AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **55** (2013), Pages 131–144

Construction of magic rectangles of odd order

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Abstract

Magic rectangles are well-known for their very interesting and entertaining combinatorics. Such magic rectangles have also been used in designing experiments. In a magic rectangle, the integers 1 to pq are arranged in an array of p rows and q columns so that each row adds to the same total P and each column to the same total Q. In the present paper we provide a systematic method for constructing a p by q magic rectangle where $p > 1$ and $q > 1$ are any odd integers.

1 Introduction

Magic rectangles are well-known for their very interesting and entertaining combinatorics. A magic rectangle is an arrangement of the integers 1 to pq in an array of p rows and q columns so that each row adds to the same total P and each column to the same total Q. The totals P and Q are termed the magic constants. Since the average value of the integers is $A = (pq + 1)/2$, we must have $P = qA$ and $Q = pA$.

The total of all the integers in the array is $pqA = pP = qQ$. If pq is even $pq+1$ is odd and so for $P = q(pq+1)/2$ and $Q = p(pq+1)/2$ to be integers p and q must both be even. On the other hand, if pq is odd then p and q must both be odd. In this case also P and Q are integers since $pq+1$ is even. Therefore, an odd by even magic rectangle is impossible.

For an update on available literature on magic rectangles we refer to Hagedorn [5], and Bier and Kleinschmidt [1]. Recently, Reyes, Das and Midha [3], and Reyes, Das, Midha and Vellaisamy [4], have provided complete solutions for constructing an even by even magic rectangle. The method of construction provided there is not extendable to the odd case. Thus the construction of odd by odd magic rectangles has been more challenging.

Das and Sarkar [2] provide constructions for a few odd by odd magic rectangles by taking a rectangle with constant column sums and transforming it into a magic rectangle via transpositions within columns. They provide tables for the constructed magic rectangles. Following the ideas of Das and Sarkar [2], in the present paper we provide complete solutions to the construction problem by providing a systematic method of constructing a p by q magic rectangle where both p and q are odd. A $p \times q$ rectangle H_p, containing integers 1, 2,..., pq once, with constant column sums, is readily produced. This rectangle can be transformed into a magic rectangle M by performing transpositions within columns to make the row sums constant. The change in a particular row sum from H_p to M is equal to a certain sum of differences of transposed entries. The method has been shaped in the form of an algorithm that is very convenient for writing a computer program.

In Section 2 we construct p by q magic rectangles. The supporting proofs related to the construction are given in Section 4. In Section 3 we illustrate our construction method through some examples of magic rectangles.

2 The construction

We construct magic rectangles with p rows and q columns ($p > 1$ and $q > 1$ both odd). In order to build our construction method, we first define some matrices. In the sequel, J_{mn} denotes a $m \times n$ matrix having all its elements equal to 1, and J_{m1} is denoted by 1_m. Also, $\otimes$ denotes the Kronecker product symbol.

Definition 2.1. For $q = 2q' - 1$, $q' \geq 1$, define

$$R = \begin{pmatrix} 1 & 2 & \cdots & q-1 & q \\ q & q-1 & \cdots & 2 & 1 \end{pmatrix} = \begin{pmatrix} R_U \\ R_L \end{pmatrix},$$

$$S = \begin{pmatrix} 1 & 2 & \cdots & q' & q'+1 & \cdots & q \\ q' & q'+1 & \cdots & q & 1 & \cdots & q'-1 \\ q & q-2 & \cdots & 1 & q-1 & \cdots & 2 \end{pmatrix},$$

and

$$T = \begin{pmatrix} q & q-2 & \cdots & 1 & q-1 & q-3 & \cdots & 2 \\ 1 & 2 & \cdots & q' & q'+1 & q'+2 & \cdots & q \\ 1 & 3 & \cdots & q & 2 & 4 & \cdots & q-1 \\ q & q-1 & \cdots & q' & q'-1 & q'-2 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} T_U \\ T_L \end{pmatrix},$$

where each of R_U and R_L are of order $1 \times q$ and each of T_U and T_L are of order $2 \times q$.

Definition 2.2. Let $p' \geq 1$. Define $G_3 = S$, $G_5 = \begin{pmatrix} R_U \\ S \\ R_L \end{pmatrix}$, and

$$G_p = \begin{cases} \begin{pmatrix} 1_{p'} \otimes T_U \\ S \\ 1_{p'} \otimes T_L \end{pmatrix} & \text{if } p = 4p' + 3, \\ \\ \begin{pmatrix} R_U \\ G_{p-2} \\ R_L \end{pmatrix} & \text{if } p = 4p' + 5. \end{cases}$$

Also, define, $H_p = (h_{ij})$, where $h_{ij} = (g_{ij} - 1)p + i$, $1 \leq i \leq p$, $1 \leq j \leq q$, and $G_p = (g_{ij})$.

