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Group distance magic labeling of Cartesian product of cycles

Dalibor Froncek

Department of Mathematics and Statistics
University of Minnesota Duluth
1117 University Drive, Duluth, MN 55812-3000
U.S.A.

Abstract

A group distance magic labeling of a graph $G(V,E)$ with $|V| = n$ is an injection from V to an abelian group Γ of order n such that the sum of labels of all neighbors of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$. We completely characterize all Cartesian products $C_k \Box C_m$ that admit a group distance magic labeling by Z_{km}.

1 Introduction

In this paper we deal with a graph labeling that belongs to a large family of magic-type labelings. Vaguely speaking, a magic-type labeling of a graph $G(V,E)$ is a mapping (usually an injection) from the set $V \cup E$ to a set of integers such that the sum of labels of elements adjacent and/or incident to an element $e \in V \cup E$ is equal to the same magic constant μ. An introductory text on magic-type labelings was published by Wallis [5]. A thorough dynamic survey of graph labelings including other kinds of labelings is being maintained by Gallian [2].

In particular, a *distance magic labeling* of a graph $G(V,E)$ with $|V| = n$ is an injection from V to the set $\{1, 2, \ldots, n\}$ such that the sum of the labels of all neighbors of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$ is equal to the same *magic constant* μ. This labeling has been also called a 1-*vertex magic vertex labeling* or a Σ-*labeling* by various authors. A graph that admits a distance magic labeling is often called a *distance magic graph.* The concept of distance magic labeling has been motivated by the construction of magic squares. Although the first results were probably obtained by Vilfred in a Ph.D. thesis [4] in 1994, it gained more attention only recently. For a survey, we refer the reader to [1].

Naturally, distance magic labelings of regular graphs have been studied by many authors. In particular, Rao, Singh and Parameswaran [3] proved the following result on Cartesian products of cycles.

Theorem 1.1. ([3]) *The graph $C_k \square C_m$ is distance magic if and only if $k = m \equiv 2 \pmod 4$.*

Based on the notion of distance magic graphs, we introduce a weaker concept called group distance magic graphs below. A *group distance magic labeling* or a Γ-*distance magic labeling* of a graph $G(V, E)$ with $|V| = n$ is an injection from V to an abelian group Γ of order n such that the *weight* of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$, called the *magic constant*. Obviously, every graph with n vertices and a distance magic labeling also admits a Z_n-distance magic labeling. The converse is not necessarily true. We exhibit a class of graphs that are Z_n-distance magic while not distance magic.

We prove that $C_k \square C_m$ is Z_{km}-distance magic if and only if km is even. Moreover, we present a Z_2^{2k}-distance magic labeling of products $C_{2^k} \square C_{2^k}$ for every $k \geq 2$.

2 Z_{km}-distance magic labeling of $C_k \square C_m$

We will assume that $k \leq m$ unless stated otherwise and denote the vertices of $C_k \square C_m$ by $x_{i,j}$ with $0 \leq i \leq k-1, 0 \leq j \leq m-1$. We will identify the vertices with their labels, hence $x_{i,j}$ will mean both "vertex $x_{i,j}$" and "the label of vertex $x_{i,j}$."

By a *diagonal* of $C_k \square C_m$ we mean a sequence of vertices $(x_{0,j}, x_{1,j+1}, \ldots, x_{k-1,j+k-1}, x_{0,j+k}, x_{1,j+k+1}, \ldots, x_{k-1,j-1})$ of length l. It is easy to observe that $l = \operatorname{lcm}(k, m)$, the least common multiple of k and m.

To simplify notation, we may denote the diagonal by $D^j = (d_0^j, d_1^j, \ldots, d_{l-1}^j)$. Similarly, by a *backward diagonal* we understand a sequence $(x_{0,j}, x_{1,j-1}, \ldots, x_{k-1,j-(k-1)}, x_{0,j-k}, x_{1,j-(k+1)}, \ldots, x_{k-1,j+1})$ denoted by $B^j = (b_0^j, b_1^j, \ldots, b_{l-1}^j)$.

First we observe that for k, m both odd, no $C_k \square C_m$ is Z_{km}-distance magic.

Lemma 2.1. *When k, m are both odd, then $C_k \square C_m$ is not Z_{km}-distance magic.*

Proof. Obviously, when k and m are both odd, then $l = \operatorname{lcm}(k, m)$ is also odd, say $l = 2s + 1$. We proceed by contradiction and assume that a Z_{km}-distance magic labeling of $C_k \square C_m$ exists. We observe that the number of diagonals is $g = km/l = \gcd(k, m)$.

