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On the ubiquity and utility of cyclic schemes*

M. Abreu M.J. Funk†

Dipartimento di Matematica e Informatica
Università della Basilicata
85100 Potenza
Italy
marien.abreu@unibas.it martin.funk@unibas.it

D. Labbate

Dipartimento di Matematica
Politecnico di Bari
Via E. Orabona, 4–70125 Bari
Italy
labbate@poliba.it

V. Napolitano

Dipartimento di Ingegneria Civile
Seconda Università di Napoli
via Roma, 29–81031 Aversa
Italy
vito.napolitano@unina2.it

Abstract

Let k, l, m, n, and μ be positive integers. A $\mathbb{Z}_\mu$-*scheme of valency* (k,l) and *order* (m,n) is an $m \times n$ array (S_{ij}) of subsets $S_{ij} \subseteq \mathbb{Z}_\mu$ such that for each row and column one has $\sum_{j=1}^{n} |S_{ij}| = k$ and $\sum_{i=1}^{m} |S_{ij}| = l$, respectively. Any such scheme is an algebraic equivalent of a (k,l)-semi-regular bipartite voltage graph with n and m vertices in the bipartition sets and voltages coming from the cyclic group $\mathbb{Z}_\mu$. We are interested in the subclass of $\mathbb{Z}_\mu$-schemes that are characterized by the property $a-b+c-d \not\equiv 0 \pmod{\mu}$ for all $a \in S_{ij}$, $b \in S_{ih}$, $c \in S_{gh}$, and $d \in S_{gj}$ where $i, g \in \{1, \dots, m\}$ and $j, h \in \{1, \dots, n\}$ need not be distinct. These $\mathbb{Z}_\mu$-schemes can be used to represent adjacency matrices of regular graphs of girth

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† Corresponding author

≥ 5 and semi-regular bipartite graphs of girth ≥ 6. For suitable $\rho, \sigma \in \mathbb{N}$ with $\rho k = \sigma l$, they also represent incidence matrices for polycyclic $(\rho\mu_k, \sigma\mu_l)$ configurations and, in particular, for all known Desarguesian elliptic semiplanes. Partial projective closures yield *mixed* $\mathbb{Z}_\mu$*-schemes*, which allow new constructions for Krčadinac's sporadic configuration of type (34_6) and Balbuena's bipartite $(q-1)$-regular graphs of girth 6 on as few as $2(q^2-q-2)$ vertices, with q ranging over prime powers. Besides some new results, this survey essentially furnishes new proofs in terms of (mixed) $\mathbb{Z}_\mu$-schemes for ad hoc constructions used thus far.

1 $\mathbb{Z}_\mu$-Schemes and Cyclic Voltage Graphs

Preliminary note. This paper deals with constructions in some classes of $(0,1)$-matrices, which turn up as incidence matrices of configurations or adjacency matrices of graphs. Even if basic notions and notations seem to be generally known and widely used, misunderstandings can arise since precise formal definitions vary slightly from author to author. So one might be tempted to fix every notion to the least detail, at the risk of distracting the reader's attention from the essentially new concepts. To overcome this dilemma, the reader will find a synopsis on $(0,1)$-matrices, graphs, and configurations in Section 9. Notions defined in the synopsis are set up in italics at their very first appearance in the paper.

Definition 1.1 A $\mathbb{Z}_\mu$**-scheme of order** (m,n) is an $m \times n$ array $M^{(\mu)} = (S_{ij})$ of subsets $S_{ij} \subseteq \mathbb{Z}_\mu$. The $\mathbb{Z}_\mu$-scheme $M^{(\mu)}$ has **valency** (k,l) if, for each row and column, the sums of the cardinalities of the entries have constant values k and l, i.e.

$$\sum_{j=1}^{n} |S_{ij}| = k \quad \text{and} \quad \sum_{i=1}^{m} |S_{ij}| = l\,.$$

If each entry has cardinality ≤ 1 and precisely 1, the scheme is called **simple** and **full**, respectively. If $m = n$ and $k = l$, we say that $M^{(\mu)}$ has **order** n and **valency** k, respectively. A $\mathbb{Z}_\mu$-scheme of order n is said to be **skew-symmetric** if $S_{ij} = -S_{ji}$ for all $1 \leq i, j \leq n$.

Notation 1.2 When writing down a $\mathbb{Z}_\mu$-scheme $M^{(\mu)}$, the curly braces of the entries will always be omitted. Accordingly, the empty set $\emptyset = \{\}$ becomes a blank entry. If necessary, μ will be mentioned as superscript $^{(\mu)}$.

A *circulant* $(0,1)$-matrix, say $\overline{C}$, is uniquely determined by the positions of the entries 1 in its first row. This gives rise to a bijective mapping, say ι, from the class of circulant $(0,1)$-matrices of order μ onto the power set of $\mathbb{Z}_\mu$, namely

$$\overline{C} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{\mu-2} & c_{\mu-1} \\ c_{\mu-1} & c_0 & c_1 & & c_{\mu-2} \\ \vdots & c_{\mu-1} & c_0 & \ddots & \vdots \\ c_2 & & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{\mu-1} & c_0 \end{pmatrix} \quad \longmapsto \quad C := \{ i \in \mathbb{Z}_\mu \mid c_i = 1 \}\,,$$

where the empty set becomes the image of the zero matrix of order μ. When speaking of positions (i,j) in circulant matrices of order μ, the indices range over $\{0,\ldots,\mu-1\}=\mathbb{Z}_\mu$. For later use, the rule determining the inverse mapping is worthwhile to be stated explicitly:

Lemma 1.3 *$\overline{C}$ has entry 1 in position (i,j) if and only if $j-i$ (mod μ) belongs to C.* □

The mapping induces the following notation for $(0,1)$-block matrices with circulant blocks ([1]):

Definition 1.4 The **blow-up** of a $\mathbb{Z}_\mu$-scheme $M^{(\mu)}=(S_{ij})$ is the block $(0,1)$-matrix $\overline{M^{(\mu)}}$ with square blocks of order μ which is obtained from $M^{(\mu)}$ by substituting the circulant $(0,1)$-matrices $\overline{S_{ij}}$ for the entries S_{ij}.

In the sequel, the position of an entry in the blow-up $\overline{M^{(\mu)}}$ will be given in terms of the position (i,j) of the block S_{ij} and the **local** position (i',j') within the circulant block S_{ij}.

Adjacency matrices of *graphs* are symmetric $(0,1)$-matrices with entries 0 on the main diagonal. These two properties can easily be translated into the language of $\mathbb{Z}_\mu$-schemes.

Proposition 1.5 *The blow-up $\overline{M^{(\mu)}}$ of a square $\mathbb{Z}_\mu$-scheme $M^{(\mu)}=(S_{ij})$ of order n is symmetric if and only if $M^{(\mu)}$ is skew-symmetric.*

Proof. Let a be an element in S_{ij}. Then an entry 1 turns up in local position (i',j') in the circulant matrix $\overline{S_{ij}}$ if and only if $j'-i'\equiv a$ (mod μ). Symmetrically, 1 appears in local position (j',i') in the circulant matrix $\overline{S_{ji}}$ if and only if $i'-j'\equiv -a\in S_{ji}$, numbers taken modulo μ. □

Corollary 1.6 *The blow-up $\overline{M^{(\mu)}}$ of a square $\mathbb{Z}_\mu$-scheme $M^{(\mu)}=(S_{ij})$ of order n has entries 0 on its main diagonal if and only if $0\notin S_{ii}$ for all $i=1,\ldots,n$.*

Proof. Apply the above Proof in the case $i=j$ and $i'=j'$ to see that 1 is an entry on the main diagonal of $\overline{S_{ii}}$ if and only if $0\in S_{ii}$. □

In the light of these two statements, we call a skew-symmetric $\mathbb{Z}_\mu$-scheme $M^{(\mu)}=(S_{ij})$ of order n **admissible** if $0\notin S_{ii}$ for all $i=1,\ldots,n$. *Cyclic voltage graphs* and *admissible cyclic voltage assignments* are surveyed in the beginning of Section 9.