Theorem 2.1. *Suppose $p > 1$, $q > 1$, $n > 1$ are odd. Let M_1 be a $p \times n$ magic rectangle. Then there exists a $p \times qn$ magic rectangle M.*

Proof. For $1 \leq i \leq p$, $1 \leq j \leq q$, let $M_j = M_1 + (j-1)pnJ_{pn}$ and m_{ij} represent the i-th row of M_j. Then M is obtained from G_p by replacing the element j in the i-th row of G_p by m_{ij}, $1 \leq i \leq p$, $1 \leq j \leq q$. Using the fact that M_1 is $p \times n$ magic rectangle and using the properties of the matrix G_p (Lemma 4.1), it is easy to see that each row sum of M equals $(\frac{n(pn+1)}{2})q + npn + n2pn + \cdots + n(q-1)pn = \frac{qn(pqn+1)}{2}$ and each column sum of M equals $\frac{p(pn+1)}{2} + (\frac{p(q+1)}{2} - p)pn = \frac{p(pqn+1)}{2}$. Hence M is a $p \times nq$ magic rectangle. □

Theorem 2.2. *For p odd, consider the $p \times p$ matrices*

$$N_1 = \begin{pmatrix} 1 & 2 & \cdots & p-1 & p \\ 2 & 3 & \cdots & p & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p-1 & p & \cdots & p-3 & p-2 \\ p & 1 & \cdots & p-2 & p-1 \end{pmatrix} \text{ and } N_2 = \begin{pmatrix} p & p-1 & \cdots & 2 & 1 \\ 1 & p & \cdots & 3 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p-2 & p-3 & \cdots & p & p-1 \\ p-1 & p-2 & \cdots & 1 & p \end{pmatrix}.$$

Then a $p \times p$ magic rectangle M is given by $M = N_1 + p(N_2 - J_{pp})$.

Proof. Let a_{ij} and b_{ij} be the (ij)-th element of N_1 and N_2, $1 \leq i, j \leq p$. Notice that N_1 and N_2 are orthogonal latin squares since both of them are latin squares and the collection $\{(a_{ij}, b_{ij}), 1 \leq i, j \leq p\}$ collects all p^2 ordered pairs. Based on the orthogonality of N_1 and N_2, the collection of all elements of M is the set $\{i + (j - 1)p, 1 \leq i, j \leq p\}$ which contains p^2 consecutive integers from 1 to p^2. Now, it is clear to see each column and row sums of M equals $p(1+p)/2 + p(p(1+p)/2 - p) = p(1 + p^2)/2$. Hence M is a $p \times p$ magic rectangle. □

Magic rectangle construction method: step by step descriptions

Suppose $p > 1$ and $q > 1$ are odd. The $p \times q$ magic rectangle M is constructed as follows.

Step I. Construct G_p and H_p (both matrices are defined in Definition 2.2). Each row of G_p is a permutation of $(1, 2, \ldots, q)$ and its column sums are all equal. H_p has the following properties. (i) each of the numbers 1 through pq appear once; (ii) each column sum equals $p(pq + 1)/2$; (iii) the i-th row sum is $pq(q - 1)/2 + qi$; (iv) for $1 \leq i \leq (p-1)/2$, the difference between the $(p+1-i)$-th row sum and the i-th row sum is $q(p + 1 - 2i)$; (v) the $(p + 1)/2$-th row sum is $q(pq + 1)/2$.

Step II. Each column sum of H_p are equal, that matches the column sum requirements of a magic rectangle. Hence, in this step, we will rearrange the element orders within columns such that the resulting H_p has the equal row sums, therefore, it is the required magic rectangle. Observe that (iii), (iv) and (v) properties of H_p in Step I, we wish to exchange some elements of the i-th row with the corresponding elements of the $(p + 1 - i)$-th row, such that the resulting row sum of the i-th row gains $q(p + 1 - 2i)/2$ and the resulting row sum of the $(p + 1 - i)$-th row loses $q(p + 1 - 2i)/2$. Then, the resulting row sum of i-th equals to $pq(q - 1)/2 + qi + q(p + 1 - 2i)/2 = q(pq + 1)/2$ and the resulting row sum of the $(p + 1 - i)$-th row equals to $pq(q - 1)/2 + q(p + 1 - i)$ - $q(p + 1 - 2i)/2 = q(pq + 1)/2$.

We now explain how this row element exchanges procedure works. For $1 \leq i \leq p$, let $g_i = (g_{i1}, g_{i2}, \ldots, g_{iq})$, $g_{(p+1-i)} = (g_{(p+1-i),1},\ g_{(p+1-i),2}, \ldots, g_{(p+1-i),q})$, $h_i = (h_{i1}, h_{i2}, \ldots, h_{iq})$ and $h_{(p+1-i)} = (h_{(p+1-i),1}, h_{(p+1-i),2}, \ldots, h_{(p+1-i),q})$ denote the i-th and $(p+1-i)$th rows of G_p and H_p. Based on $h_i = pg_i + i1'_q$, the above row element exchanges on rows of H_p can be done through the row element exchanges on the rows of G_p. We prefer to do the row element exchanges on rows of G_p, since the row differences of G_p are easier to be characterized.

Let the difference row vector $d_i = g_{(p+1-i)} - g_i = (d_{i1}, d_{i2}, \ldots, d_{iq})$, $1 \leq i \leq (p-1)/2$. Assume, in d_i, the l elements, $d_{ij_1},\ d_{ij_2}, \ldots, d_{ij_l}$, are chosen, such that $p \sum_{u=1}^{l} d_{ij_u} + l(p+1-2i) = q(p+1-2i)/2$. This suggests that, in G_p, the l elements $g_{ij_1}, g_{ij_2}, \ldots, g_{ij_l}$, of the i-th row should be exchanged with the l elements, $g_{(p+i-1),j_1}, g_{(p+i-1),j_2}, \ldots, g_{(p+i-1),j_l}$ of the $(p + i - 1)$-th row. Also, it indicates, in H_p, the l elements, $h_{ij_1}, h_{ij_2}, \ldots, h_{ij_l}$, of the i-th row should be exchanged with the l elements, $h_{(p+i-1),j_1}, h_{(p+i-1),j_2}, \ldots, h_{(p+i-1),j_l}$ of the $(p + i - 1)$-th row. Then the i-th and $(p + 1 - i)$-th row sums of the resulting H_p are both equal to $q(pq + 1)/2$.