First we assume that $g \geq 3$. We know that for any i, j we have $w(x_{i,j}) = x_{i,j-1} + x_{i-1,j} + x_{i,j+1} + x_{i+1,j} = \mu$. In particular, we have $w(x_{0,1}) = x_{0,0} + x_{k-1,1} + x_{0,2} + x_{1,1}$ and $w(x_{1,2}) = x_{1,1} + x_{0,2} + x_{1,3} + x_{2,2}$. In terms of diagonals, we have

$$w(x_{0,1}) = d_0^0 + d_{l-1}^2 + d_0^2 + d_1^0$$

and

$$w(x_{1,2}) = d_1^0 + d_0^2 + d_1^2 + d_2^0.$$

Since $w(x_{0,1}) = w(x_{1,2}) = \mu$, it follows that

$$d_0^0 + d_{l-1}^2 = d_2^0 + d_1^2.$$

By repeating the same argument for vertices d^1_{2i} and d^1_{2i+1} for $1 \leq i \leq s$, we obtain

$$d^0_{2i} + d^2_{2i-1} = d^0_{2i+2} + d^2_{2i+1}$$

for any i and consequently

$$d^0_0 + d^2_{l-1} = d^0_{2t} + d^2_{2t-1}$$

for any t. As $l = 2s+1$, for $t = s+1$ we have $d^0_{2t} + d^2_{2t-1} = d^0_{2s+2} + d^2_{2s+1}$ and we can see that $d^0_{2s+2} + d^2_{2s+1} = d^0_1 + d^2_0$ since the calculations in the subscript are performed modulo $l = 2s+1$. It follows that

$$d^0_0 + d^2_{l-1} = d^0_1 + d^2_0.$$

Now we look again at $w(x_{0,1})$ and consider the backward diagonals $B^0 = (b^0_0, b^0_1, \ldots, b^0_{l-1})$ and $B^2 = (b^2_1, b^2_2, \ldots, b^2_{l-1}, b^2_0)$. Following a similar argument as above, we obtain

$$b^0_0 + b^2_1 = b^0_{l-1} + b^2_0.$$

We have $d^0_0 = b^0_0 = x_{0,0}, d^0_1 = b^2_1 = x_{1,1}, d^2_{l-1} = b^0_{l-1} = x_{k-1,1}$, and $d^2_0 = b^2_0 = x_{0,2}$, and thus we obtain

$$x_{0,0} + x_{k-1,1} = x_{1,1} + x_{0,2} \tag{1}$$

and

$$x_{0,0} + x_{1,1} = x_{k-1,1} + x_{0,2}. \tag{2}$$

By subtracting, (1) − (2) gives $x_{k-1,1} - x_{1,1} = x_{1,1} - x_{k-1,1}$ which implies $2x_{k-1,1} = 2x_{1,1}$. However, the calculation is performed in Z_{km} where km is odd, and hence it follows that $x_{k-1,1} = x_{1,1}$. This is clearly impossible, since the labeling is an injection, and no two vertices can have the same label.

Now we check the remaining case when $l = km$ and there is only one diagonal, D^0. Then

$$w(x_{0,1}) = x_{0,0} + x_{k-1,1} + x_{0,2} + x_{1,1} = d^0_0 + d^0_z + d^0_{z+1} + d^0_1$$

for some z with $z \not\equiv 0 \pmod{l}$ and

$$w(x_{1,2}) = x_{1,1} + x_{0,2} + x_{1,3} + x_{2,2} = d^0_1 + d^0_{z+1} + d^0_{z+2} + d^0_2.$$

Since $w(x_{0,1}) = w(x_{1,2}) = \mu$, it again follows that

$$d^0_0 + d^0_z = d^0_2 + d^0_{z+2}$$

and

$$d^0_0 + d^0_z = d^0_t + d^0_{z+t}$$

for any t. Setting $t = z$, we get

$$d^0_0 + d^0_z = d^0_z + d^0_{2z}$$

and hence $d^0_0 = d^0_{2z}$. Since l is odd and $z \not\equiv 0 \pmod{l}$, $2z \not\equiv 0 \pmod{l}$. We know that $d^0_0 = x_{0,0}$ and $d^0_{2z} = x_{i,j}$ for some pair $(i,j) \neq (0,0)$. Consequently, $x_{0,0} = x_{i,j}$. This is again impossible, since the labeling is an injection. Therefore, no such labeling exists and the proof is complete. □

Now we show that when at least one of k, m is even, then $C_k \Box C_m$ is Z_{km}-distance magic.