Remark 1.7 (i) Any admissible $\mathbb{Z}_\mu$-scheme $M^{(\mu)}=(S_{ij})$ of order n arises from, and gives rise to, a cyclic voltage graph (K,α) on n vertices with an admissible cyclic voltage assignment α. Labelling the vertices of K by $1,\ldots,n$, the rules

$$S_{ij}:=\left\{\, a\in\mathbb{Z}_\mu \;\middle|\; \alpha(e)=\begin{array}{c}a\\-a\end{array} \text{ for some edge } e\in EK \text{ running from } \begin{array}{c}i\text{ to }j\\ j\text{ to }i\end{array} \right\}$$

for $i \neq j$ as well as

$$S_{ii} \ := \ \{a, -a \in \mathbb{Z}_\mu \mid \alpha(e) = a \text{ for an } i\text{–based loop } e \in EK\ \}$$

construct $M^{(\mu)}$ from (K, α). Vice versa, given $M^{(\mu)} = (S_{ij})$, let K be the general graph with vertex set $VK := \{1, \dots, n\}$ where $|S_{ij}|$ edges run from i to j with distinct voltages $a \in S_{ij}$ and eventually a vertex i is base of $\frac{1}{2}|S_{ii}|$ loops with voltage $\pm b \in S_{ii}$. Both constructions do comply with admissibility.
(ii) An arbitrary $\mathbb{Z}_\mu$-scheme $M^{(\mu)} = (S_{ij})$ of order (m, n) arises from, and gives rise to, a bipartite cyclic voltage graph on m white and n black vertices, with an admissible cyclic voltage assignment. Denote by $-M^{(\mu)}$ the $\mathbb{Z}_\mu$-scheme obtained from $M^{(\mu)}$ by substituting each entry with its opposite element in $\mathbb{Z}_\mu$, and let O_ν be the trivial $\mathbb{Z}_\mu$-scheme of order ν all of whose entries are $\emptyset$. Then

$$\begin{pmatrix} O_m & M^{(\mu)} \\ (-M^{(\mu)})^T & O_n \end{pmatrix}$$

is an admissible $\mathbb{Z}_\mu$-scheme of order $m+n$ and (i) applies, both constructions being compatible with bipartite (general) graphs.

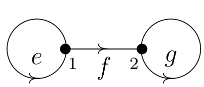
Proposition 1.8 *If $M^{(\mu)} = (S_{ij})$ is an admissible $\mathbb{Z}_\mu$-scheme with associated voltage graph (K, α), the blow-up $\overline{M^{(\mu)}}$ is an adjacency matrix of the lift of K through $\mathbb{Z}_\mu$ via α.*

Proof. Order the vertices $(i, a) \in VK \times \mathbb{Z}_\mu$ lexicographically with respect to the natural orders $1, \dots, n$ for the vertices in K and $0, 1, \dots, \mu - 1$ for the elements in $\mathbb{Z}_\mu$. □

Involving some regularity condition, Remark 1.7 reads:

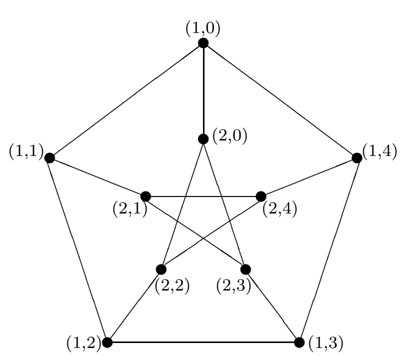
Proposition 1.9 *A $\mathbb{Z}_\mu$-scheme of order (m, n) and valency (k, l) is equivalent to a (k, l)-semi-regular bipartite voltage graph on n white and m black vertices with voltages from the cyclic group $\mathbb{Z}_\mu$, while an admissible $\mathbb{Z}_\mu$-scheme of order n and valency k is equivalent to a k-regular cyclic voltage graph on n vertices with an admissible voltage assignment.* □

Example 1.10 The celebrated Petersen graph can be seen as a lift of the dumbbell graph through $\mathbb{Z}_5$:



voltages in $\mathbb{Z}_5$:
$\alpha(e) = 1\,,\ \alpha(f) = 0\,,\ \alpha(g) = 2$

$\mathbb{Z}_5$-scheme: $M^{(5)} = \begin{pmatrix} 1,4 & 0 \\ 0 & 2,3 \end{pmatrix}^{(5)}$



2 J_2-Free $\mathbb{Z}_\mu$-Schemes

As usual, let J_n denote the square matrix all of whose entries are 1. Then J_2 is the *incidence matrix* of a **di-gon**, i.e. the structure made up by two distinct points p_1, p_2, two distinct lines L_1, L_2, and all four incidences $p_i|L_j$ with $i, j \in \{1, 2\}$. Di-gons are forbidden substructures of *configurations*. Thus, disregarding *regularity* conditions, incidence matrices of configurations are $(0, 1)$-matrices characterized by the following property:

Definition 2.1 A $(0, 1)$-matrix is called J_2**-free** if every 2×2 submatrix contains at least one entry 0. In a figurative sense, a $\mathbb{Z}_\mu$-scheme $M^{(\mu)}$ is said to be J_2**-free** if its blow-up $\overline{M^{(\mu)}}$ is so.

In [1, 2, 3, 12] such matrices were called "linear."

Criterion 2.2 *A $\mathbb{Z}_\mu$-scheme $M^{(\mu)} = (S_{ij})$ of order (m, n) is J_2-free if and only if for all (not necessarily distinct) $1 \le i, g \le m$ and $1 \le j, h \le n$ and all $a \in S_{ij}$, $b \in S_{ih}$, $c \in S_{gh}$, and $d \in S_{gj}$ one has*

$$(\dagger) \qquad a - b + c - d \not\equiv 0 \ (mod\, \mu) .$$

Proof. To prove sufficiency, assume that $\overline{M^{(\mu)}}$ has a sub-matrix J of order 2 all of whose entries are 1. By construction, the upper left 1 in J appears as an entry in local position (i', j') in the block $\overline{S_{ij}}$, for some $i', j' \in \{0, \ldots, \mu - 1\}$, $i \in \{1, \ldots, m\}$, and $j \in \{1, \ldots, n\}$. This, in turn, implies that $a :\equiv j' - i' \pmod{\mu}$ is an element of the set S_{ij}. Analogously, the upper right, lower right, and lower left entry 1 in J arise from entries 1 in local positions

$$\begin{array}{llcl} (i', h') & \text{in the block } S_{ih} & \Longrightarrow & b :\equiv h' - i' \in S_{ih} \,, \\ (g', h') & \text{in the block } S_{gh} & \Longrightarrow & c :\equiv h' - g' \in S_{gh} \,, \\ (g', j') & \text{in the block } S_{gj} & \Longrightarrow & d :\equiv j' - g' \in S_{gj} \,, \end{array}$$

differences taken modulo μ. Subtracting the second and fourth congruences from the sum of the first and third, we obtain $0 \equiv a - b + c - d \pmod{\mu}$, a contradiction.