Step III. Do the above row element exchanges of G_p and H_p, for $1 \leq i \leq (p - 1)/2$. Keep the $(p + 1)/2$-th row of H_p intact, since its row sum equals to $q(pq + 1)/2$.

Now, the resulting H_p has equal column sums and row sums, and is a $p \times q$ magic rectangle.

3 Some illustrative examples

The method provided in this paper allows one to write a simple computer program for obtaining any odd by odd magic rectangle. We provide two examples of magic rectangle of sides 7 by 11 and sides 13 by 19.

Magic rectangle of order 7 by 11

$p = 7 = 4 + 3$, $q = 11 = 2 \times 6 - 1$. Thus $p' = 1$ and $q' = 6$.

$$G_7 = \begin{pmatrix} 11 & 9 & 7 & 5 & 3 & 1 & 10 & 8 & 6 & 4 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 \\ 11 & 9 & 7 & 5 & 3 & 1 & 10 & 8 & 6 & 4 & 2 \\ 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 \\ 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

From G_7 above, we obtain H_7 as below:

$$H_7 = \begin{pmatrix} \mathbf{71} & 57 & 43 & \mathbf{29} & 15 & 1 & 64 & 50 & 36 & 22 & 8 \\ \mathbf{2} & 9 & \mathbf{16} & 23 & 30 & 37 & 44 & 51 & 58 & 65 & 72 \\ 3 & 10 & 17 & \mathbf{24} & 31 & 38 & 45 & \mathbf{52} & 59 & 66 & 73 \\ 39 & 46 & 53 & 60 & 67 & 74 & 4 & 11 & 18 & 25 & 32 \\ 75 & 61 & 47 & \mathbf{33} & 19 & 5 & 68 & \mathbf{54} & 40 & 26 & 12 \\ \mathbf{6} & 20 & \mathbf{34} & 48 & 62 & 76 & 13 & 27 & 41 & 55 & 69 \\ \mathbf{77} & 70 & 63 & \mathbf{56} & 49 & 42 & 35 & 28 & 21 & 14 & 7 \end{pmatrix}.$$

For rows i and $7 - i$, $1 \le i \le 2$, we carry out the corresponding interchanges as in Case II of Theorem 4.1. Here, $l = (11 - 7)/2 = 2 = 2 \times 0 + 2$. Thus $y = 0$. So we interchange (i) between rows 1 and 7 at positions 1 and $(7+3-2)/2 = 4$, (ii) between rows 2 and 6 at positions 1 and $(7+3-4)/2 = 3$. For rows 3 and 5, we carry out the corresponding interchanges as in Case I of Theorem 4.1. Here, $q = 11 = 6 \times 2 - 1$ (i.e., k=2), $l = 2 = 4 \times 0 + 2$. Thus $y = 0$. Hence, we interchange between rows 3 and 5 at positions $2 \times 2 = 4$ and $4 \times 2 = 8$.

From H_7 above, on carrying out the interchanges, we obtain 7×11 magic rectangle M as below:

$$M = \begin{pmatrix} 77 & 57 & 43 & 56 & 15 & 1 & 64 & 50 & 36 & 22 & 8 \\ 6 & 9 & 34 & 23 & 30 & 37 & 44 & 51 & 58 & 65 & 72 \\ 3 & 10 & 17 & 33 & 31 & 38 & 45 & 54 & 59 & 66 & 73 \\ 39 & 46 & 53 & 60 & 67 & 74 & 4 & 11 & 18 & 25 & 32 \\ 75 & 61 & 47 & 24 & 19 & 5 & 68 & 52 & 40 & 26 & 12 \\ 2 & 20 & 16 & 48 & 62 & 76 & 13 & 27 & 41 & 55 & 69 \\ 71 & 70 & 63 & 29 & 49 & 42 & 35 & 28 & 21 & 14 & 7 \end{pmatrix}.$$

Magic rectangle of order 13 by 19

$p = 13 = 4 \times 2 + 5$, $q = 19 = 2 \times 10 - 1$. Thus $p' = 2$ and $q' = 10$.

$$G_{13} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 19 & 17 & 15 & 13 & 11 & 9 & 7 & 5 & 3 & 1 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 19 & 17 & 15 & 13 & 11 & 9 & 7 & 5 & 3 & 1 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 19 & 17 & 15 & 13 & 11 & 9 & 7 & 5 & 3 & 1 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

From G_{13} above, we obtain $H_{13} = (H_{13}^1, H_{13}^2)$, where

$$H_{13}^1 = \begin{pmatrix} \mathbf{1} & 14 & 27 & 40 & 53 & 66 & \mathbf{79} & 92 & 105 & 118 & 131 & 144 \\ 236 & 210 & 184 & 158 & 132 & \mathbf{106} & 80 & 54 & 28 & \mathbf{2} & \mathbf{223} & 197 \\ 3 & 16 & 29 & 42 & \mathbf{55} & 68 & 81 & 94 & 107 & \mathbf{120} & \mathbf{133} & 146 \\ 238 & 212 & 186 & \mathbf{160} & 134 & 108 & 82 & 56 & 30 & \mathbf{4} & \mathbf{225} & 199 \\ 5 & 18 & \mathbf{31} & 44 & 57 & 70 & 83 & 96 & 109 & \mathbf{122} & \mathbf{135} & 148 \\ 6 & 19 & 32 & 45 & 58 & \mathbf{71} & 84 & \mathbf{97} & 110 & 123 & 136 & 149 \\ 124 & 137 & 150 & 163 & 176 & 189 & 202 & 215 & 228 & 241 & 7 & 20 \\ 242 & 216 & 190 & 164 & 138 & \mathbf{112} & 86 & \mathbf{60} & 34 & 8 & 229 & 203 \\ 9 & 35 & \mathbf{61} & 87 & 113 & 139 & 165 & 191 & 217 & \mathbf{243} & \mathbf{22} & 48 \\ 244 & 231 & 218 & \mathbf{205} & 192 & 179 & 166 & 153 & 140 & \mathbf{127} & \mathbf{114} & 101 \\ 11 & 37 & 63 & 89 & \mathbf{115} & 141 & 167 & 193 & 219 & \mathbf{245} & \mathbf{24} & 50 \\ 246 & 233 & 220 & 207 & 194 & \mathbf{181} & 168 & 155 & 142 & \mathbf{129} & \mathbf{116} & 103 \\ \mathbf{247} & 234 & 221 & 208 & 195 & 182 & \mathbf{169} & 156 & 143 & 130 & 117 & 104 \end{pmatrix} \text{ and}$$

