64	64	4	1
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	64	64	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$	64	32	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{17}$	68	68	34	8
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9$	72	72	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$	72	72	2	0
$\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	75	75	5	2
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{19}$	76	76	38	9
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5$	80	40	10	2
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$	80	80	10	2
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_7$	84	84	14	3
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	96	96	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	96	96	2	0
$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	96	96	4	1
$\mathbb{Z}_2 \times \mathbb{Z}_7 \times \mathbb{Z}_7$	98	98	14	3
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$	100	100	50	10
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	100	100	10	2
$\mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	100	50	10	2

Table 2: Non-Existence for $k = 16$, $p = 2$, $t = 2$

G	n	m	$\exp(J)$	f
$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	18	18	3	1
$\mathbb{Z}_5 \times \mathbb{Z}_5$	25	25	5	2
$\mathbb{Z}_3 \times \mathbb{Z}_9$	27	27	9	3
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	27	27	3	1
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	36	36	3	1
$\mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	50	50	5	2
$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	54	54	3	1
$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_9$	54	54	9	3
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_8$	72	72	3	1
$\mathbb{Z}_3 \times \mathbb{Z}_{27}$	81	81	27	9
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9$	81	81	9	3
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	81	81	3	3
$\mathbb{Z}_9 \times \mathbb{Z}_9$	81	81	9	3
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{11}$	99	99	33	5
$\mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	100	100	5	2

Table 3: Non-Existence for $k = 25$, $p = 5$, $t = 1$

G	n	m	$\exp(J)$	f
$\mathbb{Z}_3 \times \mathbb{Z}_9$	27	27	9	3
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	27	27	3	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$	28	28	14	3
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	32	8	2	0
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	32	16	2	0
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	32	32	2	0
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$	32	8	2	0
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$	36	36	18	3
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	36	36	6	1
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	36	18	6	1
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5$	40	20	2	0
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$	40	40	2	0
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	45	45	3	1
$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8$	48	12	6	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	48	24	6	1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	48	48	6	1
$\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	48	12	6	1
$\mathbb{Z}_7 \times \mathbb{Z}_7$	49	49	7	3
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{13}$	52	52	26	2

Table 3: continued

G	n	m	$\exp(J)$	f
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3$	54	54	6	1
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_9$	54	54	18	3
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_7$	56	56	14	3
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_7$	56	28	14	3
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5$	60	60	6	1
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_7$	63	63	21	3
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_8$	64	16	2	0
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$	64	32	2	0
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	64	64	2	0
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_4$	64	16	2	0
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{17}$	68	68	34	8
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_9$	72	36	18	3
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_9$	72	72	18	3
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3$	72	72	6	1
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_8$	72	18	6	1
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_4$	72	36	6	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_5$	80	40	2	0
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5$	80	80	2	0
$\mathbb{Z}_3\times\mathbb{Z}_{27}$	81	81	27	9
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_9$	81	81	9	3
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3$	81	81	3	1
$\mathbb{Z}_9\times\mathbb{Z}_9$	81	81	9	3
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_7$	84	84	42	3
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_5$	90	90	6	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{23}$	92	92	46	11
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_8$	96	24	6	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4$	96	48	6	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$	96	96	6	1
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4\times\mathbb{Z}_4$	96	24	6	1
$\mathbb{Z}_2\times\mathbb{Z}_7\times\mathbb{Z}_7$	98	98	14	3
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{25}$	100	100	2	0

Table 4: Non-Existence for $k = 36$, $p = 2$, $t = 1$

G	n	m	$\exp(J)$	f
$\mathbb{Z}_3\times\mathbb{Z}_{27}$	81	81	27	9
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_9$	81	81	9	3
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3$	81	81	3	1
$\mathbb{Z}_9\times\mathbb{Z}_9$	81	81	9	3
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_{11}$	99	99	33	5