To prove necessity, suppose that there exist (not necessarily distinct) $i, g \in \{1, \ldots, m\}$ and $j, h \in \{1, \ldots, n\}$ such that for some $a \in S_{ij}$, $b \in S_{ih}$, $c \in S_{gh}$, and $d \in S_{gj}$ one has

$$(\ddagger) \qquad a - b + c - d \equiv 0 \pmod{\mu} .$$

In the first row of $\overline{S_{ij}}$ and $\overline{S_{ih}}$, there are entries 1 in local positions $(0, a)$ and $(0, b)$, respectively. Now consider the circulant $(0, 1)$-block $\overline{S_{gj}}$. Since $d \in S_{gj}$, there exists a row index $j' \in \{0, \ldots, \mu - 1\}$, such that $\overline{S_{gj}}$ has an entry 1 in local position (j', a), namely $j' :\equiv a - d \pmod{\mu}$. Then $(\ddagger)$ implies $j' \equiv c - b \pmod{\mu}$. Hence $\overline{S_{gh}}$ has an entry 1 in position (j', b) and $\overline{M^{(\mu)}}$ contains a 2×2 submatrix all of whose entries are 1, a contradiction. □

Condition $(\dagger)$ has some repercussions on non-empty entries S_{ij}:

Corollary 2.3 *Each entry $S_{ij} \neq \emptyset$ of a J_2-free $\mathbb{Z}_\mu$-scheme $M^{(\mu)} = (S_{ij})$ is a* deficient cyclic difference set.

Proof. Apply condition (†) in the case that $i = g$ and $j = h$: all the differences $a - b$, $c - d$ with $a \neq b$ and $c \neq d$ are distinct in pairs. □

Corollary 2.4 *Suppose that the $\mathbb{Z}_\mu$-scheme $M^{(\mu)} = (S_{ij})$ of order (m, n) is J_2-free. Then, for all $i, g \in \{1, \dots, m\}$ and $j, h \in \{1, \dots, n\}$, the differences covered by either S_{ij} and S_{ih} or by S_{ij} and S_{gj} are pairwise distinct.*

Proof. Apply condition (†) in the case that either $i = g$ or $j = h$. □

3 Polycyclic Configurations and $\mathbb{Z}_\mu$-Schemes

Boben and Pisanski [7] call an (m_k, n_l) configuration $\mathcal{C}$ **polycyclic** or μ-**cyclic** if $\mathcal{C}$ admits a cyclic automorphism of order μ whose orbits partition both the point set and the line set of $\mathcal{C}$ into subsets of size μ. This definition makes sense only if $1 < \mu \mid gcd(m, n)$ and $m = \rho\mu$, $n = \sigma\mu$ for suitable $\rho, \sigma \in \mathbb{N}$. *Cyclic* configurations (n_k) are n-cyclic.

Incidence matrices reveal the polycyclic structure of a configuration if a suitable labelling matches with the orbits under the cyclic automorphism.

Proposition 3.1 *A $(\rho\mu_k, \sigma\mu_l)$ configuration $\mathcal{C}$ is polycyclic if and only if it admits an incidence matrix $\overline{M^{(\mu)}}$ obtained by blowing up a $\mathbb{Z}_\mu$-scheme $M^{(\mu)}$ of valency (k, l) and order (ρ, σ).*

Proof. Sufficiency is guaranteed by the very construction: $\overline{M^{(\mu)}}$ admits a cyclic automorphism of order μ, namely the simultaneous action of the intrinsic cyclic automorphism on each circulant block of $\overline{M^{(\mu)}}$. This automorphism induces an automorphism of $\mathcal{C}$, whose orbits partition both the point set and the line set of $\mathcal{C}$ into ρ and σ subsets of size μ, respectively. Hence $\mathcal{C}$ is polycyclic.

To prove necessity, suppose that $\mathcal{C}$ is polycyclic with respect to some automorphism φ of order μ. Choose representatives $p_0, \dots, p_{m-1}$ in the point set and $L_0, \dots, L_{m-1}$ in the line set of $\mathcal{C}$ for the orbits; i.e. using the abbreviation $(i) := \varphi^i$, one has the following cycle decompositions:

$$\begin{array}{ccc} (p_0^{(0)}\ p_0^{(1)}\ p_0^{(2)} \dots p_0^{(\mu-1)}) & \dots & (p_{\rho-1}^{(0)}\ p_{\rho-1}^{(1)}\ p_{\rho-1}^{(2)} \dots p_{\rho-1}^{(\mu-1)}) \\ (L_0^{(0)}\ L_0^{(1)}\ L_0^{(2)} \dots\ L_0^{(\mu-1)}) & \dots & (L_{\sigma-1}^{(0)}\ L_{\sigma-1}^{(1)}\ L_{\sigma-1}^{(2)} \dots\ L_{\sigma-1}^{(\mu-1)}) \end{array}$$

Let $\overline{M_\varphi}$ be the incidence matrix for $\mathcal{C}$ obtained when labelling its points and lines according to the above cycle decompositions of φ. Interpret

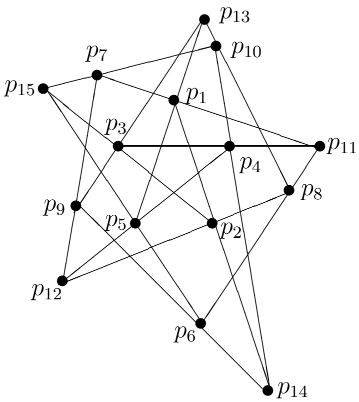
$$\overline{M_\varphi} = \begin{pmatrix} \overline{M_{0,0}} & \overline{M_{0,1}} & \dots & \overline{M_{0,m-1}} \\ \overline{M_{1,0}} & \overline{M_{1,1}} & \dots & \overline{M_{1,m-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{M_{m-1,1}} & \overline{M_{m-1,2}} & \dots & \overline{M_{m-1,m-1}} \end{pmatrix}$$

as an $m \times m$ block matrix with square blocks $\overline{M_{i,j}}$ of order μ.

Then each block $\overline{M_{i,j}}$ is a circulant $(0,1)$-matrix (which might also be a copy of the zero matrix): in fact, if, for some $i,j,s,t \in \{0,\ldots,\mu-1\}$, the point $p_i^{(s)}$ and the line $L_j^{(t)}$ are incident, so are their φ-images, i.e. $p_i^{(s+1)}$ is incident with $L_j^{(t+1)}$, apices taken modulo μ; this implies that for any entry 1 in position (s,t) in the block $M_{i,j}$, there exist entries 1 also in the positions $(s+z, t+z)$ for $z = 1,\ldots,\mu-1$, numbers again taken modulo μ; Hence $\overline{M_{i,j}}$ is a circulant matrix. A block $M_{i,j}$ has all its entries 0 if for each $s \in \{0,\ldots \mu-1\}$ the point $p_i^{(s)}$ is not incident with any line $L_j^{(t)}$ with $t \in \{0,\ldots,\mu-1\}$.

$\overline{M_\varphi}$ can be seen as the blow-up of a $\mathbb{Z}_\mu$-scheme of order (ρ,σ), say M_φ, obtained by applying ι to the blocks of $\overline{M_\varphi}$. The valency of M_φ is (k,l), since $\overline{M_\varphi}$ has exactly k and l entries 1 in each row and column, respectively. □

Example 3.2 The Cremona-Richmond configuration ([8], represented geometrically in the figure below) is a 5-cyclic (15_3) configuration.



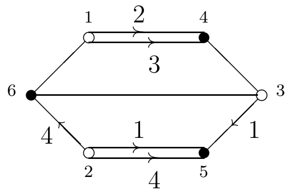
The permutation

$$(1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)(11\ 12\ 13\ 14\ 15)\,,$$

acting on the indices of the points, induces an automorphism of order 5, which partitions both the point and line sets into three orbits of length 5 each. Choose the points p_1, p_6, p_{11} and the lines $\{p_3, p_4, p_{11}\}$,$\{p_7, p_{10}, p_{15}\}$, $\{p_1, p_7, p_{11}\}$ as first elements in each orbit. Then the resulting incidence matrix is the blow-up of the $\mathbb{Z}_5$-scheme

$$M_{CR}^{(5)} \quad := \quad \begin{pmatrix} 2,3 & & 0 \\ & 1,4 & 4 \\ 0 & 1 & 0 \end{pmatrix}^{(5)} .$$

Note that the associated bipartite cyclic voltage graph differs slightly from the one given in [7, Figure 6] or in [29, Figure 4(b)]:



As in [29], orientation and voltage are omitted if an edge gets voltage $0 \in \mathbb{Z}_5$.

