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Perfect difference families and related variable-weight optical orthogonal codes

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Abstract

Perfect $(v, K, 1)$ difference families ($(v, K, 1)$-PDF in short, and $(v, k, 1)$-PDF when $K = \{k\}$) were introduced by Ge et al. for their useful application to the construction of properly centered permutations, which can be used to construct new radar arrays. Some works had been done on the existences of $(v, k, 1)$-PDFs, while little is known when $|K| \geq 2$ except for partial results by Ge et al. In 1996, Yang introduced variable-weight optical orthogonal code (variable-weight OOC in short) for multimedia optical CDMA systems with multiple quality of service requirements. Jiang et al. presented general constructions of optimal variable-weight OOCs via $(v, K, 1)$-PDFs. Suppose K is a set of of positive integers, s is a positive integer, and $s \notin K$, a $(v, \{K, s^*\}, 1)$-PDF is a $(v, K \cup \{s\}, 1)$-PDF that contains only one block of size s. In this paper, by using perfect Langford sequences, the existences of $(v, \{3, s^*\}, 1)$-PDFs are completely solved for $4 \leq s \leq 7$. New optimal variable-weight OOCs are then obtained.

1 Introduction

Optical orthogonal codes (OOCs) were introduced by Salehi, as signature sequences to facilitate multiple access in optical fibre networks [12, 13]. Most existing works on OOCs have assumed that all codewords have the same weight. In general, the

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code size of OOCs depends upon the weights of codewords. Variable-weight OOCs can generate larger code size than that of constant-weight OOCs [8]. In 1996, Yang introduced multimedia optical CDMA communication system employing variable-weight OOCs [16].

For variable-weight OOCs, we will use the notations in [4]. Let $W = \{w_1, \ldots, w_r\}$ be an ordering of a set of r integers greater than 1, $\Lambda_a = \{\lambda_a^{(1)}, \ldots, \lambda_a^{(r)}\}$ an r-tuple (*auto-correlation sequence*) of positive integers, λ_c a positive integer (*cross-correlation parameter*), and $Q = (q_1, \ldots, q_r)$ an r-tuple (*weight distribution sequence*) of positive rational numbers whose sum is 1. If $|W| = 1$, then it is certain that $Q = (1)$. The reader may refer to [4] for the detailed definition for a $(v, W, \Lambda_a, \lambda_c, Q)$ variable-weight optical orthogonal code $\mathcal{C}$, or $(v, W, \Lambda_a, \lambda_c, Q)$-OOC. If $\lambda_a^{(i)} = \lambda_a$ for every i, one simply says that $\mathcal{C}$ is a $(v, W, \lambda_a, \lambda_c, Q)$-OOC. Also, speaking of a (v, W, λ, Q)-OOC one means a $(v, W, \lambda_a, \lambda_c, Q)$-OOC where $\lambda_a = \lambda_c = \lambda$. The term variable-weight optical orthogonal code, or variable-weight OOC, is also used if there is no need to list the parameters.

In [9], perfect difference families are used to construct variable-weight OOCs. For $B = \{x_1, \ldots, x_n\} \subset Z_v$ with $x_1 < x_2 < \cdots < x_n$, define $\Delta^+ B = \{x_j - x_i : 1 \le i < j \le n\}$. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_h\}$, where $B_i = \{x_{i1}, x_{i2}, \ldots, x_{ik_i}\}$ is a collection of h subsets of Z_v with increasing orders, called *blocks*. If the differences $\Delta^+\mathcal{B} = \bigcup_{i=1}^{h} \Delta^+ B_i = \{x_{im} - x_{in} : i = 1, 2, \ldots, h, 1 \le n < m \le k_i\}$ cover the set $\{1, 2, \ldots, (v-1)/2\}$, then $\mathcal{B}$ is called a *perfect* $(v, K, 1)$ difference family, or briefly, a $(v, K, 1)$-PDF, where $K = \{k_1, k_2, \ldots, k_h\}$. $(v, K, 1)$-PDFs were introduced by Ge et al. in [6] for their useful application to the construction of properly centered permutations, which can be used to construct new radar arrays. A $(v, k, 1)$-PDF is a $(v, K, 1)$-PDF with $K = \{k\}$. An obvious necessary condition for the existence of a $(v, k, 1)$-PDF is $v \equiv 1 \pmod{k(k-1)}$.

A $(k(k-1)t+1, k, 1)$-PDF is equivalent to a graceful labeling [11] of a graph with t connected components, all isomorphic to the complete graph on k vertices. It is also equivalent to regular perfect systems of difference sets starting with 1, which have been studied by many authors (see [3] and the references therein). Such a perfect difference family is a powerful tool to construct optimal constant-weight OOCs [2, 5]. For more about perfect difference families, the reader may refer to recent papers [7], [15] and the references therein.

In the following three results in [9], we always assume that a $(g, K, 1)$-PDF exists, where $K = \{k_1, k_2, \cdots, k_n\}$ is a set of distinct positive integers, s_l is the number of blocks of size k_l, $1 \le l \le n$, and $s = s_1 + s_2 + \cdots + s_n$.

Lemma 1.1 *Let* $q = 4t+1$ *be a prime,* $q \ge \max\{k_1, k_2, \ldots, k_n\}$*, and* $\gcd(q, g+2) = 1$.

(1) *If* $k_l \ne 4$, $1 \le l \le n$, *then there exists an optimal* $((g+2)q, \{4, k_1, \cdots, k_n\}, 1, (\frac{1}{4s+1}, \frac{4s_1}{4s+1}, \cdots, \frac{4s_n}{4s+1}))$*-OOC;*

(2) *If there exists an* $l \in [1,n]$ *such that* $k_l = 4$*, and* $\gcd(4s_l+1, 4s_1, \ldots, 4s_{l-1}, 4s_{l+1}, \ldots, 4s_n) = 1$*, then there exists an optimal* $((g+2)q, \{4, k_1, \ldots, k_{l-1}, k_{l+1}, \ldots, k_n\}, 1, (\frac{4s_l+1}{4s+1}, \frac{4s_1}{4s+1}, \ldots, \frac{4s_{l-1}}{4s+1}, \frac{4s_{l+1}}{4s+1}, \ldots, \frac{4s_n}{4s+1}))$*-OOC.*

Lemma 1.2 *Let* $q = 2t+1$ *be a prime,* $q \geq \max\{k_1, k_2, \ldots, k_n\}$*, and* $\gcd(q, g+2) = 1$.