Table 5: Non-Existence for $k = 36$, $p = 3$, $t = 1$

G	n	m	$\exp(J)$	f
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_9$	36	36	2	0
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_5$	40	20	10	2
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5$	40	40	10	2
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4$	48	48	4	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$	48	48	2	0
$\mathbb{Z}_3\times\mathbb{Z}_4\times\mathbb{Z}_4$	48	48	4	1
$\mathbb{Z}_7\times\mathbb{Z}_7$	49	49	7	3
$\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}_5$	50	50	10	2
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_7$	56	56	14	3
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_7$	56	56	28	3
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5$	60	60	10	2
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_8$	64	32	4	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$	64	64	4	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	64	64	2	0
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_4$	64	64	4	1
$\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_4$	64	64	4	1
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_8$	64	32	4	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{17}$	68	68	34	8
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_9$	72	72	4	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_9$	72	72	2	0
$\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_5$	75	75	5	2
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{19}$	76	76	38	9
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_5$	80	40	10	2
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5$	80	80	10	2
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_7$	84	84	14	3
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4$	96	96	4	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$	96	96	2	0
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4\times\mathbb{Z}_4$	96	96	4	1
$\mathbb{Z}_2\times\mathbb{Z}_7\times\mathbb{Z}_7$	98	98	14	3
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{25}$	100	100	50	10
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}_5$	100	100	10	2
$\mathbb{Z}_4\times\mathbb{Z}_5\times\mathbb{Z}_5$	100	50	10	2

Table 6: Non-Existence for $k = 49$, $p = 7$, $t = 1$

G	n	m	exp(J)	f
$\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}_5$	50	50	10	2
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{13}$	52	52	26	6
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_7$	56	56	2	0
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_7$	56	56	4	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5$	60	20	10	2
$\mathbb{Z}_2\times\mathbb{Z}_{32}$	64	16	8	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{16}$	64	32	8	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_8$	64	64	8	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$	64	64	4	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	64	64	2	0
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_4$	64	64	4	1
$\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_4$	64	64	4	1
$\mathbb{Z}_4\times\mathbb{Z}_{16}$	64	32	8	1
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_8$	64	64	8	1
$\mathbb{Z}_8\times\mathbb{Z}_8$	64	64	8	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{17}$	68	68	34	8
$\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_5$	75	25	5	2
$\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}_8$	80	20	10	2
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_5$	80	40	10	2
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5$	80	80	10	2
$\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_5$	80	20	10	2
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_7$	84	28	2	0
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_{11}$	88	88	44	5
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{11}$	88	88	22	5
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{23}$	92	92	46	11
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_8$	96	32	8	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4$	96	32	4	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$	96	32	2	0
$\mathbb{Z}_3\times\mathbb{Z}_4\times\mathbb{Z}_8$	96	32	8	1
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4\times\mathbb{Z}_4$	96	32	4	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{25}$	100	100	50	2
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}_5$	100	100	10	2
$\mathbb{Z}_4\times\mathbb{Z}_5\times\mathbb{Z}_5$	100	50	10	2

Table 7: Non-Existence for $k = 64$, $p = 2$, $t = 3$

G	n	m	$\exp(J)$	f
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{17}$	68	34	17	4
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_9$	72	36	9	3
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_8$	72	72	3	1
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_4$	72	36	3	1
$\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_5$	75	25	5	2
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{19}$	76	38	19	9
$\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}_8$	80	40	5	2
$\mathbb{Z}_3\times\mathbb{Z}_{27}$	81	81	27	9
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_9$	81	81	9	3
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3$	81	81	3	1
$\mathbb{Z}_9\times\mathbb{Z}_9$	81	81	9	3
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_{11}$	88	44	11	5
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_{16}$	96	48	3	1
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_{11}$	99	99	33	5
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{25}$	100	50	25	10
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}_5$	100	50	5	2
$\mathbb{Z}_4\times\mathbb{Z}_5\times\mathbb{Z}_5$	100	100	5	2

Table 8: Non-Existence for $k = 81$, $p = 3$, $t = 2$

G	n	m	$\exp(J)$	f
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_7$	84	84	14	3
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_5$	90	30	10	2
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_{16}$	96	24	4	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_8$	96	48	4	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4$	96	96	4	1
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$	96	96	2	0
$\mathbb{Z}_3\times\mathbb{Z}_4\times\mathbb{Z}_8$	96	48	4	1
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4\times\mathbb{Z}_4$	96	96	4	1
$\mathbb{Z}_2\times\mathbb{Z}_7\times\mathbb{Z}_7$	98	98	14	3
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{25}$	100	100	50	10
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}_5$	100	100	10	2
$\mathbb{Z}_4\times\mathbb{Z}_5\times\mathbb{Z}_5$	100	50	10	2