Example 3.3 Reye's $(12_4, 16_3)$ configuration (cf. e.g. [7, Figure 2], [18, Footnote on P. 140]) is represented by the $\mathbb{Z}_4$-scheme

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0,1 & & 2 & 3 \\ & 0,3 & 2 & 1 \end{pmatrix}^{(4)} .$$

Inverting the approach, (n_k) configurations can be constructed for whose parameters n, k no instances were known so far:

Example 3.4 [12] The $\mathbb{Z}_7$-scheme

$$T_{98}^{(7)} = \begin{pmatrix}
0,1,3 & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& 0,1,3 & & & & & & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
& & 0,1,3 & & & & & 0 & 2 & 4 & 6 & 1 & 3 & 5 \\
& & & 0,1,3 & & & & 0 & 3 & 6 & 2 & 5 & 1 & 4 \\
& & & & 0,1,3 & & & 0 & 4 & 1 & 5 & 2 & 6 & 3 \\
& & & & & 0,1,3 & & 0 & 5 & 3 & 1 & 6 & 4 & 2 \\
& & & & & & 0,1,3 & 0 & 6 & 5 & 4 & 3 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2,3,5 & & & & & & \\
0 & 6 & 5 & 4 & 3 & 2 & 1 & & 2,3,5 & & & & & \\
0 & 5 & 3 & 1 & 6 & 4 & 2 & & & 2,3,5 & & & & \\
0 & 4 & 1 & 5 & 2 & 6 & 3 & & & & 2,3,5 & & & \\
0 & 3 & 6 & 2 & 5 & 1 & 4 & & & & & 2,3,5 & & \\
0 & 2 & 4 & 6 & 1 & 3 & 5 & & & & & & 2,3,5 & \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & & & & & & & 2,3,5
\end{pmatrix}^{(7)}$$

represents a (98_{10}) configuration. Criterion 2.2 guarantees that the blow-up $\overline{T_{98}^{(7)}}$ is J_2-free: all non-simple full 2×2 sub-schemes are of type $\begin{pmatrix} 0,1,3 & z \\ -z & 2,3,5 \end{pmatrix}^{(7)}$ for some $z \in \mathbb{Z}_7$ and

$$a - z + c + z \not\equiv 0 \pmod 7 \text{ for all } a \in \{0, 1, 3\} \text{ and } c \in \{2, 3, 5\},$$

whereas all simple full 2×2 sub-schemes arise from the full multiplication table of $GF(7)$, for details see [12].

However, not every configuration which has found some consideration in the literature is polycyclic.

Example 3.5 The unique $(9_4, 12_3)$ configuration cannot be represented by any $\mathbb{Z}_3$-scheme. Geometrically, it turns up as the configuration of the nine points of inflection of a third-order plane curve without double points in the complex projective plane, see e.g. [18, P. 102]. It can also be seen as the affine plane over $GF(3)$. Its automorphism group has order 432. A representation which exhibits a maximum polycyclic subconfiguration isomorphic to the Pappian (9_3) reads

$$\begin{pmatrix} 0 & 0 & 0 & \mathbf{c_1} \\ 0 & 1 & 2 & \mathbf{c_2} \\ 0 & 2 & 1 & \mathbf{c_3} \end{pmatrix}^{(3)}$$

where the blow-up $\overline{\mathbf{c_i}}$ of the symbol $\mathbf{c_i}$ is the 3×3 matrix whose entries in the i^{th} column are 1, and 0 otherwise (cf. Definition 6.1).

4 Elliptic Semiplanes as Polycyclic Configurations

(*Desarguesian*) *elliptic semiplanes* are surveyed in the very last paragraph of Section 9. Let $q = p^\nu$ be a prime power. In [1] it is pointed out that Desarguesian elliptic semiplanes of types C and L admit incidence matrices of orders q^2 and $q^2 - 1 = (q+1)(q-1)$, respectively, which are $q \times q$ and $(q+1) \times (q+1)$ block matrices with square blocks of orders q and $q-1$. The blocks are related to certain addition and multiplication tables of the finite field $GF(q)$. This result has been obtained by choosing suitable coordinates, which, in turn, depend on the choice of a suitable labelling for the elements of $GF(q)$. In general, however, the matrices constructed in [1] cannot be represented by $\mathbb{Z}_p$-schemes or $\mathbb{Z}_q$-schemes. In this Section we show how this can be achieved by fine-tuning the choice of the labelling.

Recurrently we will use the following tool:

Definition 4.1 Let M be a matrix of order (m, n) with entries in $GF(q)$. For each $x \in GF(q)$, we extract its **position matrix** P_x, i.e. the $(0,1)$-matrix of order (m, n) whose entry in position (i, j) is defined by

$$(P_x)_{i,j} := \begin{cases} 1 & \text{if } x \text{ appears as an entry in position } (i,j)\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

Construction 4.2 The multiplicative group $(GF(q)^*, \cdot)$ is a cyclic group of order $q-1$, hence one has

$$GF(q)^* = \langle y \rangle = \{y, y^2, \ldots, y^{q-2}, y^{q-1} = 1\}$$

for a fixed generator $y \in GF(q)^*$. Write down the quotient table of $(GF(q)^*, \cdot)$ with respect to the canonical order $1, y, y^2, \ldots, y^{q-2}$ for the elements:

:	1	y	y^2	y^3	y^4	$\ldots$	y^{q-2}
1	1	y^{-1}	y^{-2}	y^{-3}	y^{-4}	$\ldots$	y^{-q+2}
y	y	1	y^{-1}	y^{-2}	y^{-3}	$\ldots$	y^{-q+3}
y^2	y^2	y	1	y^{-1}	y^{-2}	$\ldots$	y^{-q+4}
y^3	y^3	y^2	y	1	y^{-1}	$\ldots$	y^{-q+5}
y^4	y^4	y^3	y^2	y	1	$\ldots$	y^{-q+6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
y^{q-2}	y^{q-2}	y^{q-3}	y^{q-4}	y^{q-5}	y^{q-6}	$\ldots$	1

Taking into account that $y^{q-1} = 1$ and hence $y^{-q+2} = y$, $y^{-q+3} = y^2$, etc, this quotient table reveals itself as a circulant matrix of order $q-1$. Since an element $x \in GF(q)^*$ appears in each row and column of the quotient table precisely once, its position matrix P_x is a permutation matrix; in particular, P_x is a circulant $(0,1)$-matrix, which can be characterised by the only entry 1 in its first row using the bijection ι; this leads to the rule

$$P_{y^{-i}} = \overline{\{i\}} \quad \text{for } i \in \mathbb{Z}_{q-1}\,.$$

Construction 4.3 Consider the additive group $(GF(q), +)$ and label its elements, say $x_0, x_1, \ldots, x_{q-1}$, such that $x_0 = 0$. Write down the difference table of $(GF(q), +)$ with respect to this labelling. Note that an entry of this difference table is equal to 0 if and only if it lies in its main diagonal, whereas all the other entries are actually elements of $GF(q)^*$. Let $L^{(q-1)} := (\lambda_{ij})_{0 \le i,j \le q}$ be the $\mathbb{Z}_{q-1}$-scheme of order $q+1$ defined by

$$\lambda_{i,j} := \begin{cases} \text{blank} & \text{if } \ i = j; \\ \quad 0 & \text{if } \ i = q \ \text{ or } \ j = q\,, \text{ but not both;} \\ \quad z & \text{if } \ i, j \in \{0, \ldots, q-1\} \text{ with } i \neq j \text{ such that } x_i - x_j = y^z. \end{cases}$$

Obviously, $L^{(q-1)}$ is a simple $\mathbb{Z}_\mu$-scheme of valency q which has blank entries on its main diagonal.