(1) *If* $k_l \neq 3$, $1 \leq l \leq n$*, then there exists an optimal* $((g+2)q, \{3, k_1, \ldots, k_n\}, 1, (\frac{1}{2s+1}, \frac{2s_1}{2s+1}, \cdots, \frac{2s_n}{2s+1}))$*-OOC;*

(2) *If there exists an* $l \in [1,n]$ *such that* $k_l = 3$*, and* $\gcd(2s_l+1, 2s_1, \ldots, 2s_{l-1}, 2s_{l+1}, \ldots, 2s_n) = 1$*, then there exists an optimal* $((g+2)q, \{3, k_1, \ldots, k_{l-1}, k_{l+1}, \ldots, k_n\}, 1, (\frac{2s_l+1}{2s+1}, \frac{2s_1}{2s+1}, \ldots, \frac{2s_{l-1}}{2s+1}, \frac{2s_{l+1}}{2s+1}, \ldots, \frac{2s_n}{2s+1}))$*-OOC.*

Lemma 1.3 *Let* $q = 6t+1$ *be a prime,* $q \geq \max\{k_1, k_2, \ldots, k_n\}$*, and* $\gcd(q, g+1) = 1$.

(1) *If* $k_l \neq 4$, $1 \leq l \leq n$*, then there exist an optimal* $((g+1)q, \{4, k_1, \ldots, k_n\}, 1, (\frac{1}{6s+1}, \frac{6s_1}{6s+1}, \ldots, \frac{6s_n}{6s+1}))$*-OOC;*

(2) *If there exists an* $l \in [1,n]$ *such that* $k_l = 4$*, and* $\gcd(6s_l+1, 6s_1, \ldots, 6s_{l-1}, 6s_{l+1}, \ldots, 6s_n) = 1$*, then there exists an optimal* $((g+1)q, \{4, k_1, \ldots, k_{l-1}, k_{l+1}, \ldots, k_n\}, 1, (\frac{6s_l+1}{6s+1}, \frac{6s_1}{6s+1}, \ldots, \frac{6s_{l-1}}{6s+1}, \frac{6s_{l+1}}{6s+1}, \ldots, \frac{6s_n}{6s+1}))$*-OOC.*

Lemmas 1.1–1.3 provide constructions of variable-weight OOCs via perfect difference families. In the following, new infinite classes of perfect difference families are constructed, and new optimal variable-weight OOCs are then obtained.

2 Preliminaries and Nonexistence Results

Let K be a set of positive integers, s a positive integer such that $s \notin K$. A $(v, K \cup \{s^*\}, 1)$-PDF $\mathcal{B}$ is a $(v, K \cup \{s\}, 1)$-PDF with the property that $\mathcal{B}$ contains one and only one block of size s. When $K = \{k\}$, we write simply $(v, \{k, s^*\}, 1)$-PDF. It is easy to see that the necessary conditions for the existence of a $(v, \{k, s^*\}, 1)$-PDF is $v - s(s-1) \equiv 1 \pmod{k(k-1)}$ (thus v is odd), and $v \geq s(s-1) + k(k-1) + 1$.

Example 1 The following are some examples of $(v, \{3, s^*\}, 1)$-PDFs for $s = 4, 5, 6, 7$.
$(v, s) = (25, 4)$ $\{0, 1, 3, 10\}$, $\{0, 4, 12\}$, $\{0, 5, 11\}$
$(v, s) = (33, 5)$ $\{0, 1, 6, 14, 16\}$, $\{0, 3, 12\}$, $\{0, 4, 11\}$.
$(v, s) = (43, 6)$ $\{0, 1, 3, 11, 16, 20\}$, $\{0, 6, 18\}$, $\{0, 7, 21\}$.
$(v, s) = (49, 6)$ $\{0, 1, 5, 7, 19, 22\}$, $\{0, 8, 24\}$, $\{0, 9, 20\}$, $\{0, 10, 23\}$.
$(v, s) = (73, 7)$ $\{0, 3, 14, 21, 27, 31, 36\}$, $\{0, 2, 25\}$, $\{0, 8, 34\}$, $\{0, 12, 32\}$, $\{0, 16, 35\}$, $\{0, 29, 30\}$.

The existence of $(v, 3, 1)$-PDFs was completely solved (see [1]). Partial results were obtained for $(v, k, 1)$-PDFs when $k = 4, 5$ (see [1], [7]), and a significant advance for $(v, 4, 1)$-PDFs can be found in [7]. It was also proved that there do not exist $(v, k, 1)$-PDFs for $k \geq 6$ (see [10]). For $|K| > 1$, little is known about the existence of $(v, K, 1)$-PDFs except for the $(v, \{4, 5\}, 1)$-PDFs in [6], where $57 \leq v \leq 149$.

In this paper, we focus our attention on the existence of $(v, \{3, s^*\}, 1)$-PDFs for $4 \leq s \leq 7$. By using perfect Langford sequences, the existence of $(v, \{3, s^*\}, 1)$-PDFs for $4 \leq s \leq 7$ is completely solved. The following results are obtained.

Theorem 2.1

(1) *There exists a* $(v, \{3, 4^*\}, 1)$-PDF *if and only if* $v \equiv 1 \pmod 6$, *and* $v \geq 19$*;*

(2) *There exists a* $(v, \{3, 5^*\}, 1)$-PDF *if and only if* $v \equiv 9, 15 \pmod{24}$, *and* $v \geq 33$*;*

(3) *There exists a* $(v, \{3, 6^*\}, 1)$-PDF *if and only if* $v \equiv 1 \pmod 6$, *and* $v \geq 43$*;*

(4) *There exists a* $(v, \{3, 7^*\}, 1)$-PDF *if and only if* $v \equiv 1, 7 \pmod{24}$, *and* $v \geq 73$.

In the rest of this section, some nonexistence results are given.

Lemma 2.2 *There exists no* $(v, \{3, s^*\}, 1)$-PDF *if one of the following conditions holds:*
(1) $v - s(s-1) \equiv 13, 19 \pmod{24}$ *when* $s \equiv 1 \pmod 8$*;*
(2) $v - s(s-1) \equiv 7, 13 \pmod{24}$ *when* $s \equiv 3 \pmod 8$*;*
(3) $v - s(s-1) \equiv 1, 7 \pmod{24}$ *when* $s \equiv 5 \pmod 8$*;*
(4) $v - s(s-1) \equiv 1, 19 \pmod{24}$ *when* $s \equiv 7 \pmod 8$.