$$H_{13}^2 = \begin{pmatrix} 157 & 170 & 183 & 196 & 209 & 222 & \mathbf{235} \\ 171 & 145 & 119 & 93 & 67 & 41 & 15 \\ 159 & 172 & 185 & 198 & 211 & 224 & 237 \\ 173 & 147 & 121 & 95 & 69 & 43 & 17 \\ 161 & 174 & 187 & 200 & 213 & 226 & 239 \\ \mathbf{162} & 175 & 188 & 201 & 214 & 227 & 240 \\ 33 & 46 & 59 & 72 & 85 & 98 & 111 \\ \mathbf{177} & 151 & 125 & 99 & 73 & 47 & 21 \\ 74 & 100 & 126 & 152 & 178 & 204 & 230 \\ 88 & 75 & 62 & 49 & 36 & 23 & 10 \\ 76 & 102 & 128 & 154 & 180 & 206 & 232 \\ 90 & 77 & 64 & 51 & 38 & 25 & 12 \\ 91 & 78 & 65 & 52 & 39 & 26 & \mathbf{13} \end{pmatrix}.$$

For rows 1 and 13, we carry out the corresponding interchanges as in Case III of Theorem 4.1. Here, $l = (19 - 13)/2 = 3 = 2 \times 1 + 1$. Thus $y = 1$. So for row 1 we carry out the corresponding interchanges which is between rows 1 and 13 at positions 1, 19 and 7. For rows i and $14-i$, $2 \leq i \leq 5$, we carry out the corresponding interchanges as in Case II of Theorem 4.1. Again, $l = 2y + 1$ with $y = 1$. So we interchange (i) between rows 2 and 12 at positions 10, 11 and 6, (ii) between rows 3 and 11 at positions 10, 11 and 5, (iii) between rows 4 and 10 at positions 10, 11 and 4, (iv) between rows 5 and 9 at positions 10, 11 and 3.

Finally, for rows 6 and 8, we carry out the corresponding interchanges as in Case I of Theorem 4.1. Here, $q = 19 = 6 \times 3 + 1$ (i.e., k=3), $l = 3 = 4 \times 0 + 3$. Thus $y = 0$. Hence we interchange between rows 6 and 8 at positions 13, 6 and 8.

From H_{13} above, on carrying out the interchanges, we obtain 13×19 magic rectangle $M = (M^1, M^2)$, where

$$M^1 = \begin{pmatrix} 247 & 14 & 27 & 40 & 53 & 66 & 169 & 92 & 105 & 118 & 131 & 144 \\ 236 & 210 & 184 & 158 & 132 & 181 & 80 & 54 & 28 & 129 & 116 & 197 \\ 3 & 16 & 29 & 42 & 115 & 68 & 81 & 94 & 107 & 245 & 24 & 146 \\ 238 & 212 & 186 & 205 & 134 & 108 & 82 & 56 & 30 & 127 & 114 & 199 \\ 5 & 18 & 61 & 44 & 57 & 70 & 83 & 96 & 109 & 243 & 22 & 148 \\ 6 & 19 & 32 & 45 & 58 & 112 & 84 & 60 & 110 & 123 & 136 & 149 \\ 124 & 137 & 150 & 163 & 176 & 189 & 202 & 215 & 228 & 241 & 7 & 20 \\ 242 & 216 & 190 & 164 & 138 & 71 & 86 & 97 & 34 & 8 & 229 & 203 \\ 9 & 35 & 31 & 87 & 113 & 139 & 165 & 191 & 217 & 122 & 135 & 48 \\ 244 & 231 & 218 & 160 & 192 & 179 & 166 & 153 & 140 & 4 & 225 & 101 \\ 11 & 37 & 63 & 89 & 55 & 141 & 167 & 193 & 219 & 120 & 133 & 50 \\ 246 & 233 & 220 & 207 & 194 & 106 & 168 & 155 & 142 & 2 & 223 & 103 \\ 1 & 234 & 221 & 208 & 195 & 182 & 79 & 156 & 143 & 130 & 117 & 104 \end{pmatrix} \text{ and}$$

$$M^2 = \begin{pmatrix} 157 & 170 & 183 & 196 & 209 & 222 & 13 \\ 171 & 145 & 119 & 93 & 67 & 41 & 15 \\ 159 & 172 & 185 & 198 & 211 & 224 & 237 \\ 173 & 147 & 121 & 95 & 69 & 43 & 17 \\ 161 & 174 & 187 & 200 & 213 & 226 & 239 \\ 177 & 175 & 188 & 201 & 214 & 227 & 240 \\ 33 & 46 & 59 & 72 & 85 & 98 & 111 \\ 162 & 151 & 125 & 99 & 73 & 47 & 21 \\ 74 & 100 & 126 & 152 & 178 & 204 & 230 \\ 88 & 75 & 62 & 49 & 36 & 23 & 10 \\ 76 & 102 & 128 & 154 & 180 & 206 & 232 \\ 90 & 77 & 64 & 51 & 38 & 25 & 12 \\ 91 & 78 & 65 & 52 & 39 & 26 & 235 \end{pmatrix}.$$

4 Proof of the validity of the construction

We first establish some preliminary results. Recall that matrices R, S, T, G_p and H_p are defined in Section 2.