Lemma 4.4 *The $\mathbb{Z}_{q-1}$-scheme $L^{(q-1)}$ is J_2-free.*

Proof. Apply Criterion 2.2: let $\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right)^{(q-1)}$ be a full sub-scheme of $L^{(q-1)}$ and distinguish two cases.

(i) The entries b, c, and d lie neither in the last column nor in the last row of $L^{(q-1)}$. Then, by construction, there exist elements $x_i, x_j, x_g, x_h \in GF(q)$ with $x_i \neq x_g$ and $x_j \neq x_h$ such that

$$x_i - x_j = y^a\,, \ \ x_i - x_h = y^b\,, \ \ x_g - x_j = y^d\,, \ \ x_g - x_h = y^c\,.$$

Then

$$a - b + c - d \not\equiv 0 \pmod{q-1}$$

if and only if

$$1 \neq y^{a-b+c-d} = \frac{y^a y^c}{y^b y^d} = \frac{(x_i - x_j)(x_g - x_h)}{(x_i - x_h)(x_g - x_j)} = \frac{x_i x_g - x_i x_h - x_j x_g + x_j x_h}{x_i x_g - x_i x_j - x_h x_g + x_h x_j},$$

which holds true if and only if

$$x_i x_h + x_j x_g \neq x_i x_j + x_h x_g ,$$

or, equivalently,

$$(x_i - x_g)(x_h - x_j) \neq 0 .$$

(*ii*) The entries b and c lie in the last column or c and d lie in the last row of $L^{(q-1)}$. Then the full 2×2 sub-scheme reads either $\left(\begin{smallmatrix} a & 0 \\ d & 0 \end{smallmatrix}\right)^{(q-1)}$ or $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)^{(q-1)}$ and one has $a - b + c - d \not\equiv 0 \pmod{q-1}$ since otherwise either $y^a = y^d$ or $y^a = y^b$ would appear twice in one and the same column or row of the difference table $(GF(q), +)$, a contradiction. □

Proposition 4.5 *The Desarguesian elliptic semiplane $\mathcal{S}^L_{q^2-1}$ of type L derived from $PG(2,q)$ is isomorphic to the $(q-1)$-cyclic configuration of type $\left((q^2-1)_q\right)$ represented by the J_2-free simple $\mathbb{Z}_{q-1}$-scheme $L^{(q-1)}$ of order $q+1$ and valency q.*

Proof. It is sufficient to check that the construction of [1] applies also for the above labelling for the elements of $GF(q)^*$. The multiplication table of $(GF(q)^*, \times)$ used in [1] is actually a quotient table and matches with the above way of writing it down. □

Example 4.6 For later application, construct the $\mathbb{Z}_6$-scheme $L^{(6)}$ representing an incidence matrix for the Desarguesian elliptic semiplane $\mathcal{S}^L_{48}$ on 48 points. The tables

:	1	3	2	6	4	5
1	1	5	4	6	2	3
3	3	1	5	4	6	2
2	2	3	1	5	4	6
6	6	2	3	1	5	4
4	4	6	2	3	1	5
5	5	4	6	2	3	1

and

−	0	1	2	3	4	5	6
0	0	6	5	4	3	2	1
1	1	0	6	5	4	3	2
2	2	1	0	6	5	4	3
3	3	2	1	0	6	5	4
4	4	3	2	1	0	6	5
5	5	4	3	2	1	0	6
6	6	5	4	3	2	1	0

are a quotient table of $GF(7)^* = \langle 3 \rangle$ and a difference table of $GF(7)$ according to Constructions 4.2 and 4.3 , respectively. Then the position matrices of the elements $1 = 3^0,\ 2 = 3^2,\ 3 = 3^1,\ 4 = 3^4,\ 5 = 3^5$, and $6 = 3^3$ in $GF(7)^*$ extracted from the quotient table read $\overline{0}$, $\overline{4}$, $\overline{5}$, $\overline{2}$, $\overline{1}$, and $\overline{3}$, respectively, and the difference table gives rise to the the following $\mathbb{Z}_6$-scheme:

$$L^{(6)} = \begin{pmatrix} & 3 & 1 & 2 & 5 & 4 & 0 & 0 \\ 0 & & 3 & 1 & 2 & 5 & 4 & 0 \\ 4 & 0 & & 3 & 1 & 2 & 5 & 0 \\ 5 & 4 & 0 & & 3 & 1 & 2 & 0 \\ 2 & 5 & 4 & 0 & & 3 & 1 & 0 \\ 1 & 2 & 5 & 4 & 0 & & 3 & 0 \\ 3 & 1 & 2 & 5 & 4 & 0 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}^{(6)}$$

Construction 4.7 Consider the finite field $GF(q)$ as $GF(p)[t]/(f(t))$ for some irreducible polynomial $f(t) \in GF(p)[t]$ of degree ν. Then each element in $GF(q)$ can be represented as a polynomial of degree at most $\nu-1$ with coefficients in $GF(p)$. Label all the polynomials $\sum_{i=1}^{\nu-1} a_i t^i$ with zero constant terms by $\pi_1 = 0, \pi_2, \ldots, \pi_{p^{\nu-1}}$; they form a subgroup S of $(GF(q), +)$, which has a copy of $GF(p)$ as direct complement, namely the constant polynomials. Hence each element in $GF(q)$ may be written as $\pi_i + z$ for some $\pi_i \in S$ and $z \in GF(p)$. Choose the canonical order $0, 1, \ldots, p-1$ for the elements of $GF(p)$ and introduce a lexicographic order for $PG(q)$ by the rule

$$\pi_i + z < \pi_j + w \quad \text{if and only if} \quad \begin{cases} \text{either } i < j \\ \text{or } \ i = j \text{ and } z < w \end{cases}.$$

Write down the difference table of $(GF(q), +)$. Then the block, say B_{ij}, corresponding to minuends in $\pi_i + GF(p)$ and subtrahends in $\pi_j + GF(p)$ reads:

$-$	π_j	$\pi_j + 1$	$\ldots$	$\pi_j + p - 1$
π_i	$\pi_i - \pi_j$	$\pi_i - \pi_j - 1$	$\ldots$	$\pi_i - \pi_j - p + 1$
$\pi_i + 1$	$\pi_i - \pi_j + 1$	$\pi_i - \pi_j$	$\ldots$	$+\pi_i - \pi_j - p + 2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\pi_i + p - 1$	$\pi_i - \pi_j + p - 1$	$\pi_i - \pi_j + p - 2$	$\ldots$	$\pi_i - \pi_j$

The block B_{ij} is a circulant matrix, which is immediately seen by introducing the block

$$A := \boxed{\begin{matrix} 0 & -1 & \ldots & -p+1 \\ 1 & 0 & \ldots & -p+2 \\ \vdots & \vdots & \ddots & \vdots \\ p-1 & p-2 & \ldots & 0 \end{matrix}} \equiv \boxed{\begin{matrix} 0 & p-1 & \ldots & 1 \\ 1 & 0 & \ldots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ p-1 & p-2 & \ldots & 0 \end{matrix}}$$

(entries taken modulo p) and re-writing B_{ij} as $B_{ij} = \pi_i - \pi_j + A$. With these data, the difference table becomes a $p^{\nu-1} \times p^{\nu-1}$ block matrix with circulant blocks of order p, namely:

$-$	$GF(p)$	$\pi_1 + GF(p)$	$\ldots$	$\pi_{p^{\nu-1}} + GF(p)$
$GF(p)$	A	$-\pi_1 + A$	$\ldots$	$-\pi_{p^{\nu-1}} + A$
$\pi_1 + GF(p)$	$\pi_1 + A$	A	$\ldots$	$\pi_1 - \pi_{p^{\nu-1}} + A$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\pi_{p^{\nu-1}} + GF(p)$	$\pi_{p^{\nu-1}} + A$	$\pi_{p^{\nu-1}} - \pi_1 + A$	$\ldots$	A

The block structure reveals the difference table $D_S = (\pi_i - \pi_j)_{1 \leq i,j \leq p^{\nu-1}}$ for the subgroup S, seen as a set of representatives for the factor group $GF(q)/GF(p)$ where $GF(p)$ plays the rôle of the kernel under the epimorphism

$$\epsilon \quad : \quad \begin{cases} (GF(q), +) & \longrightarrow & S \\ \sum_{i=0}^{\nu-1} a_i t^i \; + \; (f(t)) & \longmapsto & \sum_{i=1}^{\nu-1} a_i t^i \end{cases}.$$

We use this fact to construct a $\mathbb{Z}_p$-scheme P_{π_i+z} of order $p^{\nu-1}$ for each $\pi_i + z \in GF(q)$: extract the position matrix, say Q_{π_i}, from the difference table D_S for the group S; then P_{π_i+z} is obtained from Q_{π_i} by substituting $\{z\}$ and a blank entry for each entry 1 and 0 in Q_{π_i}, respectively.

Lemma 4.8 *The blow-up of the $\mathbb{Z}_p$-scheme P_{π_i+z} is the position matrix of the element $\pi_i + z$ extracted from the above difference table for $GF(q)$.* □

Construction 4.9 Take up the quotient table for $GF(q)^*$ from Construction 4.2 and add a new q^{th} row and column all of whose entries are 0. Denote the resulting matrix by $G = (\gamma_{ij})_{0\le i,j\le q-1}$. Compose a block $\mathbb{Z}_p$-scheme $C^{(p)} := (\mathit{\Gamma}_{ij})_{0\le i,j\le q-1}$ following the rule

$$\mathit{\Gamma}_{ij} \ := \ P_{\pi+z} \quad \text{if and only if} \quad \gamma_{ij} = \pi + z \in S \oplus GF(p)\,.$$

Seen as a $\mathbb{Z}_p$-scheme, $C^{(p)}$ is simple and has order $qp^{\nu-1} = p^{2\nu-1}$ and valency q.

Lemma 4.10 *The $\mathbb{Z}_p$-scheme $C^{(p)}$ is J_2-free.*

Proof. Apply Criterion 2.2: let $\left(\begin{smallmatrix} a & b \\ d & c\end{smallmatrix}\right)^{(p)}$ be a full sub-scheme of $C^{(p)}$. By construction, a, b, c, and d lie in precisely four distinct blocks of $C^{(p)}$, say in $\mathit{\Gamma}_{ij}$, $\mathit{\Gamma}_{ih}$, $\mathit{\Gamma}_{gh}$, and $\mathit{\Gamma}_{gj}$, respectively, for $i, j, g, h \in \{0, \ldots, q-1\}$ with $i \ne g$ and $j \ne h$. Then there exist elements, say $\pi_a, \pi_b, \pi_c, \pi_d \in S$, such that

$$\mathit{\Gamma}_{ij} = P_{\pi_a+a}\,, \quad \mathit{\Gamma}_{ih} = P_{\pi_b+b}\,, \quad \mathit{\Gamma}_{gh} = P_{\pi_c+c}\,, \quad \text{and} \quad \mathit{\Gamma}_{gj} = P_{\pi_d+d}\,.$$

The upper left element a in $\left(\begin{smallmatrix} a & b \\ d & c\end{smallmatrix}\right)^{(p)}$ turns up in P_{π_a+a} in local position, say (i', j') for some $i', j' \in \{1, \ldots, p^{\nu-1}\}$. Analogously, b, c, and d appear in P_{π_b+b}, P_{π_c+c}, and P_{π_d+d} in local positions (i', h'), (g', h'), and (g', j'), respectively, for some $g', h' \in \{1, \ldots, p^{\nu-1}\}$. This implies that π_a, π_b, π_c, and π_d, turn up as entries in the difference table D_S in positions (i', j'), (i', h'), (g', h'), and (g', j'), respectively.

Hence

$$\pi_a = \pi_{i'} - \pi_{j'}\,, \quad \pi_b = \pi_{i'} - \pi_{h'}\,, \quad \pi_c = \pi_{g'} - \pi_{h'}\,, \quad \pi_d = \pi_{g'} - \pi_{j'}\,,$$

and one has

$$\pi_a - \pi_b + \pi_c - \pi_d \ = \ \pi_{i'} - \pi_{j'} - \pi_{i'} + \pi_{h'} + \pi_{g'} - \pi_{h'} - \pi_{g'} + \pi_{j'} \ = \ 0\,.$$

Now we distinguish two cases.

(i) The entries b, c, and d lie neither in the last column nor in the last row of blocks of $C^{(p)}$, i.e. $i, j, g, h \in \{0, \ldots, q-2\}$. Hence, by construction,

$$\pi_a + a = \gamma_{ij} = \frac{y^i}{y^j},\ \pi_b + b = \gamma_{ih} = \frac{y^i}{y^h},\ \pi_c + c = \gamma_{gh} = \frac{y^g}{y^h},\ \pi_d + d = \gamma_{gj} = \frac{y^g}{y^j}\,.$$

But then $i \ne g$ and $j \ne h$ imply

$$0 \ \ne \ \frac{(y^i - y^g)(y^h - y^j)}{y^j y^h} \ = \ \frac{y^i}{y^j} - \frac{y^i}{y^h} + \frac{y^g}{y^h} - \frac{y^g}{y^j} \ =$$

$$= \pi_a + a - \pi_b - b + \pi_c + c - \pi_d - d \ = \ a - b + c - d\,.$$

(ii) The entries $b = 0$ and $c = 0$ lie in the last column or $c =$ and $d = 0$ lie in the last row of blocks of $C^{(p)}$, i.e. either $h = q - 1$ or $g = q - 1$. If $h = q - 1$, one has

$$\pi_a + a = \gamma_{ij} = \frac{y^i}{y^j}, \quad \pi_b + b = \pi_c + c = 0, \quad \pi_d + d = \gamma_{gj} = \frac{y^g}{y^j}\,,$$

and $i \neq g$ implies

$$0 \ \neq \ \frac{(y^i - y^g)}{y^j} \ = \ \frac{y^i}{y^j} - \frac{y^g}{y^j} \ =$$
$$= \ \pi_a + a - \pi_b - b + \pi_c + c - \pi_d - d \ = \ a - b + c - d\,.$$

An analogous reasoning works for $g = q - 1$. □

Proposition 4.11 *The Desarguesian elliptic semiplane $\mathcal{S}^C_{q^2}$ of type C derived from $PG(2, q)$ is isomorphic to the p-cyclic configuration of type $\big((q^2)_q\big)$ represented by the J_2-free simple $\mathbb{Z}_p$-scheme $C^{(p)}$ of order $p^{2\nu-1}$ and valency q.*

Proof. Again it is sufficient to check that the construction of [1] applies also for the above labelling for the elements of $GF(q)$. □

Remark 4.12 In [1] it has been pointed out that Desarguesian elliptic semiplanes $\big((q^4 - q)_{q^2}\big)$ of type D and Baker's elliptic semiplane (45_7) of type B do admit representations by $\mathbb{Z}_{q^2+q+1}$-schemes and a $\mathbb{Z}_3$-scheme, respectively.