Table 9: Kronecker Products for $k = 9$

G-developed $W(n,k)$ Generated	Construction
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 36, 9)$	$I_2 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 18, 9)$
$(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8, 48, 9)$	$I_2 \otimes (\mathbb{Z}_{24}, 24, 9)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{13}, 52, 9)$	$I_2 \otimes (\mathbb{Z}_{26}, 26, 9)$
$(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 54, 9)$	$I_2 \otimes (\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 27, 9)$
$(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_9, 54, 9)$	$I_2 \otimes (\mathbb{Z}_3 \times \mathbb{Z}_9, 27, 9)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 72, 9)$	$I_2 \otimes I_2 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 18, 9)$
$(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4, 72, 9)$	$I_4 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 18, 9)$
$(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 81, 9)$	$I_3 \otimes (\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 27, 9)$
$(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5, 90, 9)$	$I_5 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 18, 9)$
$(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{16}, 96, 9)$	$I_2 \otimes (\mathbb{Z}_{48}, 48, 9)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8, 96, 9)$	$I_2 \otimes I_2 \otimes (\mathbb{Z}_{24}, 24, 9)$
$(\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_8, 96, 9)$	$I_4 \otimes (\mathbb{Z}_{24}, 24, 9)$

Table 10: Kronecker Products for $k = 16$

G-developed $W(n,k)$ Generated	Construction
$(\mathbb{Z}_4 \times \mathbb{Z}_4, 16, 16)$	$(\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, 16, 16)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4, 24, 16)$	$(\mathbb{Z}_6, 6, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, 24, 16)$	$(\mathbb{Z}_6, 6, 4) \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4, 32, 16)$	$I_2 \otimes (\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, 32, 16)$	$I_2 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, 32, 16)$	$I_2 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8, 32, 16)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_8, 8, 4)$
$(\mathbb{Z}_4 \times \mathbb{Z}_8, 32, 16)$	$(\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_8, 8, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 36, 16)$	$(\mathbb{Z}_6, 6, 4) \otimes (\mathbb{Z}_6, 6, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5, 40, 16)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_{10}, 10, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4, 48, 16)$	$I_2 \otimes (\mathbb{Z}_6, 6, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, 48, 16)$	$I_2 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_6, 6, 4)$
$(\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4, 48, 16)$	$I_3 \otimes (\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_7 \times \mathbb{Z}_7, 49, 16)$	$(\mathbb{Z}_7, 4, 4) \otimes (\mathbb{Z}_7, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7, 56, 16)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_{14}, 14, 4)$
$(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7, 63, 16)$	$I_3 \otimes (\mathbb{Z}_{21}, 21, 16)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{16}, 64, 16)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_{16}, 16, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8, 64, 16)$	$I_2 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_8, 8, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, 64, 16)$	$I_2 \otimes I_2 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, 64, 16)$	$I_2 \otimes I_2 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4, 64, 16)$	$I_2 \otimes (\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4, 64, 16)$	$I_4 \otimes (\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_4 \times \mathbb{Z}_{16}, 64, 16)$	$(\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_{16}, 16, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8, 64, 16)$	$I_2 \otimes (\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_8, 8, 4)$
$(\mathbb{Z}_8 \times \mathbb{Z}_8, 64, 16)$	$(\mathbb{Z}_8, 8, 4) \otimes (\mathbb{Z}_8, 8, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9, 72, 16)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_{18}, 18, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 72, 16)$	$I_2 \otimes (\mathbb{Z}_6, 6, 4) \otimes (\mathbb{Z}_6, 6, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4, 72, 16)$	$(\mathbb{Z}_6, 6, 4) \otimes (\mathbb{Z}_{12}, 12, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5, 80, 16)$	$I_2 \otimes (\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_{10}, 10, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5, 80, 16)$	$I_2 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_{10}, 10, 4)$
$(\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_5, 80, 16)$	$I_5 \otimes (\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{11}, 88, 16)$	$(\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_{22}, 22, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{11}, 88, 16)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_{22}, 22, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8, 96, 16)$	$I_6 \otimes (\mathbb{Z}_6, 6, 4) \otimes (\mathbb{Z}_8, 8, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4, 96, 16)$	$I_2 \otimes I_2 \otimes (\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_6, 6, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, 96, 16)$	$I_2 \otimes I_2 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_6, 6, 4)$
$(\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_8, 96, 16)$	$I_3 \otimes (\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_8, 8, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4, 96, 16)$	$I_2 \otimes I_3 \otimes (\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_7 \times \mathbb{Z}_7, 98, 16)$	$I_2 \otimes (\mathbb{Z}_7, 7, 4) \otimes (\mathbb{Z}_7, 7, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5, 100, 16)$	$(\mathbb{Z}_{10}, 10, 4) \otimes (\mathbb{Z}_{10}, 10, 4)$