Lemma 4.1. *The rows of R, S, T and G_p are permutations of $(1, 2, 3, \ldots, q)$ and the column sums of R, S, T and G_p are $q+1$, $3(q+1)/2$, $2(q+1)$ and $p(q+1)/2$ respectively.*

Proof. For R, S and T, it is easy to see that each row being a permutation of $(1, 2, \ldots, q)$ has row sum $q(q+1)/2$ and each column has column sum $q+1$, $3(q+1)/2$, $2(q+1)$, respectively. G_p is formed as the combination of copies of R, S and T, has the above row properties and equal column sums $p(q+1)/2$.

The following quantities are needed in Lemma 4.2. For a positive integer k, let
$e_2 = (-1\ 1)$, $e_3 = (-1\ 0\ 1)$, $e_4 = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$, $v_1 = (6k{-}2\ 6k{-}5\ \cdots\ 3k{+}10)$,
$v_2 = (3k-2\ 3k-5\ \cdots\ 1)$, $v_3 = (-2\ -5\ \cdots\ -3k+1)$, $v_4 = (3k-3\ 3k-6\ \cdots\ 3)$,
$v_5 = (-3\ -6\ \cdots\ -3k+3)$, $v_6 = (-3k\ -3k-3\ \cdots\ -6k+3)$,
$w_1 = (6k\ 6k-3\ \cdots\ 3k+3)$, $w_2 = (3k\ 3k-3\ \cdots\ 3)$, $w_3 = (-3\ -6\ \cdots\ -3k)$,
$w_4 = (3k-2\ 3k-5\ \cdots\ 1)$, $w_5 = (-2\ -5\ \cdots\ -3k+1)$,
$w_6 = (-3k-2\ -3k-5\ \cdots\ -6k+1)$.
Observe that v_4 and v_5 are of length $k-1$, while the others are of length k.

Lemma 4.2. *Let $q = 2q' - 1$. Then we have*
(i) $e_2 R = \begin{pmatrix} q-1 & q-3 & \ldots & 4 & 2 & 0 & -2 & -4 & \ldots & -q+3 & -q+1 \end{pmatrix}$,

(ii) $e_3 S = \begin{cases} \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & 0 & v_5 & v_6 \end{pmatrix} & \text{if } q = 6k-1 \\ \begin{pmatrix} w_1 & w_2 & 0 & w_3 & w_4 & w_5 & w_6 \end{pmatrix} & \text{if } q = 6k+1, \end{cases}$

(iii) $e_4T = \begin{pmatrix} 0 & 1 & \cdots & q'-1 & -q'+1 & -q'+2 & \cdots & -2 & -1 \\ 0 & 1 & \cdots & q'-1 & -q'+1 & -q'+2 & \cdots & -2 & -1 \end{pmatrix}.$

Proof. This follows from straight observations on row differences in R, S and T. □

The following two lemmas describe detailed element relations among v_i's and w_i's, $1 \le i \le 6$, which appear in Lemma 4.2 (ii).

Lemma 4.3. *For* $u = 1, 2, 3, 6$, *let* $v_u = \begin{pmatrix} v_{u1} & v_{u2} & \cdots & v_{uk} \end{pmatrix}$ *and for* $u = 4, 5$, *let* $v_u = \begin{pmatrix} v_{u1} & v_{u2} & \cdots & v_{u,k-1} \end{pmatrix}$. *Then,*
(i) $v_{1j} = 6k - 2 - 3(j-1)$, $1 \le j \le k$,
(ii) $v_{2j} = 3k - 2 - 3(j-1)$, $1 \le j \le k$,
(iii) $v_{3j} = -2 - 3(j-1)$, $1 \le j \le k$,
(iv) $v_{4j} = 3k - 3 - 3(j-1)$, $1 \le j \le k-1$,
(v) $v_{5j} = -3 - 3(j-1)$, $1 \le j \le k-1$,
(vi) $v_{6j} = -3k - 3(j-1)$, $1 \le j \le k$.

Also, the following hold:
(a) For $1 \le j \le k$, $v_{1j} + v_{6,k+1-j} = 1$ *and* $v_{2j} + v_{3,k+1-j} = -1$,
(b) For $1 \le j \le k-1$, $v_{4j} + v_{5,k-j} = 0$,
(c) For $1 \le j \le k$, $v_{1j} + v_{2j} + v_{3,k+1-j} + v_{6,k+1-j} = 0$,
(d) $v_{2k} = 1$; $v_{4,k-1} + v_{51} = 0$.

Lemma 4.4. *For* $1 \le u \le 6$, *let* $w_u = \begin{pmatrix} w_{u1} & w_{u2} & \cdots & w_{uk} \end{pmatrix}$. *Then* $1 \le j \le k$,
(i) $w_{1j} = 6k - 3(j-1)$,
(ii) $w_{2j} = 3k - 3(j-1)$,
(iii) $w_{3j} = -3 - 3(j-1)$,
(iv) $w_{4j} = 3k - 3j + 1$,
(v) $w_{5j} = -2 - 3(j-1)$,
(vi) $w_{6j} = -3k - 2 - 3(j-1)$.