5 Regular Graphs of Girth 5 with Few Vertices

A (k, g)-**cage** is a k-regular graph of *girth* g with a minimum number of vertices. For a survey on the known cages, see e.g. in [32]. For parameters k, g for which the (k, g)-cage problem is unsolved, some interest has been given to constructing k-regular graphs of girth g with as few vertices as possible. For $g = 5$, the results given in [3] have been outdated by a paper of Jørgensen's [20], but the methods based on $\mathbb{Z}_\mu$-schemes presented in [3] succeed in tying up with results of [20]:

Example 5.1 The smallest known 9-regular graph of girth 5 has 96 vertices, see [20], Corollary 9. To construct such a graph, start with the elliptic semiplane $\mathcal{S}^L_{48}$ on 48 points. Represent it by the $\mathbb{Z}_6$-scheme $L^{(6)}$ of order 8 and valency 7 constructed in Example 4.6 and compose the following simple $\mathbb{Z}_6$-scheme of order 16 and valency 9:

$$T^{(6)}_{96} \quad = \quad \left(\begin{array}{cccccccc|cccccccc}
1,5 & & & & & & & & & 3 & 1 & 2 & 5 & 4 & 0 & 0 \\
 & 1,5 & & & & & & & 0 & & 3 & 1 & 2 & 5 & 4 & 0 \\
 & & 1,5 & & & & & & 4 & 0 & & 3 & 1 & 2 & 5 & 0 \\
 & & & 1,5 & & & & & 5 & 4 & 0 & & 3 & 1 & 2 & 0 \\
 & & & & 1,5 & & & & 2 & 5 & 4 & 0 & & 3 & 1 & 0 \\
 & & & & & 1,5 & & & 1 & 2 & 5 & 4 & 0 & & 3 & 0 \\
 & & & & & & 1,5 & & 3 & 1 & 2 & 5 & 4 & 0 & & 0 \\
 & & & & & & & 1,5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\hline
 & 0 & 2 & 1 & 4 & 5 & 3 & 0 & 2,4 & & & & & & & \\
3 & & 0 & 2 & 1 & 4 & 5 & 0 & & 2,4 & & & & & & \\
5 & 3 & & 0 & 2 & 1 & 4 & 0 & & & 2,4 & & & & & \\
4 & 5 & 3 & & 0 & 2 & 1 & 0 & & & & 2,4 & & & & \\
1 & 4 & 5 & 3 & & 0 & 2 & 0 & & & & & 2,4 & & & \\
2 & 1 & 4 & 5 & 3 & & 0 & 0 & & & & & & 2,4 & & \\
0 & 2 & 1 & 4 & 5 & 3 & & 0 & & & & & & & 2,4 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & 2,4
\end{array}\right)^{(6)}$$

The upper right block is a copy of $L^{(6)}$ and the lower left block is obtained by transposing $L^{(6)}$ and substituting each entry with its opposite value in $(\mathbb{Z}_6, +)$. Hence both blocks are J_2-free. To check that the whole scheme $T_{96}^{(6)}$ is J_2-free, it is enough to see that each full 2×2 subscheme not lying completely in one of these two blocks is of type $\left(\begin{smallmatrix} 1,5 & z \\ -z & 2,4 \end{smallmatrix}\right)^{(6)}$ for some $z \in \mathbb{Z}_6$ and

$$a - z + c + z \not\equiv 0 \pmod 6 \quad \text{for all} \quad a \in \{1,5\} \quad \text{and} \quad c \in \{2,4\}\,.$$

Since $T_{96}^{(6)}$ is skew-symmetric and 0 does not turn up as entry on its main diagonal, the blow-up $\overline{T_{96}^{(6)}}$ is the adjacency matrix of a C_4-free 9-regular graph G on 96 vertices. A short argument shows that G is also C_3-free (cf [3], Lemma 2.5). Hence G has girth ≥ 5. Equality holds since a 5-cycle in G is made up by the vertices corresponding to the $1^{st}, 2^{nd}, 3^{rd}, 91^{st}$, and 93^{rd} rows of $\overline{T_{96}^{(6)}}$.

Remark 5.2 The above Example shows that a $\mathbb{Z}_\mu$-scheme not only qualifies when major emphasis is laid on an immediate access to adjacency matrices, but also reveals hidden geometric structures: consider the Levi graph $\Lambda(\mathcal{S}_{48}^L)$, whose adjacency matrix is represented by the $\mathbb{Z}_6$-scheme $T_{96}^{(6)}$ without its diagonal entries; this graph is 7-regular and has girth 6; then G is obtained by suitably gluing in 6-cycles with adjacency matrices represented by $(1,5)^{(6)}$ and $(2,4)^{(6)}$.

Example 5.3 Hoffman-Singleton's celebrated $(7,5)$-cage [19], say G_{HS}, can be obtained in a similar way from $\Lambda(\mathcal{S}_{25}^C)$. In order to construct an adjacency matrix for G_{HS}, we use the representation of G_{HS} due to Robertson [30]: take five copies $P_0, \ldots, P_4$ of the pentagram with vertices $0, \ldots, 4$ and edges $02, 24, 41, 13, 30$, as well as five copies $Q_0, \ldots, Q_4$ of the pentagon with vertices $0, \ldots, 4$ and edges $01, 12, 23, 34, 40$. They make up the 50 vertices and the first 50 edges; add further edges according to the following rule: the vertex i of P_j is joined to the vertex l of Q_k if, and only if,

$$l \equiv i + jk \pmod 5\,.$$

Displaying the copies of the pentagrams and pentagons in the order

$$P_1\,, \ldots, P_4\,, P_0\,, Q_1\,, \ldots, Q_4\,, Q_0$$

such that the vertices within each P_i and Q_j maintain the natural order $0, 1, 2, 3, 4$, the corresponding adjacency matrix turns out to be the blow-up of the following $\mathbb{Z}_5$-scheme:

$$T_{50}^{(5)} \quad = \quad \left(\begin{array}{ccccc|ccccc}
2,3 & & & & & 1 & 2 & 3 & 4 & 0 \\
 & 2,3 & & & & 2 & 4 & 1 & 3 & 0 \\
 & & 2,3 & & & 3 & 1 & 4 & 2 & 0 \\
 & & & 2,3 & & 4 & 3 & 2 & 1 & 0 \\
 & & & & 2,3 & 0 & 0 & 0 & 0 & 0 \\
\hline
4 & 3 & 2 & 1 & 0 & 1,4 & & & & \\
3 & 1 & 4 & 2 & 0 & & 1,4 & & & \\
2 & 4 & 1 & 3 & 0 & & & 1,4 & & \\
1 & 2 & 3 & 4 & 0 & & & & 1,4 & \\
0 & 0 & 0 & 0 & 0 & & & & & 1,4
\end{array}\right)^{(5)}$$

The Levi graph $\Lambda(\mathcal{S}_{25}^C)$ has an adjacency matrix which is represented by the $\mathbb{Z}_5$-scheme $T_{50}^{(5)}$ without its diagonal entries; this graph is 5-regular and has girth 6, and G_{HS} is obtained by suitably gluing in 5-cycles with adjacency matrices represented by $(2,3)^{(5)}$ and $(1,4)^{(5)}$.

6 Mixed Simple $\mathbb{Z}_\mu$-Schemes

A configuration $\mathcal{C}$ represented by a simple J_2-free $\mathbb{Z}_\mu$-scheme $M^{(\mu)}$ can be partitioned into μ-sets of pairwise parallel points and lines, say Π_i and Λ_j. A standard construction in finite geometries applies, namely a kind of projective closure: new lines L_i and new points p_j may be added to $\mathcal{C}$ such that L_i and p_j are incident with each element in Π_i and Λ_j, respectively. Eventually, a new point may also be incident with some new line. The following notion renders this construction compatible with the representation of incidence matrices as blow-ups of $\mathbb{Z}_\mu$-schemes.