Proof We only prove that the conclusion is true for condition (1). For conditions (2)–(4), the proof is similar.

Suppose $\mathcal{B}$ is a $(v, \{3, s^*\}, 1)$-PDF. For each block $C = \{c_1, c_2, \ldots, c_h\} \in \mathcal{B}$, $h \in \{3, s\}$, let N_C^e be the number of even numbers in $\Delta^+ C = \{c_j - c_i : 1 \leq i < j \leq h\}$, and N the number of even numbers in $\Delta^+\mathcal{B}$. For condition (1), let $s = 8t + 1$ and $v = 24f + s(s-1) + x = 24f + 64t^2 + 8t + x$, where $x = 13,\ 19$. Let $A = \{x_1, x_2, \ldots, x_s\}$ be the unique block of size s, and let a be the number of even numbers in A; then we have $N_A^e = a(a-1)/2 + (s-a)(s-a-1)/2 = a[a - (8t+1)] + 32t^2 + 4t$. It is clear that N_A^e is even. For each block $B = \{y_1, y_2, y_3\}$ of size 3, N_B^e is 1 or 3. Let n be the number of blocks of size 3 in $\mathcal{B}$. Then $n = 4f + (x-1)/6$. Also, we have $N = N_A^e + \sum_{B \in \mathcal{B} \setminus A} N_B^e$. If $x = 13$, then we have $v = 24f + 64t^2 + 8t + 13$, $\frac{v-1}{2} = 12f + 32t^2 + 4t + 6$, and thus $N = 6f + 16t^2 + 2t + 3$ is odd. In this case $n = 4f + 2$ is even, and hence $\sum_{B \in \mathcal{B} \setminus A} N_B^e$ is even. Since N_A^e is even, $N = N_A^e + \sum_{B \in \mathcal{B} \setminus A} N_B^e$ is even, a contradiction.

If $x = 19$, then we have $v = 24f + 64t^2 + 8t + 19$, $\frac{v-1}{2} = 12f + 32t^2 + 4t + 9$; thus $N = 6f + 16t^2 + 2t + 4$ is even. In this case $n = 4f + 3$ is odd, and hence $\sum_{B \in \mathcal{B} \setminus A} N_B^e$

is odd. Since N_A^e is even, $N = N_A^e + \sum_{B \in \mathcal{B} \setminus A} N_B^e$ is odd, also a contradiction. This completes the proof. ∎

3 A Construction via Perfect Langford Sequences

For a given subset $B = \{x_1, x_2, \ldots, x_n\}$ of Z_v with $x_1 < x_2 < \cdots < x_n$, and a family $\mathcal{B}$ of subsets of Z_v with increasing orders, $\Delta B^+ = \{x_j - x_i : 1 \le i < j \le n\}$ and $\Delta \mathcal{B}^+ = \bigcup_{B \in \mathcal{B}} \Delta B^+$ are defined to be the same as in Section 1.

To construct perfect difference families, perfect Langford sequences will be used. The following definition of a perfect Langford sequence is from [14]. A sequence $\{c, c+1, \ldots, c+m-1\}$ is a *perfect Langford sequence* (PLS(m, c) in short) starting with c if the set $\{1, 2, \ldots, 2m\}$ can be arranged in disjoint pairs (a_i, b_i), where $i = 1, \ldots, m$ so that $\{b_1 - a_1, b_2 - a_2, \ldots, b_m - a_m\} = \{c, c+1, \ldots, c+m-1\}$.

The existence of perfect Langford sequences had been completely solved in [14]. The following result can be found in [14, Theorem 1].

Theorem 3.1 *A* PLS(m, c) *exists if and only if*
(1) $m \ge 2c - 1$*;*
(2) $m \equiv 0, 1 \pmod 4$ *when c is odd;* $m \equiv 0, 3 \pmod 4$ *when c is even.*

The following result presents a construction of perfect difference families from a given $(v, K, 1)$-PDF and PLS$(m, (v+1)/2)$s.

Lemma 3.2 *Suppose that there exists a* $(u, K, 1)$-PDF *and a* PLS$(m, (u+1)/2)$*, then there exists a* $(6m + u, K \cup \{3\}, 1)$-PDF.

Proof Let $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ be a $(u, K, 1)$-PDF, then $\Delta^+\mathcal{B} = \{1, 2, \ldots, (u-1)/2\}$. Let (a_i, b_i), $i = 1, \ldots, m$, be the pairs arrangement of the PLS$(m, (u+1)/2)$. Put $x_{i0} = 0$, $x_{i1} = a_i + m + (u+1)/2 - 1$, $x_{i2} = b_i + m + (u+1)/2 - 1$, $S_i = \{x_{i0}, x_{i1}, x_{i2}\}$, $i = 1, \ldots, m$, $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$, then $\Delta^+\mathcal{S} = \bigcup_{1 \le i \le m} \Delta^+ S_i = \bigcup_{1 \le i \le m} \{x_{i1} - x_{i0}, x_{i2} - x_{i1}, x_{i2} - x_{i0}\} = \bigcup_{1 \le i \le m} \{b_i - a_i, a_i + m + (u+1)/2 - 1, b_i + m + (u+1)/2 - 1\} = \{(u+1)/2, \ldots, (u+1)/2 + 3m - 1\}$, and hence $\Delta^+\mathcal{B} \cup \Delta^+\mathcal{S} = \{1, 2, \ldots, (u+1)/2 + 3m - 1\}$. Thus, $\mathcal{B} \cup \mathcal{S}$ forms a $(6m + u, K \cup \{3\}, 1)$-PDF. ∎

Example 2 A $(91, \{3, 4^*\}, 1)$-PDF from a $(13, 4, 1)$-PDF $\mathcal{B}_{13} = \{\{0, 1, 4, 6\}\}$, and a PLS$(13, 7)$.

The following is a pairs arrangement of a PLS$(13, 7)$:
$(8, 15)$, $(13, 21)$, $(7, 16)$, $(4, 14)$, $(9, 20)$, $(10, 22)$, $(11, 24)$, $(12, 26)$, $(3, 18)$, $(1, 17)$, $(2, 19)$, $(5, 23)$, $(6, 25)$.