Table 11: Kronecker Products for $k = 25$

G-developed $W(n,k)$ Generated	Construction
$(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{11}, 99, 25)$	$I_3 \otimes (\mathbb{Z}_{33}, 33, 25)$

Table 12: Kronecker Product for $k = 36$

G-developed $W(n,k)$ Generated	Construction
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{13}, 52, 36)$	$(\mathbb{Z}_2 \otimes \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_{13}, 13, 9)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 72, 36)$	$(\mathbb{Z}_2 \otimes \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 18, 9)$
$(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4, 72, 36)$	$(\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 18, 9)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8, 96, 36)$	$(\mathbb{Z}_2 \otimes \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_{24}, 24, 9)$
$(\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_8, 96, 36)$	$(\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_{24}, 24, 9)$

Table 13: Kronecker Product for $k = 64$

G-developed $W(n,k)$ Generated	Construction
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8, 64, 64)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_2 \times \mathbb{Z}_8, 16, 16)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, 64, 64)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, 64, 64)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4, 64, 64)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4, 64, 64)$	$(\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_4, 4, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8, 64, 64)$	$(\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_2 \times \mathbb{Z}_8, 16, 16)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5, 80, 64)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5, 20, 16)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_7, 84, 64)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_{21}, 21, 16)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4, 96, 64)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_6, 6, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, 96, 64)$	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2, 4, 4) \otimes (\mathbb{Z}_6, 6, 4)$
$(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4, 96, 64)$	$(\mathbb{Z}_6, 6, 4) \otimes (\mathbb{Z}_4, 4, 4) \otimes (\mathbb{Z}_4, 4, 4)$

Table 14: Table of G-developed $W(n, k)$

G	k=4	9	16	25	36	49	64	81	100
$\mathbb{Z}_4$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2$	Y	.	.	.	.	.	.	.	.
$\mathbb{Z}_5$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_6$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_7$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_8$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_4$	Y	.	.	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	Y	.	.	.	.	.	.	.	.
$\mathbb{Z}_9$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_3\times\mathbb{Z}_3$	N	.	.	.	.	.	.	.	.
$\mathbb{Z}_{10}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_{11}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{12}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_6$	Y	N	.	.	.	.	.	.	.
$\mathbb{Z}_{13}$	.	+	.	.	.	.	.	.	.
$\mathbb{Z}_{14}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_{15}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{16}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_4\times\mathbb{Z}_4$	Y	N	K	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_8$	Y	N	Y	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$	Y	N	K	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	Y	N	+	.	.	.	.	.	.
$\mathbb{Z}_{17}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{18}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3$	Y	+	N	.	.	.	.	.	.
$\mathbb{Z}_{19}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{20}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5$	Y	N	Y	.	.	.	.	.	.
$\mathbb{Z}_{21}$	+	.	+	.	.	.	.	.	.
$\mathbb{Z}_{22}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_{23}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{24}$	+	+	.	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4$	Y	N	K	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$	Y	N	K	.	.	.	.	.	.
$\mathbb{Z}_{25}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_5\times\mathbb{Z}_5$	N	N	N	.	.	.	.	.	.
$\mathbb{Z}_{26}$	+	+	.	.	.	.	.	.	.
$\mathbb{Z}_{27}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_3\times\mathbb{Z}_9$	N	Y	N	N	.	.	.	.	.
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3$	N	Y	N	N	.	.	.	.	.
$\mathbb{Z}_{28}$	+	.	+	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_7$	Y	N	Y	N	.	.	.	.	.