Also, the following hold:
(a) $w_{1j} + w_{6,k+1-j} = 1$; $w_{2j} + w_{3,k+1-j} = 0$; $w_{4j} + w_{5,k+1-j} = -1$,
(b) $w_{1j} + w_{4j} + w_{5,k+1-j} + w_{6,k+1-j} = 0$,
(c) $w_{4k} = 1$; $w_{2k} + w_{31} = 0$.

Lemma 4.5. *The* $p \times q$ *matrix* H_p *has the following properties,*
(i) each of the numbers 1 through pq *appear once,*
(ii) each column sum equals $\frac{p(pq+1)}{2}$,
(iii) the i*-th row sum is* $\frac{pq(q-1)}{2} + qi$,
(iv) for $1 \le i \le \frac{p-1}{2}$, *the difference between the* $(p+1-i)$*-th row sum and the* i*-th row sum is* $q(p+1-2i)$,
(v) the $(\frac{p+1}{2})$*-th row sum is* $\frac{q(pq+1)}{2}$.

Proof. This follows from Lemma 4.1.

Theorem 4.1. *For given odd integers $p > 1$ and $q > 1$, there exist a $p \times q$ magic rectangle M.*

Proof. By the application of Theorems 2.1 and 2.2 and without loss of generality, we can assume that $p < q$ and q does not have a factor 3. First, construct G_p and H_p. From Lemma 4.5, in H_p, each column sum equals $\frac{p(pq+1)}{2}$ and the $(\frac{p+1}{2})$-th row sum equals $\frac{q(pq+1)}{2}$. Therefore we will rearrange the elementorders within columns of H_p such that the resulting H_p has equal row sums. More specific, our objective, for $i \leq \frac{p-1}{2}$, is to interchange some elements of the i-th row of H_p with the corresponding elements of $(p+1-i)$-th row such that it results in the two row sums being equal to $\frac{q(pq+1)}{2}$. The row elements interchanges strategy in H_p as follows.

Suppose we interchange l elements $h_{ij_1}, h_{ij_2}, \ldots, h_{ij_l}$ of the i-th row with corresponding elements $h_{p+1-i,j_1}, h_{p+1-i,j_2}, \ldots, h_{p+1-i,j_l}$ of the $(p+1-i)$-th row of H_p. Then, by Lemma 4.5 (ii), (iii) and (iv), we must have $\sum_{u=1}^{l}[h_{p+1-i,j_u} - h_{ij_u}] = \frac{q(p+1-2i)}{2}$ in order to make the corresponding row sums equal to $\frac{q(pq+1)}{2}$. This implies

$$p\sum_{u=1}^{l}(g_{p+1-i,j_u} - g_{ij_u}) + (p+1-2i)l = \frac{q(p+1-2i)}{2},$$

which has a solution $\sum_{u=1}^{l}[g_{p+1-i,j_u} - g_{ij_u}] = \frac{p+1-2i}{2}$ and $l = \frac{q-p}{2}$ for $1 \leq i \leq \frac{p-1}{2}$. Thus, for $l = (q-p)/2$ we seek $j_1, j_2, \ldots, j_l$ such that $\sum_{u=1}^{l}[g_{p+1-i,j_u} - g_{ij_u}] = \frac{(p+1-2i)}{2}$.

Based on the different row positions of H_p, there are three cases of positions interchanges, Cases I, II and III, which are shown at the end of the proof.

Let the desired magic rectangle M be the matrix obtained from H_p after carrying out interchanges at positions as below:

(A) If p=3, then do the positions interchanges as in Case I.

(B) If p=5, then the first and the 5-th rows do the positions interchanges as in Case III and the rest of rows do the positions interchanges as in Case I.

(C) If $p = 4p' + 3$ and $p' \geq 1$, then the $(p-1)/2$th and $(p+1)/2$-th do the positions interchanges as in Case I and the rest rows do the positions interchanges as in Case II with $i = 1, 3, 5, \ldots, 2p' - 1$.

(D) If $p = 4p' + 5$ and $p' \geq 1$, then the first and the last rows do the positions interchanges as in Case III, the $(p-1)/2$th and $(p+1)/2)$-th rows do the positions interchanges as in Case I and the rest of rows do the positions interchanges as in Case II with $i = 2, 4, 6, \ldots, 2p'$.

Then, following the effects of interchange of elements in H_p, the $p \times q$ matrix M has the following properties,
(i) each of the numbers 1 through pq appear once,
(ii) each row sum equals $\frac{q(pq+1)}{2}$,

(iii) each column sum equals $\frac{p(pq+1)}{2}$.
Thus M is a magic rectangle.

Case I: Interchanges between row elements of H_p corresponding to S
The three rows of S appear as $(\frac{p-1}{2})$-th, $(\frac{p+1}{2})$-th and $(\frac{p+3}{2})$-th rows of G_p. Also, the difference between the $(\frac{p+3}{2})$-th and $(\frac{p-1}{2})$-th row sums of H_p is $2q$. Suppose l corresponding elements between the $(\frac{p+3}{2})$-th and $(\frac{p-1}{2})$-th rows of H_p are interchanged. Let the sum of the differences between these l elements in G_p equals s (say). Then l and s satisfies $ps + 2l = q$ with a solution $s = 1$ and $l = (q - p)/2$.