From the construction in Lemma 3.2, one can get

$$\mathcal{S} = \{\{0,27,34\},\{0,32,40\},\{0,26,35\},\{0,23,33\},\{0,28,39\},\{0,29,41\},\{0,30,43\},\{0,31,45\},\{0,22,37\},\{0,20,36\},\{0,21,38\},\{0,24,42\},\{0,25,44\}\}.$$

It is easy to check that $\mathcal{B}_{13} \cup \mathcal{S}$ forms a $(91, \{3, 4^*\}, 1)$-PDF.

The following result can be obtained from Theorem 3.1 and Lemma 3.2.

Lemma 3.3 *Suppose that there exists a* $(u, K, 1)$-PDF*, then there exists a* $(6m + u, K \cup \{3\}, 1)$-PDF *provided that the following two conditions holds:*
(1) $m \geq u$;
(2) $m \equiv 0, 1 \pmod 4$ *when* $u \equiv 1 \pmod 4$; $m \equiv 0, 3 \pmod 4$ *when* $u \equiv 3 \pmod 4$.

4 Proof of Theorem 2.1

The following result is clear.

Lemma 4.1 *If there exists a* $(v, \{w, s^*\}, 1)$-PDF*, then*

$$v \equiv 1 \pmod 2,\ v - s(s-1) \equiv 1 \pmod{w(w-1)}, \text{ and } v \geq s(s-1) + w(w-1) + 1.$$

One can obtain the following result from Lemma 4.1.

Lemma 4.2 (1) *For a* $(v, \{3, 4^*\}, 1)$-PDF*, we have* $v \equiv 1 \pmod 6$*, and* $v \geq 19$; (2) *for a* $(v, \{3, 5^*\}, 1)$-PDF*, we have* $v \equiv 3 \pmod 6$*, and* $v \geq 27$; (3) *for a* $(v, \{3, 6^*\}, 1)$-PDF*, we have* $v \equiv 1 \pmod 6$*, and* $v \geq 37$; (4) *for a* $(v, \{3, 7^*\}, 1)$-PDF*, we have* $v \equiv 1 \pmod 6$*, and* $v \geq 49$.

4.1 $s = 4$

Lemma 4.3 *There exists a* $(v, \{3, 4^*\}, 1)$-PDF *for each* $v \equiv 13, 19 \pmod{24}$*, and* $v \geq 91$.

Proof A $(13, 4, 1)$-PDF exists from Example 2. Let $m \equiv 0, 1 \pmod 4$, and $m \geq 13$, then a PLS$(m, 7)$ exists from Lemma 3.1. One can obtain the result by applying Lemma 3.3 with $u = 13$, $m \equiv 0, 1 \pmod 4$, and $m \geq 13$. □

Lemma 4.4 *There exists a* $(v, \{3, 4^*\}, 1)$-PDF *for each* $v \equiv 1, 7 \pmod{24}$*, and* $v \geq 175$.

Proof A $(25, \{3, 4^*\}, 1)$-PDF exists from Example 1. Similar to the proof of Lemma 4.3, the result can be obtained by applying Lemma 3.3 with $u = 25$, $m \equiv 0, 1 \pmod 4$, and $m \geq 25$. □

Lemma 4.5 *There exists a* $(v, \{3, 4^*\}, 1)$-PDF *for each* $v \equiv 1 \pmod 6$*, and* $19 \le v < 175$.

Proof For each $v \equiv 1 \pmod 6$, and $19 \le v < 175$, with the aid of a computer, one can find a $(v, \{3, 4^*\}, 1)$-PDF. To save space, we only list the base blocks of $(v, \{3, 4^*\}, 1)$-PDFs in Appendix A for $19 \le v < 80$. For other values of v, we omit it, the interested reader may contact the first author to have a copy. □

From Lemmas 4.2–4.5, one can obtain the following result.

Lemma 4.6 *There exists a* $(v, \{3, 4^*\}, 1)$-PDF *if and only if* $v \equiv 1 \pmod 6$*, and* $v \ge 19$.

4.2 $s = 5$

The following result can be obtained by applying Lemma 2.2 with $s = 5$.

Lemma 4.7 *There does not exist a* $(v, \{3, 5^*\}, 1)$-PDF *for each* $v \equiv 3, 21 \pmod{24}$.

Lemma 4.8 *There exists a* $(v, \{3, 5^*\}, 1)$-PDF *for each* $v \equiv 9, 15 \pmod{24}$*, and* $v \ge 231$.

Proof A $(33, \{3, 5^*\}, 1)$-PDF exists from Example 1. Similar to the proof of Lemma 4.3, the result can be obtained by applying Lemma 3.3 with $u = 33$, $m \equiv 0, 1 \pmod 4$, and $m \ge 33$. □

Lemma 4.9 *There exists a* $(v, \{3, 5^*\}, 1)$-PDF *for each* $v \equiv 9, 15 \pmod{24}$*, and* $33 \le v < 231$.

Proof For each $v \equiv 9, 15 \pmod{24}$, and $33 \le v < 231$, with the aid of a computer, one can find a $(v, \{3, 5^*\}, 1)$-PDF. We list the base blocks of $(v, \{3, 5^*\}, 1)$-PDFs in Appendix B.

From Lemmas 4.2, 4.7-4.9, one can obtain the following result.

Lemma 4.10 *There exists a* $(v, \{3, 5^*\}, 1)$-PDF *if and only if* $v \equiv 9, 15 \pmod{24}$*, and* $v \ge 33$.

4.3 $s = 6$

Lemma 4.11 *There exists a* $(v, \{3, 6^*\}, 1)$-PDF *for each* $v \equiv 1 \pmod 6$*, and* $v \geq 343$.

Proof A $(43, \{3, 6^*\}, 1)$-PDF exists from Example 1. Similar to the proof of Lemma 4.3, one can obtain that there exists a $(v, \{3,6^*\}, 1)$-PDF for each $v \equiv 13,19 \pmod{24}$, and $v \geq 301$ by applying Lemma 3.3 with $u = 43$, $m \equiv 0, 3 \pmod 4$, and $m \geq 43$.

A $(49, \{3, 6^*\}, 1)$-PDF exists from Example 1, one can obtain a $(v, \{3, 6^*\}, 1)$-PDF for each $v \equiv 1, 7 \pmod{24}$, and $v \geq 343$ by applying Lemma 3.3 with $u = 49$, $m \equiv 0, 1 \pmod 4$, and $m \geq 49$. This completes the proof. ∎

Lemma 4.12 *There exists a* $(v, \{3, 6^*\}, 1)$-PDF *for each* $v \equiv 1 \pmod 6$*, and* $43 \leq v < 343$*. There does not exists a* $(37, \{3, 6^*\}, 1)$-PDF.