Table 14: continued

G	k=4	9	16	25	36	49	64	81	100
$\mathbb{Z}_{29}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{30}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_{31}$	.	.	+	+	.	.	.	.	.
$\mathbb{Z}_{32}$	+	.	.		.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_{16}$	Y		Y		.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_4$	Y	N	K	N	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$	Y	N	K	N	.	.	.	.	.
$\mathbb{Z}_2^5$	Y	N	K	N	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_8$	Y	N	K	N	.	.	.	.	.
$\mathbb{Z}_4\times\mathbb{Z}_8$	Y	N	K		.	.	.	.	.
$\mathbb{Z}_{33}$	.	.	.	+	.	.	.	.	.
$\mathbb{Z}_{34}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_{35}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_{36}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_9$	Y	N	Y	N	N	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3$	Y	K	K	N	+	.	.	.	.
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_4$	Y	Y	N	N	+	.	.	.	.
$\mathbb{Z}_{37}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{38}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_{39}$	.	+	.	.	.	.	.	.	.
$\mathbb{Z}_{40}$	+	.	.		.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_5$	Y	N	Y	N	N	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5$	Y	N	K	N	N	.	.	.	.
$\mathbb{Z}_{41}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{42}$	+	.	+	.	.	.	.	.	.
$\mathbb{Z}_{43}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{44}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{11}$	Y		Y			.	.	.	.
$\mathbb{Z}_{45}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_5$	N	Y		N		.	.	.	.
$\mathbb{Z}_{46}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_{47}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{48}$	+	+	.	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_8$	Y	K	Y	N		.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4$	Y	N	K	N	N	.	.	.	.
$\mathbb{Z}_2^4\times\mathbb{Z}_3$	Y	N	K	N	N	.	.	.	.
$\mathbb{Z}_3\times\mathbb{Z}_4\times\mathbb{Z}_4$	Y	N	K	N	N	.	.	.	.
$\mathbb{Z}_{49}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_7\times\mathbb{Z}_7$	Y	N	K	N	N	.	.	.	.
$\mathbb{Z}_{50}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}_5$	Y	N	N	+	N	N	.	.	.
$\mathbb{Z}_{51}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{52}$	+	+	.	.	+	.	.	.	.

Table 14: continued

G	k=4	9	16	25	36	49	64	81	100
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{13}$	Y	K	Y	N	K	N	.	.	.
$\mathbb{Z}_{53}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{54}$	+	.	.	.	.		.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3$	Y	K	N	N	Y		.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_9$	Y	K	N	N			.	.	.
$\mathbb{Z}_{55}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{56}$	+	.	+	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_7$	Y	N	K	N	N	N	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_7$	Y	N	Y	N	N	N	.	.	.
$\mathbb{Z}_{57}$	.	.	.	.	.	+	.	.	.
$\mathbb{Z}_{58}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_{59}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{60}$	+	.	.	.		.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5$	Y	N	Y	N	N	N	.	.	.
$\mathbb{Z}_{61}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{62}$	+	.	+	+	.	.	.	.	.
$\mathbb{Z}_{63}$	+	.	+	.	.		.	.	.
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_7$	Y	Y	K	N			.	.	.
$\mathbb{Z}_{64}$	+	.	.		.		.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_{32}$	Y		Y			N		.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{16}$	Y		K			N	Y	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_8$	Y	N	K	N	N	N	K	.	.
$\mathbb{Z}_2^4\times\mathbb{Z}_4$	Y	N	K	N	N	N	K	.	.
$\mathbb{Z}_2^6$	Y	N	K	N	N	N	K	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_4$	Y	N	K	N	N	N	K	.	.
$\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_4$	Y	N	K		N	N	K	.	.
$\mathbb{Z}_4\times\mathbb{Z}_{16}$	Y		K			N		.	.
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_8$	Y	N	K		N	N		.	.
$\mathbb{Z}_8\times\mathbb{Z}_8$	Y		K			N		.	.
$\mathbb{Z}_{65}$	.	+	.	.	.	.	.	.	.
$\mathbb{Z}_{66}$	+	.	.	+	.	.	.	.	.
$\mathbb{Z}_{67}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{68}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{17}$	Y	N	Y	N		N	N	.	.
$\mathbb{Z}_{69}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{70}$	+	.	+	.		.	.	.	.
$\mathbb{Z}_{71}$	.	.	.	+	.	.	.	.	.
$\mathbb{Z}_{72}$	+	+	.	.	.		.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_9$	Y	N	Y	N	N		N	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_9$	Y	N	K	N	N		Y	.	.
$\mathbb{Z}_2^3\times\mathbb{Z}_3\times\mathbb{Z}_3$	Y	K	K	N	K			.	.
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_8$	Y	Y	N	N			N	.	.
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_4$	Y	K	K	N	K		N	.	.