Let $l = (q - p)/2 = 4y + z$ (say), $y \geq 0$, $1 \leq z \leq 4$. We have

Case I (a) $q = 6k - 1$. The choice of the l interchanges, $g_{(p+1)/2,j_u} - g_{(p-1)/2,j_u}$, $1 \leq u \leq l$, means the choice of the l elements from $(v_1, v_2, v_3, v_4, 0, v_5, v_6)$. Let $V_0 = \phi$ and $V_j = \{v_{1j}, v_{2j}, v_{3,k+1-j}, v_{6,k+1-j}\}$, $1 \leq j \leq k-1$. The l interchanges are as follows.
(1) If $l = 4y + 1$, then choose V_j, $0 \leq j \leq y$ and v_{2k}.
(2) If $l = 4y + 2$, then choose V_j, $0 \leq j \leq y$ and v_{2k}, 0.
(3) If $l = 4y + 3$, then choose V_j, $0 \leq j \leq y$ and v_{2k}, $v_{4,k-1}$, v_{51}.
(4) If $l = 4y + 4$, then choose V_j, $0 \leq j \leq y$ and v_{2k}, $v_{4,k-1}$, 0, v_{51}.

By Lemma 4.3, we got $\sum\limits_{l \in V_j} l = 0, j = 0, 1, \ldots, y \leq k-1$, $v_{2k} = 1$ and $v_{4,k-1} + v_{51}$=0, hence, $\sum_{u=1}^{l}[g_{(p+1)/2,j_u} - g_{(p-1)/2,j_u}] = 1$ for (1) to (4).

Case I (b) $q = 6k + 1$. The choice of the l interchanges, $g_{(p+1)/2,j_u} - g_{(p-1)/2,j_u}$, $1 \leq u \leq l$, means the choice of the l elements from $(w_1, w_2, 0, w_3, w_4, w_5, w_6)$. Let $W_0 = \phi$ and $W_j = \{w_{1j}, w_{4j}, w_{5,k+1-j}, w_{6,k+1-j}\}$, $1 \leq j \leq k-1$. The l interchanges are as follows.
(1) If $l = 4y + 1$, then choose W_j, $0 \leq j \leq y$ and w_{4k}.
(2) If $l = 4y + 2$, then choose W_j, $0 \leq j \leq y$ and w_{4k}, 0.
(3) If $l = 4y + 3$, then choose W_j, $0 \leq j \leq y$ and w_{4k}, w_{2k}, w_{31}.
(4) If $l = 4y + 4$, then choose W_j, $0 \leq j \leq y$ and w_{4k}, w_{2k}, 0, w_{31}.

By Lemma 4.4, we obtained $\sum\limits_{l \in U_j} l = 0, j = 0, 1, \ldots, y \leq k-1$, $w_{4k} = 1$ and $v_{2k}+v_{31}$=0, hence, $\sum_{u=1}^{l}[g_{(p+1)/2,j_u} - g_{(p-1)/2,j_u}] = 1$ for (1) to (4).

Let c_j, $1 \leq j \leq q$ denote the j-th column position of H_p. Then the above solution suggests interchange between $(\frac{p-1}{2})$-th and $(\frac{p+3}{2})$-th rows of H_p at l positions as below:

For $q = 6k - 1$, let $C_0^1 = \phi$ and $C_j^1 = \{c_j, c_{k+j}, c_{3k+1-j}, c_{6k-j}\}$, $1 \leq j \leq (k-1)$.

(1) If $l = 4y + 1$, then choose C_j^1, $0 \leq j \leq y$ and c_{2k}.

(2) If $l = 4y + 2$, then choose C_j^1, $0 \leq j \leq y$ and c_{2k}, c_{4k}.

(3) If $l = 4y + 3$, then choose C_j^1, $0 \le j \le y$ and c_{2k}, c_{4k-1}, c_{4k+1}.

(4) If $l = 4y + 4$, then choose C_j^1, $0 \le j \le y$ and c_{2k}, c_{4k-1}, c_{4k}, c_{4k+1}.

For $q = 6k + 1$, let $C_0^2 = \phi$ and $C_j^2 = \{c_j, c_{3k+1+j}, c_{5k+2-j}, c_{6k+2-j}\}$, $1 \le j \le k - 1$.

(1) If $l = 4y + 1$, C_j^2, $0 \le j \le y$ and c_{4k+1}.

(2) If $l = 4y + 2$, C_j^2, $0 \le j \le y$ and c_{4k+1}, c_{2k+1}.

(3) If $l = 4y + 3$, C_j^2, $0 \le j \le y$ and c_{4k+1}, c_{2k}, c_{2k+2}.

(4) If $l = 4y + 4$, C_j^2, $0 \le j \le y$ and c_{4k+1}, c_{2k}, c_{2k+1}, c_{2k+2}.

Case II: Interchanges between row elements of H_p corresponding to T_U and T_L

Let $T_U = \begin{pmatrix} T_{U1} \\ T_{U2} \end{pmatrix}$ and $T_L = \begin{pmatrix} T_{L1} \\ T_{L2} \end{pmatrix}$. Let $\lfloor z \rfloor$ denote the largest integer less than or equal to z. As rows of G_p, the four rows of T appears as follows. For $\frac{p-1}{2} - 2\lfloor \frac{p-3}{4} \rfloor \le i \le \frac{p-3}{2} - 1$ and i taking alternate values (i.e., $1, 3, \ldots$ or $2, 4, \ldots$), T_{U1} appear as i-th row, T_{U2} appear as $(i+1)$-th row, T_{L1} appear as $(p+1-i-1)$-th row and T_{L2} appear as $(p+1-i)$-th row of G_p. For $\frac{p-1}{2} - 2\lfloor \frac{p-3}{4} \rfloor \le i \le \frac{p-3}{2}$, the difference between the $(p+1-i)$-th and i-th row sums of H_p is $q(p+1-2i)$. Suppose l corresponding elements between the $(p+1-i)$-th and i-th rows of H_p are interchanged. Again, let the sum of the differences between these l elements in G_p equals s. Then l and s satisfies $ps + (p+1-2i)l = q(p+1-2i)/2$ with a solution $s = (p+1-2i)/2$ and $l = (q-p)/2$ for carrying out interchanges between $(p+1-i)$-th and i-th rows of G_p.