Proof A $(37, \{3, 6^*\}, 1)$-PDF does not exist by computer searching. For each $v \equiv 1 \pmod 6$, and $43 \leq v < 343$, with the aid of a computer, one can find a $(v, \{3, 6^*\}, 1)$-PDF. To save space, we only list the base blocks of $(v, \{3, 6^*\}, 1)$-PDFs in Appendix C for $43 \leq v < 100$. For other values of v, we omit it; the interested reader may contact the first author to have a copy. ∎

From Lemmas 4.2, 4.11–4.12, one can obtain the following result.

Lemma 4.13 *There exists a* $(v, \{3, 6^*\}, 1)$-PDF *if and only if* $v \equiv 1 \pmod 6$*, and* $v \geq 43$.

4.4 $s = 7$

The following result can be obtained by applying Lemma 2.2 with $s = 7$.

Lemma 4.14 *There does not exist a* $(v, \{3,7^*\}, 1)$-PDF *for each* $v \equiv 13,19 \pmod{24}$.

Lemma 4.15 *There exists a* $(v, \{3, 7^*\}, 1)$-PDF *for each* $v \equiv 1, 7 \pmod{24}$*, and* $v \geq 511$.

Proof A $(73, \{3, 7^*\}, 1)$-PDF exists from Example 1. Similar to the proof of Lemma 4.3, one can obtain that there exists a $(v, \{3, 7^*\}, 1)$-PDF for each $v \equiv 1, 7 \pmod{24}$, and $v \geq 511$ by applying Lemma 3.3 with $u = 73$, $m \equiv 0, 1 \pmod 4$, and $m \geq 73$. This completes the proof. ∎

Lemma 4.16 *There exists a* $(v, \{3, 7^*\}, 1)$-PDF *for each* $v \equiv 1, 7 \pmod{24}$*, and* $73 \leq v < 511$*. There does not exist a* $(v, \{3, 7^*\}, 1)$-PDF *for* $v \in \{49, 55\}$.

Proof For $v \in \{49, 55\}$, there does not exist a $(v, \{3, 7^*\}, 1)$-PDF by computer searching. For each $v \equiv 1, 7 \pmod{24}$, and $73 \le v < 511$, with the aid of a computer, one can find a $(v, \{3, 7^*\}, 1)$-PDF. To save space, we only list the base blocks of $(v, \{3, 7^*\}, 1)$-PDFs in Appendix D for $73 \le v < 200$. For other values of v, we omit it, the interested reader may contact the first author to have a copy. □

From Lemmas 4.2, 4.14–4.16, one can obtain the following result.

Lemma 4.17 *There exists a* $(v, \{3, 7^*\}, 1)$-PDF *if and only if* $v \equiv 1, 7 \pmod{24}$, *and* $v \ge 73$.

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1 The conclusion comes from Lemmas 4.6, 4.10, 4.13, and 4.17. □

5 Concluding Remark

In this paper, by using perfect Langford sequences, the existence of $(v, \{3, s^*\}, 1)$-PDFs is completely solved for $4 \le s \le 7$. Many new optimal $(u, W, 1, Q)$-OOCs can be obtained from Theorem 2.1 and Lemmas 1.1–1.3, where $W \in \{\{3, 4\}, \{3, 5\}, \{3, 6\}, \{3, 7\}, \{3, 4, 5\}, \{3, 4, 6\}, \{3, 4, 7\}\}$. For example, let $g = 24t + h$ be an integer such that $t \ge 1$, and $h \in \{9, 15\}$; then a $(g, \{3, 5^*\}, 1)$-PDF exists from Theorem 2.1 (2), and this PDF contains $s_1 = \frac{g-21}{6}$ blocks of size 3 and $s_2 = 1$ block of size 5. Thus $s = s_1 + s_2 = \frac{g-21}{6} + 1$. From Lemma 1.1 (1), an optimal $((g + 2)q, \{3, 4, 5\}, 1, (\frac{4s_1}{4s+1}, \frac{1}{4s+1}, \frac{4}{4s+1}))$-OOC exists for each prime $q \equiv 1 \pmod 4$, and $q \ge 5$ provided that $\gcd(g+2, q) = 1$. For $1 \le t \le 5$, $h = 9$, one can obtain an optimal $((g+2)q, \{3, 4, 5\}, 1, Q)$-OOC for each prime $q \equiv 1 \pmod 4$, and $q > 5$, where $(g + 2, Q) \in \{(35, (\frac{8}{13}, \frac{1}{13}, \frac{4}{13})), (59, (\frac{24}{29}, \frac{1}{29}, \frac{4}{29})), (83, (\frac{40}{45}, \frac{1}{45}, \frac{4}{45})), (107, (\frac{56}{61}, \frac{1}{61}, \frac{4}{61})), (131, (\frac{72}{77}, \frac{1}{77}, \frac{4}{77}))\}$.