Table 14: continued

G	k=4	9	16	25	36	49	64	81	100
$\mathbb{Z}_{73}$	.	.	.	.	.	.	+	.	.
$\mathbb{Z}_{74}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_{75}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	N	N		Y	N	N	N	.	.
$\mathbb{Z}_{76}$	+	.	.	.	.		.	.	.
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{19}$	Y	N	Y		N		N	.	.
$\mathbb{Z}_{77}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_{78}$	+	+	.	.	+	.	.	.	.
$\mathbb{Z}_{79}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{80}$	+	.	.		.		.	.	.
$\mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_8$	Y		Y			N	N	.	.
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5$	Y	N	K	N	N	N	Y	.	.
$\mathbb{Z}_2^4 \times \mathbb{Z}_5$	Y	N	K	N	N	N	K	.	.
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_5$	Y		K			N		.	.
$\mathbb{Z}_{81}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_3 \times \mathbb{Z}_{27}$	N	Y	N	N	N		N	.	.
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9$	N	Y	N	N	N		N	.	.
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	N	K	N	N	N		N	.	.
$\mathbb{Z}_9 \times \mathbb{Z}_9$	N	Y	N	N	N		N	.	.
$\mathbb{Z}_{82}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_{83}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{84}$	+	.	+	.	.	.	+	.	.
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_7$	Y	N	Y	N	N	N	K	N	.
$\mathbb{Z}_{85}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{86}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_{87}$	.	.	.	.	.	+	.	.	.
$\mathbb{Z}_{88}$	+	.	.		.	.	.	.	.
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{11}$	Y		K			N	N		.
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{11}$	Y		K			N	Y		.
$\mathbb{Z}_{89}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{90}$	+	.	.	.	.			.	.
$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	Y	K		N				N	.
$\mathbb{Z}_{91}$	+	+	.	.	+	.	.	+	.
$\mathbb{Z}_{92}$	+	.	.	.	.	.	.		.
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{23}$	Y		Y	N		N			.
$\mathbb{Z}_{93}$	.	.	+	+	.	.	.	.	.
$\mathbb{Z}_{94}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_{95}$	.	.	.	.	.		.	.	.
$\mathbb{Z}_{96}$	+	+	.		+		.	.	.
$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{16}$	Y	K	Y				N	N	.
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8$	Y	K	K	N	K	N	Y	N	.
$\mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	Y	N	K	N	N	N	K	N	.
$\mathbb{Z}_2^5 \times \mathbb{Z}_3$	Y	N	K	N	N	N	K	N	.

Table 14: continued

G	k=4	9	16	25	36	49	64	81	100
$\mathbb{Z}_3\times\mathbb{Z}_4\times\mathbb{Z}_8$	Y	K	K		K	N		N	.
$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4\times\mathbb{Z}_4$	Y	N	K	N	N	N	K	N	.
$\mathbb{Z}_{97}$	.	.	.	.	.	.	.	.	.
$\mathbb{Z}_{98}$	+	.	+	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_7\times\mathbb{Z}_7$	Y	N	K	N	N	+		N	.
$\mathbb{Z}_{99}$	.	.	.	+	.	.	.	.	.
$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_{11}$	N	Y	N	K	N		N		.
$\mathbb{Z}_{100}$	+	.	.	.	.	.	.	.	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{25}$	Y	N	Y	N	N	N	N	N	.
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}_5$	Y	N	K		N	N	N	N	.
$\mathbb{Z}_4\times\mathbb{Z}_5\times\mathbb{Z}_5$	Y	N	N	Y	N	N	N	N	.

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(Received 11 Jan 2012; revised 18 July 2012, 9 Nov 2012)