Lemma 2.2. *$C_k \Box C_m$ is Z_{km}-distance magic whenever at least one of k, m is even.*

Proof. Since at least one of k, m is even, $l = \mathrm{lcm}(k, m)$ is even, say $l = 2s$. Denote $g = km/l = \gcd(k, m)$ and $h = 2g$ and notice that g is the number of diagonals of $C_k \Box C_m$. Denote by H the subgroup of Z_{km} generated by h. Obviously, H is of order $l/2 = s \geq 2$ since $m \geq k \geq 3$. Now take the diagonal D^0 and fill the even entries by the elements of H in increasing order, and the odd entries by the elements of the coset $H - 1$ in decreasing order; namely

$$d_0^0 = 0, d_2^0 = h, d_4^0 = 2h, \ldots, d_{l-2}^0 = (s-1)h$$

and

$$d_1^0 = km - 1, d_3^0 = km - h - 1, d_5^0 = km - 2h - 1, \ldots, d_{l-1}^0 = h - 1.$$

If $g > 1$, then fill the even entries of diagonal D^1 similarly with the elements of the coset $H + 1$ in increasing order and the odd entries with the elements of the coset $H - 2$ in decreasing order. For each diagonal $D^i, 2 \leq i < g$ (if more diagonals exist) fill the even entries with the elements of the coset $H + i$ in increasing order and the odd entries with the elements of the coset $H - i - 1$ in decreasing order.

Recall that $d_{2c}^a = ch + a$ and $d_{2c+1}^a = -(ch + a + 1)$. When $g > 2$, every vertex $x_{i,j}$ has in $C_k \Box C_m$ neighbors $d_b^a, d_{b+1}^a, d_{b-1}^{a+2}, d_b^{a+2}$ for some values a, b.

When b is even, then

$$d_b^a + d_{b+1}^a = d_{2c}^a + d_{2c+1}^a = (ch + a) - (ch + a + 1) = km - 1 \tag{3}$$

and

$$d_{b-1}^{a+2} + d_b^{a+2} = d_{2c-1}^{a+2} + d_{2c}^{a+2} = -((c-1)h + (a+2) + 1) + (ch + a + 2) = h - 1. \tag{4}$$

When b is odd, then

$$d_b^a + d_{b+1}^a = d_{2c+1}^a d_{2c+2}^a = -(ch + a + 1) + (c+1)h + a = h - 1 \tag{5}$$

and

$$d_{b-1}^{a+2} + d_b^{a+2} = d_{2c}^{a+2} + d_{2c+1}^{a+2} + = (ch + a + 2) - (ch + (a+2) + 1) = km - 1. \tag{6}$$

Adding (3) + (4) and (5) + (6), in both cases we get $w(x_{i,j}) = h - 2$. Hence, the labeling is Z_{km}-distance magic as desired.

When $g \leq 2$, then a vertex $x_{i,j}$ has in $C_k \Box C_m$ neighbors $d_b^a, d_{b+1}^a, d_{t-1}^a, d_t^a$ for $0 \leq a \leq 1$ and $0 \leq b < t \leq l - 1$. We know that at least one of k, m is even, so we can assume that $k = 2s$. Since $d_b^a = x_{i,j-1}$ and $d_t^a = x_{i,j+1}$, it is clear that $t = b + qk$ for some $1 \leq q < l/k$. But $k = 2s$ and $t = b + 2qs$ and hence b and t have the same parity.

When b and t are even, say $b = 2c$ and $t = 2r$, then

$$d^a_b + d^a_{b+1} = d^a_{2c} + d^a_{2c+1} = (ch + a) - (ch + a + 1) = km - 1 \tag{7}$$

and

$$d^a_{t-1} + d^a_t = d^a_{2r-1} + d^a_{2r} = -((r - 1)h + a + 1) + (rh + a) = h - 1. \tag{8}$$

When b and t are odd, say $b = 2c + 1$ and $t = 2r + 1$, then

$$d^a_b + d^a_{b+1} = d^a_{2c+1} + d^a_{2c+2} = -(ch + a + 1) + (c + 1)h + a = h - 1 \tag{9}$$

and

$$d^a_{t-1} + d^a_t = d^a_{2r} + d^a_{2r+1} + = (rh + a) - (rh + a + 1) = km - 1. \tag{10}$$

Adding again (7)+(8) and (9)+(10), we get in both cases $w(x_{i,j}) = h-2$. Hence, the labeling is Z_{km}-distance magic even in this case, which completes the construction. □

Combining Lemmas 2.1 and 2.2, we immediately get the complete characterization of Z_n-distance magic cycle products.