Let $l = (q-p)/2 = 2y + z$ (say), $y \ge 0$, $z = 1, 2$. The choice of the l interchanges, $g_{(p+1-i),j_u} - g_{i,j_u}$, $1 \le u \le l$, means the choice of the l elements from $(0, 1, 2, \ldots, q'-2, q'-1, -q'+1, -q'+2, \ldots, -2, -1)$. Let $V_0 = \phi$ and $V_j = \{q'-j, -q'+j\}$, $1 \le j \le q'-2$. The l interchanges are as follow.

(1) If $l = 2y + 1$, then choose V_j, $0 \le j \le y$ and 1.

(2) If $l = 2y + 2$, then choose V_j, $0 \le j \le y$ and 0, 1.

It is clear to see $\sum_{l \in V_j} l = 0, j = 0, 1, \ldots, y \le q'-2$, hence, $\sum_{u=1}^{l} [g_{(p+1-i),j_u} - g_{i,j_u}] = 1$ for (1) and (2).

Let c_j, $1 \le j \le q$ denote the j-th column position of H_p. Then the above solution suggests interchange between $(p+1-i)$-th and i-th rows of H_p at l positions as below:

Let $C_0 = \phi$ and $C_j = \{c_{\frac{q+3}{2}-j}, c_{\frac{q+1}{2}-j}\}$, $1 \le j \le q'-2$.

(1) If $l = 2y + 1$, then choose C_j, $0 \le j \le y$ and $c_{\frac{p+3-2i}{2}}$.

(2) If $l = 2y + 2$, then choose C_j, $0 \le j \le y$ and c_1, $c_{\frac{p+3-2i}{2}}$.

Case III: Interchanges between row elements of H_p corresponding to R_U and R_L
The two rows of R appear as 1-st and p-th rows of G_p. Also, the difference between the p-th and 1-st row sums of H_p is $q(p-1)$. Suppose l corresponding elements between the p-th and 1-st rows of H_p are interchanged. As earlier, let the sum of the differences between these l elements in G_p equals s. Then l and s satisfies $ps+(p-1)l = q(p-1)/2$ with a solution $s = (p-1)/2$ and $l = (q-p)/2$ for carrying out interchanges between p-th and 1-st rows of G_p.

Let $l = (q-p)/2 = 2y + z$ (say), $y \geq 0$, $z = 1, 2$. The choice of the l interchanges, $g_{p,j_u} - g_{1,j_u}$, $1 \leq u \leq l$, means the choice of the l elements from e_2R=$(q-1, q-3, \ldots, 4, 2, 0, -2, -4, \ldots, -q+3, -q+1)$. Let $V_0 = \phi$ and $V_j = \{q'-j, -q'+j\}$, 1 $\leq j \leq q'-2$. The l interchanges are as follow.

(1) If $l = 2y+1$, then choose V_j, $0 \leq j \leq y$ and $(p-1)/2$.

(2) If $l = 2y+2$, then choose V_j, $0 \leq j \leq y$ and $(p-1)/2$, 0.

It can be checked that $(p-1)/2$ is the $(2q-p+3)/4$-th element of e_2R and

$$\sum_{l\in V_j} l \;=\; 0, \quad j = 0, 1, \ldots, y \leq q'-2;$$

hence $\sum_{u=1}^{l}[g_{p,j_u} - g_{1,j_u}] = (p-1)/2$ for (1) and (2).

Let c_j, $1 \leq j \leq q$, denote the j-th column position of H_p. Then the above solution suggests interchange between p-th and 1-th rows of H_p at l positions as below:

Let $C_0 = \phi$ and C_j=$\{c_j,\, c_{q+1-j}\}$, $1 \leq j \leq q'-2$.

(1) If $l = 2y+1$, then choose C_j, $0 \leq j \leq y$ and $c_{(2q+3-p)/4}$.

(2) If $l = 2y+2$, then choose C_j, $0 \leq j \leq y$ and $c_{(2q+3-p)/4}$, $c_{(q+1)/2}$.

Acknowledgements

Thanks are due to a referee for some constructive suggestions. Part of this work was carried out while the second author was visiting The University of Akron.

References

[1] T. Bier and A. Kleinschmidt, Centrally symmetric and magic rectangles, *Discrete Math.* **176** (1997), 29–42.

[2] M. N. Das and P. Sarkar, Construction of magic rectangles with tables when row and column numbers are both odd, *J. Statist. Comp. Applic.* **4** (2006), 55–68.

[3] J. P. De Los Reyes, A. Das and C. K. Midha, A matrix approach to construct magic rectangles of even order, *Australas. J. Combin.* **40** (2008), 293–300.

[4] J. P. De Los Reyes, A. Das, C. K. Midha and P. Vellaisamy, On a method to construct magic rectangles of even order, *Utilitas Math.* **80** (2009), 277–284.

[5] T. Hagedorn, Magic rectangles revisited, *Discrete Math.* **207** (1999), 65–72.

(Received 30 Aug 2011; revised 3 Aug 2012)