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Appendices

Appendix A

$v = 19$ $\{0,3,5,9\}$, $\{0,1,8\}$.
$v = 25$ $\{0,1,3,10\}$, $\{0,4,12\}$, $\{0,5,11\}$.
$v = 31$ $\{0,4,9,15\}$, $\{0,1,8\}$, $\{0,2,14\}$, $\{0,3,13\}$.
$v = 37$ $\{0,1,3,15\}$, $\{0,4,11\}$, $\{0,5,18\}$, $\{0,6,16\}$, $\{0,8,17\}$.
$v = 43$ $\{0,6,8,17\}$, $\{0,1,21\}$, $\{0,3,16\}$, $\{0,4,18\}$, $\{0,5,15\}$, $\{0,7,19\}$.
$v = 49$ $\{0,1,4,21\}$, $\{0,2,11\}$, $\{0,5,19\}$, $\{0,6,18\}$, $\{0,7,22\}$, $\{0,8,24\}$, $\{0,10,23\}$.
$v = 55$ $\{0,2,13,25\}$, $\{0,1,8\}$, $\{0,4,22\}$, $\{0,5,19\}$, $\{0,6,27\}$, $\{0,9,24\}$, $\{0,10,26\}$, $\{0,17,20\}$.
$v = 61$ $\{0,2,3,22\}$, $\{0,4,30\}$, $\{0,5,13\}$, $\{0,6,24\}$, $\{0,7,28\}$, $\{0,14,23\}$, $\{0,15,25\}$, $\{0,16,27\}$, $\{0,17,29\}$.
$v = 67$ $\{0,3,20,32\}$, $\{0,1,16\}$, $\{0,2,27\}$, $\{0,6,14\}$, $\{0,7,30\}$, $\{0,9,19\}$, $\{0,11,33\}$, $\{0,13,31\}$, $\{0,21,26\}$,$\{0,24,28\}$.
$v = 73$ $\{0,6,13,27\}$, $\{0,1,31\}$, $\{0,2,36\}$, $\{0,3,25\}$, $\{0,4,32\}$, $\{0,8,26\}$, $\{0,9,19\}$, $\{0,11,23\}$, $\{0,15,35\}$, $\{0,17,33\}$, $\{0,24,29\}$.
$v = 79$ $\{0,6,7,37\}$, $\{0,3,20\}$, $\{0,4,32\}$, $\{0,8,29\}$, $\{0,9,19\}$, $\{0,11,36\}$, $\{0,12,35\}$, $\{0,13,27\}$, $\{0,15,33\}$, $\{0,16,38\}$, $\{0,24,26\}$, $\{0,34,39\}$

Appendix B

$v = 33$ $\{0,1,6,14,16\}$, $\{0,3,12\}$, $\{0,4,11\}$.
$v = 39$ $\{0,1,3,10,16\}$, $\{0,4,18\}$, $\{0,5,17\}$, $\{0,8,19\}$.
$v = 57$ $\{0,1,3,7,22\}$, $\{0,5,23\}$, $\{0,8,28\}$,$\{0,9,25\}$, $\{0,10,27\}$, $\{0,11,24\}$, $\{0,12,26\}$.
$v = 63$ $\{0,1,22,25,27\}$, $\{0,4,11\}$, $\{0,6,18\}$, $\{0,9,28\}$, $\{0,13,29\}$, $\{0,15,23\}$, $\{0,17,31\}$, $\{0,20,30\}$.
$v = 81$ $\{0,1,5,12,37\}$, $\{0,2,24\}$, $\{0,3,34\}$, $\{0,6,26\}$, $\{0,9,39\}$, $\{0,10,38\}$, $\{0,13,27\}$, $\{0,15,33\}$, $\{0,16,35\}$, $\{0,17,40\}$, $\{0,21,29\}$.
$v = 87$ $\{0,2,26,30,33\}$, $\{0,1,36\}$, $\{0,5,42\}$,$\{0,6,38\}$, $\{0,8,18\}$, $\{0,9,21\}$, $\{0,13,27\}$, $\{0,15,34\}$, $\{0,22,39\}$, $\{0,23,43\}$, $\{0,25,41\}$, $\{0,29,40\}$.
$v = 105$ $\{0,2,3,12,43\}$, $\{0,4,11\}$, $\{0,6,33\}$, $\{0,13,45\}$, $\{0,16,51\}$, $\{0,17,36\}$,$\{0,18,47\}$, $\{0,20,42\}$, $\{0,25,48\}$, $\{0,26,50\}$, $\{0,28,49\}$, $\{0,30,44\}$, $\{0,34,39\}$, $\{0,37,52\}$, $\{0,38,46\}$.
$v = 111$ $\{0,6,9,10,47\}$, $\{0,2,45\}$, $\{0,5,40\}$, $\{0,11,18\}$, $\{0,12,33\}$, $\{0,14,53\}$,$\{0,15,46\}$, $\{0,22,48\}$, $\{0,23,36\}$, $\{0,25,42\}$, $\{0,27,55\}$, $\{0,29,49\}$, $\{0,30,54\}$, $\{0,32,51\}$, $\{0,34,50\}$, $\{0,44,52\}$.
$v = 129$ $\{0,7,12,52,60\}$, $\{0,1,56\}$, $\{0,2,13\}$, $\{0,3,28\}$, $\{0,4,38\}$, $\{0,6,64\}$, $\{0,9,50\}$, $\{0,10,37\}$, $\{0,15,47\}$, $\{0,19,54\}$, $\{0,21,51\}$, $\{0,23,59\}$, $\{0,24,44\}$,$\{0,26,42\}$, $\{0,31,49\}$, $\{0,33,62\}$, $\{0,39,61\}$, $\{0,43,57\}$, $\{0,46,63\}$.
$v = 135$ $\{0,1,23,64,66\}$, $\{0,4,33\}$, $\{0,7,62\}$, $\{0,8,45\}$, $\{0,10,48\}$, $\{0,13,24\}$,$\{0,14,31\}$, $\{0,15,67\}$, $\{0,18,46\}$, $\{0,20,60\}$, $\{0,25,59\}$, $\{0,27,36\}$,$\{0,30,56\}$, $\{0,32,53\}$,

{0,35,51}, {0, 39, 58},{0,42,47}, {0, 44, 50},{0,49,61}, {0, 54, 57}.

$v = 153$ {0, 4, 9, 11, 67}, {0, 23, 68},{0, 8, 18}, {0, 13, 27},{0, 33, 76},{0, 26, 70},{0, 20, 41}, {0, 16, 62}, {0, 17, 64}, {0, 31, 73}, {0, 19, 71}, {0, 3, 51}, {0, 28, 50}, {0, 39, 74}, {0, 15, 72}, {0, 36, 60}, {0, 30, 55}, {0, 37, 75}, {0, 40, 69}, {0, 1, 54}, {0, 6, 65}, {0, 12, 61}, {0, 32, 66}.

$v = 159$ {0, 4, 26, 63, 77}, {0, 2, 76}, {0, 5, 71}, {0, 6, 60}, {0, 8, 57}, {0, 10, 58},{0, 13, 75}, {0, 17, 61}, {0, 18, 70}, {0, 19, 65}, {0, 20, 35}, {0, 21, 32}, {0, 24, 47}, {0, 25, 41}, {0, 27, 34}, {0, 29, 67}, {0, 30, 72}, {0, 31, 64}, {0, 36, 45}, {0, 40, 79}, {0, 43, 55}, {0, 50, 78}, {0, 53, 56}, {0, 68, 69}.