Theorem 2.3. *The product $C_k \Box C_m$ is Z_{km}-distance magic if and only if at least one of k, m is even.*

We observe that while the magic constant μ in the above construction was uniquely determined by $\mu = h - 2$, we can obtain other values of μ as well. If we simply increase each label by any $p \in Z_{km}$, then the weight of every vertex increases by $4p$. We also notice that when both k and m are even, then $h \equiv 0 \pmod 4$ since $h = 2\gcd(k, m)$. Then $\mu = h-2 \equiv 2 \pmod 4$ and by adding any p to all labels we always get a new magic constant $\mu^* \equiv 2 \pmod 4$.

Similarly, when $k \equiv 0 \pmod 4$ and m is odd, then $h \equiv 2 \pmod 4$ and $\mu = h - 2 \equiv 0 \pmod 4$. Then by adding any p to all labels we get a new magic constant $\mu^* \equiv 0 \pmod 4$.

Finally, when $k \equiv 2 \pmod 4$ and m is odd, then again $h \equiv 2 \pmod 4$ and $\mu = h - 2 \equiv 0 \pmod 4$. This time, however, by adding all possible values of p to the vertex labels we get all even elements of Z_{km} as the magic constant μ^*, since $km \equiv 2 \pmod 4$.

We summarize the findings as follows.

Observation 2.4. *When both k and m are even, then there exists a Z_{km}-distance magic labeling of $C_k \Box C_m$ for any magic constant $\mu \in Z_{km}, \mu \equiv 2 \pmod 4$. When $k \equiv 0 \pmod 4$ and m is odd, then there exists a Z_{km}-distance magic labeling of $C_k \Box C_m$ for any magic constant $\mu \in Z_{km}, \mu \equiv 0 \pmod 4$. When $k \equiv 2 \pmod 4$ and m is odd, then there exists a Z_{km}-distance magic labeling of $C_k \Box C_m$ for any magic constant $\mu \in Z_{km}, \mu \equiv 0 \pmod 2$.*

3 Z_2^{2k}-distance magic labeling of $C_{2^k}\square C_{2^k}$

For the product $C_m\square C_m$ we have found one more instance of a group distance labeling. Namely, when $m=2^k$, then we can consider group Z_2^{2k}. We construct the labeling recursively and start with the following.

Observation 3.1. *The graph $C_4\square C_4$ has a Z_2^4-distance magic labeling for the magic constant $\mu=(0,0,0,0)$.*

Proof. $C_4\square C_4$ can be labeled as follows:

0000	1000	0010	0110
0101	0011	0111	1101
1110	1010	1100	0100
1011	0001	1001	1111

□

Now we use the labeling of $C_4\square C_4$ to construct the labeling of $C_{2^k}\square C_{2^k}$ for $k>2$ recursively.

Theorem 3.2. *The graph $C_{2^k}\square C_{2^k}$ has a Z_2^{2k}-distance magic labeling for $k\geq 2$ and the magic constant $\mu=(0,0,\ldots,0)$.*

Proof. Let $k\geq 2$. We observe that finding a labeling of $C_{2^k}\square C_{2^k}$ is equivalent to finding a $2^k\times 2^k$ array $A=(a_{i,j})$ with entries being all distinct elements of Z_2^{2k} such that for every i,j with $1\leq i,j\leq 2^k$ it holds that $a_{i-1,j}+a_{i+1,j}+a_{i,j-1}+a_{i,j+1}=\mu$ where the subscript operations are performed in Z_{2^k} with 0 replaced by 2^k.

Now let A be a $2^k\times 2^k$ array corresponding to a Z_2^{2k}-distance magic labeling of $C_{2^k}\square C_{2^k}$. We construct two $2^{k+1}\times 2^{k+1}$ arrays, B and C with entries $b_{i,j}^{00},b_{i,j}^{01},b_{i,j}^{10},b_{i,j}^{11}$ and $c_{i,j}^{00},c_{i,j}^{01},c_{i,j}^{10},c_{i,j}^{11}$ for every $1\leq i,j\leq 2^k$, respectively. The entries of B are elements of Z_2^{2k} while the entries of C are elements of Z_2^2.