$v = 177$ {0, 22, 46, 55, 85}, {0, 2, 82}, {0, 3, 74}, {0, 4, 60}, {0, 7, 68}, {0, 10, 76},{0, 11, 69}, {0, 12, 77}, {0, 13, 88}, {0, 18, 35}, {0, 19, 51}, {0, 20, 67}, {0, 21, 57}, {0, 25, 52}, {0, 26, 49}, {0, 28, 87}, {0, 29, 44}, {0, 31, 79}, {0, 37, 53}, {0, 38, 83}, {0, 40, 81}, {0, 42, 43}, {0, 50, 84}, {0, 54, 62}, {0, 64, 70}, {0, 72, 86}, {0, 73, 78}.

$v = 183$ {0, 1, 3, 7, 21}, {0, 8, 73}, {0, 11, 72},{0, 12, 89}, {0, 13, 56},{0, 15, 68}, {0, 17, 88}, {0, 19, 83}, {0, 22, 67}, {0, 25, 74}, {0, 28, 90}, {0, 29, 63}, {0, 30, 82}, {0, 31, 57}, {0, 32, 86}, {0, 35, 85}, {0, 37, 75}, {0, 40, 84}, {0, 41, 46}, {0, 42, 66}, {0, 47, 80}, {0, 48, 87}, {0, 51, 78}, {0, 55, 91}, {0, 58, 81}, {0, 59, 69}, {0, 60, 76}, {0, 70, 79}.

$v = 201$ {0, 1, 3, 7, 25}, {0, 5, 92}, {0, 9, 77}, {0, 10, 98}, {0, 11, 71}, {0, 13, 97}, {0, 16, 86}, {0, 17, 74}, {0, 19, 85}, {0, 21, 82}, {0, 26, 76}, {0, 30, 58}, {0, 31, 75}, {0, 33, 89}, {0, 34, 83}, {0, 37, 69}, {0, 38, 52}, {0, 39, 90}, {0, 41, 95}, {0, 42, 62}, {0, 43, 79}, {0, 46, 81}, {0, 48, 93}, {0, 53, 100}, {0, 55, 67}, {0, 59, 99}, {0, 63, 78}, {0, 64, 91}, {0, 65, 94}, {0, 72, 80}, {0, 73, 96}.

$v = 207$ {0, 1, 3, 7, 15}, {0, 10, 69}, {0, 13, 77}, {0, 16, 88}, {0, 18, 96}, {0, 22, 66},{0, 23, 84}, {0, 24, 86}, {0, 28, 95}, {0, 29, 97}, {0, 32, 41}, {0, 33, 85}, {0, 34, 81}, {0, 35, 100}, {0, 38, 74}, {0, 39, 92}, {0, 40, 89}, {0, 42, 79}, {0, 43, 94}, {0, 46, 57}, {0, 50, 76}, {0, 54, 99}, {0, 55, 103}, {0, 56, 87}, {0, 58, 83}, {0, 60, 80}, {0, 63, 82}, {0, 70, 91}, {0, 71, 101}, {0, 73, 90}, {0, 75, 102}, {0, 93, 98}.

$v = 225$ {0, 1, 3, 7, 21}, {0, 5, 87}, {0, 10, 79}, {0, 13, 55}, {0, 15, 106}, {0, 17, 75},{0, 19, 86}, {0, 23, 85}, {0, 24, 105}, {0, 25, 109}, {0, 27, 103}, {0, 29, 107}, {0, 30, 96}, {0, 32, 92}, {0, 33, 70}, {0, 34, 73}, {0, 35, 83}, {0, 38, 94}, {0, 40, 99}, {0, 41, 90}, {0, 43, 54}, {0, 47, 104}, {0, 51, 95}, {0, 52, 68}, {0, 53, 98}, {0, 61, 111}, {0, 63, 89}, {0, 64, 110}, {0, 65, 101}, {0, 71, 93}, {0, 72, 80}, {0, 74, 102}, {0, 77, 108}, {0, 88, 97}, {0, 100, 112}.

Appendix C

$v = 43$ {0, 1, 3, 11, 16, 20}, {0, 6, 18}, {0, 7, 21}.

$v = 49$ {0, 1, 5, 7, 19, 22}, {0, 8, 24}, {0, 9, 20}, {0, 10, 23}.

$v = 55$ {0, 1, 3, 8, 19, 23}, {0, 6, 27}, {0, 9, 26}, {0, 10, 24}, {0, 12, 25}.

$v = 61$ {0, 3, 10, 22, 24, 28}, {0, 1, 27}, {0, 8, 23}, {0, 9, 29}, {0, 11, 16}, {0, 13, 30}.

$v = 67$ {0, 1, 5, 20, 26, 29}, {0, 2, 32}, {0, 7, 18}, {0, 10, 22}, {0, 14, 27}, {0, 16, 33}, {0, 23, 31}.

$v = 73$ {0, 1, 6, 21, 24, 34}, {0, 4, 16}, {0, 7, 32}, {0, 9, 26}, {0, 19, 30}, {0, 22, 36}, {0, 27, 35},

$\{0,29,31\}$.

$v = 79$ $\{0,1,6,9,35,37\}$, $\{0,4,21\}$, $\{0,7,27\}$, $\{0,10,32\}$, $\{0,11,30\}$, $\{0,12,25\}$, $\{0,16,39\}$, $\{0,18,33\}$, $\{0,24,38\}$.

$v = 85$ $\{0,5,9,37,38,40\}$, $\{0,6,25\}$, $\{0,7,34\}$, $\{0,8,20\}$, $\{0,11,21\}$, $\{0,13,30\}$, $\{0,14,36\}$, $\{0,16,39\}$, $\{0,18,42\}$,$\{0,26,41\}$.

$v = 91$ $\{0,9,10,26,33,45\}$, $\{0,2,39\}$, $\{0,3,18\}$, $\{0,4,31\}$, $\{0,5,25\}$, $\{0,8,42\}$, $\{0,11,40\}$, $\{0,14,44\}$, $\{0,21,43\}$, $\{0,28,41\}$, $\{0,32,38\}$.

$v = 97$ $\{0,1,4,6,35,42\}$, $\{0,9,22\}$, $\{0,10,24\}$, $\{0,11,43\}$, $\{0,12,39\}$, $\{0,16,33\}$, $\{0,19,47\}$, $\{0,20,46\}$, $\{0,23,44\}$, $\{0,25,40\}$, $\{0,30,48\}$, $\{0,37,45\}$.