Informally speaking, to construct B or C, we replace an entry $a_{i,j}$ in A by four entries of a 2×2 matrix $B_{i,j}$ or $C_{i,j}$ defined as $\begin{pmatrix} b_{i,j}^{00} & b_{i,j}^{01}\\ b_{i,j}^{10} & b_{i,j}^{11}\end{pmatrix}$ or $\begin{pmatrix} c_{i,j}^{00} & c_{i,j}^{01}\\ c_{i,j}^{10} & c_{i,j}^{11}\end{pmatrix}$, respectively. More formally, for any i with $1\leq i\leq 2^{k+1}$, a row $2i-1$ in B consists of entries $b_{i,1}^{00},b_{i,1}^{01},b_{i,2}^{00},b_{i,2}^{01},\ldots,b_{i,2^k}^{00},b_{i,2^k}^{01}$ and a row $2i$ in B consists of entries $b_{i,1}^{10},b_{i,1}^{11},b_{i,2}^{10},b_{i,2}^{11},\ldots,b_{i,2^k}^{10},b_{i,2^k}^{11}$. Array C is constructed in a similar manner.

For even $i+j$ we set

$$b_{i,j}^{00}=b_{i,j}^{11}=b_{i,j+1}^{00}=b_{i-1,j}^{11}=a_{i,j}.$$

It means that all neighbors of the vertex corresponding to entry $b_{i,j}^{01}$ in $C_{2^{k+1}}\square C_{2^{k+1}}$ receive temporarily the same label as the vertex corresponding to entry $a_{i,j}$ in $C_{2^k}\square C_{2^k}$.

For odd $i+j$ we set

$$b_{i,j}^{01}=b_{i,j}^{10}=b_{i+1,j}^{01}=b_{i,j+1}^{10}=a_{i,j}.$$

Here the neighbors of the vertex corresponding to entry $b^{11}_{i,j}$ in $C_{2^{k+1}}\square C_{2^{k+1}}$ receive temporarily the same label as the vertex corresponding to entry $a_{i,j}$ in $C_{2^k}\square C_{2^k}$.

One can verify that for even $i+j$ we have the sum of neighbors of $b^{00}_{i,j}$ equal to

$$b^{10}_{i-1,j} + b^{01}_{i,j} + b^{01}_{i,j-1} + b^{10}_{i,j} = (0,0,\ldots,0)$$

because $b^{10}_{i-1,j} = b^{01}_{i,j} = a_{i,j-1}$ implies $b^{10}_{i-1,j} + b^{01}_{i,j} = (0,0,\ldots,0)$ and $b^{01}_{i,j-1} = b^{10}_{i,j} = a_{i+1,j-1}$ implies $b^{01}_{i,j-1} + b^{10}_{i,j} = (0,0,\ldots,0)$.

Again for even $i+j$ we have the sum of neighbors of $b^{01}_{i,j}$ equal to $(0,0,\ldots,0)$ because the neighbors are

$$b^{00}_{i,j} = b^{11}_{i,j} = b^{00}_{i,j+1} = b^{11}_{i-1,j} = a_{i,j}.$$

The sum of neighbors of $b^{10}_{i,j}$ for even $i+j$ is equal to

$$b^{00}_{i,j} + b^{11}_{i,j} + b^{00}_{i,j-1} + b^{11}_{i+1,j} = (0,0,\ldots,0)$$

because $b^{00}_{i,j} = b^{11}_{i,j} = a_{i,j}$ implies $b^{00}_{i,j} + b^{11}_{i,j} = (0,0,\ldots,0)$ and $b^{00}_{i,j-1} = b^{11}_{i+1,j} = a_{i+1,j-1}$ implies $b^{00}_{i,j-1} + b^{11}_{i+1,j} = (0,0,\ldots,0)$.

The sum of neighbors of $b^{11}_{i,j}$ for even $i+j$ is equal to

$$b^{01}_{i,j} + b^{10}_{i,j+1} + b^{01}_{i,j} + b^{10}_{i+1,j} = a_{i-1,j} + a_{i,j+1} + a_{i,j-1} + a_{i+1,j} = (0,0,\ldots,0)$$

which follows from the construction of A.

Similarly, for $i+j$ odd we have the sum of neighbors of $b^{00}_{i,j}$ equal to $(0,0,\ldots,0)$ because $b^{01}_{i,j-1} = b^{10}_{i-1,j} = a_{i-1,j-1}$ implies $b^{01}_{i,j-1} + b^{10}_{i-1,j} = (0,0,\ldots,0)$ and $b^{01}_{i,j} = b^{10}_{i,j} = a_{i,j}$ implies $b^{01}_{i,j} = b^{10}_{i,j} = (0,0,\ldots,0)$.

The sum of neighbors of $b^{01}_{i,j}$ is now equal to

$$b^{00}_{i,j} + b^{00}_{i,j+1} + b^{11}_{i-1,j} + b^{11}_{i,j} = a_{i,j-1} + a_{i,j+1} + a_{i-1,j} + a_{i+1,j} = (0,0,\ldots,0)$$

as follows from the construction of A.

The sum of neighbors of $b^{10}_{i,j}$ is equal to $(0,0,\ldots,0)$ because $b^{11}_{i,j-1} = b^{00}_{i,j} = a_{i,j-1}$ implies $b^{11}_{i,j-1} + b^{00}_{i,j} = (0,0,\ldots,0)$ and $b^{11}_{i,j} = b^{00}_{i+1,j} = a_{i+1,j}$ implies $b^{11}_{i,j} = b^{00}_{i+1,j} = (0,0,\ldots,0)$.

Finally, the sum of neighbors of $b^{11}_{i,j}$ is equal to $(0,0,\ldots,0)$ because the neighbors are

$$b^{01}_{i,j} = b^{10}_{i,j} = b^{01}_{i,j+1} = b^{10}_{i+1,j} = a_{i,j}.$$

In array C, for every i,j with $1 \leq i,j \leq 2^k$, we set $c^{00}_{i,j} = c^{01}_{i,j} = (0,0), c^{10}_{i,j} = (0,1), c^{11}_{i,j} = (1,1)$ if $i+j$ is odd and $c^{00}_{i,j} = c^{01}_{i,j} = (1,0), c^{10}_{i,j} = (1,1), c^{11}_{i,j} = (0,1)$ if $i+j$ is even. It is clear that the neighbors of every entry sum up to $(0,0)$, since the neighbors of every (r,s) are $(0,0),(0,1),(1,0),(1,1)$.

Now we define a $2^{k+1} \times 2^{k+1}$ array $D = (d_{i,j})$ with entries $d^{rs}_{i,j} = (b^{rs}_{i,j}, c^{rs}_{i,j})$. This means that we attach the pair of entries of $c^{rs}_{i,j}$ after the $(2k)$-tuple of entries of $b^{rs}_{i,j}$ to obtain $d^{rs}_{i,j}$.

It now follows from the constructions of B and C that the neighbors of every $d_{i,j}^{rs}$ have the sum equal to $(0,0,\ldots,0) \in Z_2^{2k+2}$. It can be verified that the entries of D are all distinct elements of Z_2^{2k+2}. $\square$

4 Conclusion

We have completely characterized all Cartesian products $C_k \square C_m$ that allow a group distance magic labeling by Z_{km}. On the other hand, we found a class of cycle products allowing a group distance magic labeling by the group $Z_2 \times Z_2 \times \cdots \times Z_2$. It would be interesting to find other classes of $C_k \square C_m$ that admit group distance magic labeling by other abelian groups. Hence, we pose the following problem.

Problem 4.1. *For a given graph $C_k \square C_m$, determine all abelian groups Γ such that the graph $C_k \square C_m$ admits a Γ-distance magic labeling.*

References

[1] S. Arumugam, D. Froncek and N. Kamatchi, Distance Magic Graphs—A Survey, *J. Indones. Math. Soc.* Special Ed. (2011), 1–9.

[2] J. Gallian, A Dynamic Survey of Graph Labeling, *Electr. J. Combin.* **18** (2011), # DS6.

[3] S.B. Rao, T. Singh and V. Parameswaran, Some sigma labelled graphs I, in *Graphs, Combinatorics, Algorithms and Applications*, eds. S. Arumugam, B.D. Acharya and S.B. Rao, Narosa Publishing House, New Delhi (2004), 125–133.

[4] V. Vilfred, *Σ-labelled Graph and Circulant Graphs*, Ph.D. Thesis, University of Kerala, Trivandrum, India, 1994.

[5] W.D. Wallis, *Magic Graphs*, Birkhäuser, Boston-Basel-Berlin, 2001.

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