Appendix D

$v = 73$ $\{0,3,14,21,27,31,36\}$, $\{0,2,25\}$, $\{0,8,34\}$, $\{0,12,32\}$, $\{0,16,35\}$, $\{0,29,30\}$.

$v = 79$ $\{0,1,3,7,16,30,35\}$, $\{0,8,33\}$, $\{0,10,31\}$, $\{0,11,37\}$, $\{0,12,36\}$, $\{0,17,39\}$, $\{0,18,38\}$.

$v = 97$ $\{0,7,19,34,36,37,42\}$, $\{0,4,44\}$, $\{0,9,48\}$, $\{0,10,38\}$, $\{0,11,43\}$, $\{0,13,33\}$, $\{0,14,45\}$, $\{0,16,41\}$, $\{0,21,47\}$, $\{0,24,46\}$.

$v = 103$ $\{0,1,3,7,12,34,44\}$, $\{0,8,47\}$, $\{0,13,48\}$, $\{0,14,42\}$, $\{0,16,45\}$, $\{0,17,40\}$, $\{0,18,38\}$, $\{0,19,49\}$, $\{0,21,46\}$, $\{0,26,50\}$, $\{0,36,51\}$.

$v = 121$ $\{0,1,3,7,12,32,51\}$, $\{0,8,46\}$, $\{0,10,53\}$, $\{0,13,47\}$, $\{0,17,57\}$, $\{0,18,60\}$, $\{0,22,45\}$, $\{0,24,54\}$, $\{0,26,59\}$, $\{0,27,55\}$, $\{0,35,49\}$, $\{0,36,52\}$, $\{0,37,58\}$, $\{0,41,56\}$.

$v = 127$ $\{0,1,3,7,12,31,51\}$, $\{0,10,56\}$, $\{0,13,47\}$, $\{0,16,49\}$, $\{0,17,52\}$, $\{0,18,61\}$, $\{0,22,58\}$, $\{0,23,60\}$, $\{0,25,63\}$, $\{0,26,55\}$, $\{0,27,59\}$, $\{0,40,54\}$, $\{0,41,62\}$, $\{0,42,57\}$, $\{0,45,53\}$.

$v = 145$ $\{0,1,3,7,12,27,60\}$, $\{0,13,64\}$, $\{0,14,63\}$, $\{0,18,61\}$, $\{0,28,72\}$, $\{0,30,68\}$, $\{0,31,71\}$, $\{0,32,54\}$, $\{0,35,69\}$, $\{0,36,65\}$, $\{0,37,56\}$, $\{0,39,62\}$, $\{0,41,58\}$, $\{0,42,52\}$, $\{0,45,70\}$, $\{0,46,67\}$, $\{0,47,55\}$, $\{0,50,66\}$.

$v = 151$ $\{0,1,3,7,12,25,62\}$, $\{0,8,60\}$, $\{0,10,56\}$, $\{0,15,54\}$, $\{0,16,57\}$, $\{0,17,66\}$, $\{0,19,67\}$, $\{0,20,73\}$, $\{0,21,68\}$, $\{0,28,72\}$, $\{0,29,74\}$, $\{0,30,64\}$, $\{0,31,69\}$, $\{0,32,58\}$, $\{0,33,75\}$, $\{0,35,71\}$, $\{0,40,63\}$, $\{0,43,70\}$, $\{0,51,65\}$.

$v = 169$ $\{0,1,3,7,12,20,67\}$, $\{0,10,69\}$, $\{0,14,65\}$, $\{0,16,68\}$, $\{0,22,61\}$, $\{0,24,80\}$, $\{0,25,78\}$, $\{0,26,75\}$, $\{0,28,74\}$, $\{0,31,76\}$, $\{0,32,70\}$, $\{0,35,62\}$, $\{0,36,79\}$, $\{0,37,77\}$, $\{0,41,71\}$, $\{0,42,57\}$, $\{0,44,73\}$, $\{0,48,82\}$, $\{0,50,83\}$, $\{0,54,72\}$, $\{0,58,81\}$, $\{0,63,84\}$.

$v = 175$ $\{0,1,3,7,12,20,68\}$, $\{0,10,70\}$, $\{0,14,58\}$, $\{0,16,69\}$, $\{0,18,84\}$, $\{0,22,85\}$, $\{0,23,72\}$, $\{0,24,71\}$, $\{0,25,79\}$, $\{0,26,81\}$, $\{0,28,59\}$, $\{0,29,80\}$, $\{0,33,83\}$, $\{0,34,73\}$, $\{0,35,76\}$, $\{0,36,74\}$, $\{0,37,64\}$, $\{0,42,87\}$, $\{0,46,86\}$, $\{0,43,75\}$, $\{0,52,82\}$, $\{0,57,78\}$, $\{0,62,77\}$.

$v = 193$ $\{0,1,3,7,12,20,68\}$, $\{0,10,82\}$, $\{0,14,94\}$, $\{0,16,92\}$, $\{0,18,81\}$, $\{0,21,73\}$, $\{0,22,71\}$, $\{0,23,74\}$, $\{0,26,79\}$, $\{0,27,96\}$, $\{0,29,64\}$, $\{0,31,86\}$, $\{0,32,89\}$, $\{0,34,78\}$, $\{0,36,90\}$, $\{0,37,75\}$, $\{0,39,85\}$, $\{0,40,87\}$, $\{0,41,84\}$, $\{0,42,70\}$, $\{0,45,95\}$, $\{0,58,88\}$, $\{0,59,83\}$, $\{0,60,93\}$, $\{0,62,77\}$, $\{0,66,91\}$.

$v = 199$ $\{0,1,3,7,12,20,69\}$, $\{0,18,89\}$, $\{0,21,79\}$, $\{0,23,98\}$, $\{0,28,84\}$, $\{0,30,85\}$, $\{0,31,94\}$, $\{0,33,77\}$, $\{0,34,93\}$, $\{0,35,82\}$, $\{0,37,91\}$, $\{0,38,78\}$, $\{0,39,65\}$, $\{0,41,70\}$, $\{0,42,92\}$, $\{0,43,67\}$, $\{0,48,80\}$, $\{0,45,96\}$, $\{0,52,88\}$, $\{0,53,99\}$, $\{0,60,87\}$, $\{0,61,76\}$, $\{0,64,86\}$, $\{0,72,97\}$, $\{0,73,83\}$, $\{0,74,90\}$, $\{0,81,95\}$.

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