Definition 6.1 For $s \geq 1$, we introduce the symbol $\mathbf{r}_i^s$ whose blow-up is understood to be the $(0,1)$-matrix $\overline{\mathbf{r}_i^s}$ of order (s,μ) having entries 1 in its i^{th} row and entries 0 elsewhere. The transpose, denoted by $\overline{\mathbf{c}_i^s} := \left(\overline{\mathbf{r}_i^s}\right)^T$, is interpreted as the blow-up of the symbol $\mathbf{c}_i^s$. Let $M^{(\mu)} = (z_{ij})$ be a simple $\mathbb{Z}_\mu$-scheme of order (m,n) with $z_{ij} \in \mathbb{Z}_\mu \cup \{\emptyset\}$. For permutations $\pi \in S_m$ and $\rho \in S_n$, the scheme

$$M_{mix}^{(\mu)} \quad := \quad \left(\begin{array}{cccc|c} z_{11} & z_{12} & \dots & z_{1n} & \mathbf{c}_{1^\pi}^m \\ z_{21} & z_{22} & \dots & z_{2n} & \mathbf{c}_{2^\pi}^m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{m1} & z_{m2} & \dots & z_{mn} & \mathbf{c}_{m^\pi}^m \\ \hline \mathbf{r}_{1^\rho}^n & \mathbf{r}_{2^\rho}^n & \dots & \mathbf{r}_{n^\rho}^n & \mathbf{e} \end{array}\right)^{(\mu)}$$

is called a **mixed** $\mathbb{Z}_\mu$-scheme, where the blow-up $\overline{\mathbf{e}}$ of the symbol $\mathbf{e}$ is a $(0,1)$-matrix of order (n,m).

Note that the parameter μ is not explicitly mentioned in the symbols $\mathbf{r}_i^s$ and $\mathbf{c}_j^t$ since its value coincides with the parameter μ of the $\mathbb{Z}_\mu$-scheme under consideration. In the cases $s = \mu$ and $t = \mu$, we shortly write $\mathbf{r}_i$ and $\mathbf{c}_j$ instead of $\mathbf{r}_i^\mu$ and $\mathbf{c}_j^\mu$, respectively. Suitable matrices $\overline{\mathbf{e}}$ are characterized in the following

Lemma 6.2 *Let $M^{(\mu)} = (z_{ij})$ be a simple J_2-free $\mathbb{Z}_\mu$-scheme of order (m,n), where $z_{ij} \in \mathbb{Z}_\mu \cup \{\emptyset\}$. Then the following are equivalent*

- *(i) the blow-up of the mixed scheme $M_{mix}^{(\mu)}$ is still J_2-free;*
- *(ii) the blow-up $\overline{\mathbf{e}}$ may have entry 1 in position $(\pi(j),\rho(i))$ only if $z_{ij} = \emptyset$.*

Proof. Let the rows and the columns of the blow-up $\overline{M^{(\mu)}}$ correspond, as usual, to the points and lines of a configuration $\mathcal{C}$. Then the i^{th} column of $M^{(\mu)}$ gives rise to

μ columns in the blow-up $\overline{M^{(\mu)}}$. Since $M^{(\mu)}$ is simple, these columns can be seen as a block matrix made up by just one column of blocks each of which being either a permutation or a zero matrix of order μ. Therefore, at most one entry 1 turns up in each row of these columns, i.e. any two of the corresponding lines do not have any point of $\mathcal{C}$ in common. Hence these lines make up a μ-set Λ_i of pairwise parallel lines in $\mathcal{C}$. An analogous reasoning holds for any μ-set Π_j of points represented by the j^{th} row of $M^{(\mu)}$. Perform the above construction and add a new point p_i and a new line L_j such that p_i and L_j are incident with each element in Λ_i and Π_j, respectively. In terms of incidence matrices, this means, for each set Λ_i and Π_j, to add a new row and column to $\overline{M^{(\mu)}}$ which have entries 1 in precisely those μ positions which correspond to the elements in Λ_i and Π_j, respectively. We distinguish two cases:

First suppose that $\overline{\mathbf{e}}$ is the zero matrix of order (n, m). This means that no new point lies on any new line. Then the resulting incidence table is still J_2-free. Since this construction works independently for each row and column of $M^{(\mu)}$, any permutation $\rho \in S_n$ and $\pi \in S_m$ acting on the indices of the sets Λ_i and Π_j, respectively, will do. Hence the resulting incidence matrix can be represented as the blow-up of $M^{(\mu)}_{mix}$ and the equivalence is clear in this case.

Now suppose that the blow-up $\overline{\mathbf{e}}$ has entry 1 in position (i^ρ, j^π), i.e. the new point p_{i^ρ} is incident with the new line L_{j^π}. Then the blow-up $\overline{M^{(\mu)}_{mix}}$ is J_2-free if and only if no line in Λ_i is incident with any point in Π_j. This, in turn, is equivalent with $z_{ij} = \emptyset$. Clearly, $\overline{\mathbf{e}}$ is not uniquely determined. □

Example 6.3 In Proposition 4.11 it has been shown that the Desarguesian elliptic semiplane $\mathcal{S}^C_{q^2}$ of type C can be seen as a p-cyclic configuration of type $\big((q^2)_q\big)$. The above Lemma provides a second representation for $\mathcal{S}^C_{q^2}$ in terms of a mixed $\mathbb{Z}_{q-1}$-scheme. Let $C^{(q-1)}$ be the $\mathbb{Z}_{q-1}$-scheme obtained by deleting the last row and column in the simple $\mathbb{Z}_{q-1}$-scheme $L^{(q-1)}$ constructed in the proof of Construction 4.3. Since $C^{(q-1)}$ has blank entries in its main diagonal, Lemma 6.2 implies that the mixed $\mathbb{Z}_{q-1}$-scheme $C^{(q-1)}_{mix}$ is J_2-free if the blow-up of $\mathbf{e}$ is chosen to be the unit matrix of order q. The blow-up $\overline{C^{(q-1)}_{mix}}$ has valency q and order $q(q-1)+q = q^2$.

Remark 6.4 The reader will have noticed that the valency of mixed $\mathbb{Z}_\mu$-schemes has not yet been taken into account. Obviously, $M^{(\mu)}_{mix}$ has valencies μ and $\mu+1$ only if $M^{(\mu)}$ had valencies $\mu-1$ and μ, respectively, and $\overline{\mathbf{e}}$ is chosen to be the zero matrix in the former case and a suitable permutation matrix in the latter case. On the other hand, *partially* mixed $\mathbb{Z}_\mu$-schemes (i.e. new points and lines are added only for some μ-sets of pairwise parallel points and lines) can yield $\mathbb{Z}_\mu$-schemes of valency k even if $M^{(\mu)}$ did not have a valency. Instances will be discussed in the following Sections.

7 Regular Graphs of Girth 6 with Few Vertices

All the known $(k, 6)$-cages but one are Levi graphs of finite projective planes of order $k-1$, the exception being the $(7, 6)$-cage (settled by O'Keefe and Wong [27]). This cage revealed itself to be the Levi graph of the elliptic semiplane (45_7) discovered by Baker some years earlier [4]. Again, for values k for which the $(k, 6)$-cage problem is unsolved, some interest has been given to finding k-regular graphs of girth 6 with as few vertices as possible. Levi graphs of Desarguesian elliptic semiplanes reveal themselves to be good candidates: for $k = 11, 13, 16, 19, 23$, and 25, instances of smallest known k-regular graphs of girth 6 are $\Lambda(\mathcal{S}^L_{120})$, $\Lambda(\mathcal{S}^L_{168})$, $\Lambda(\mathcal{S}^D_{252})$, $\Lambda(\mathcal{S}^L_{360})$, $\Lambda(\mathcal{S}^L_{528})$, and $\Lambda(\mathcal{S}^D_{620})$, respectively, see [2] (cf. also [13, 25]). In [2], further instances have been obtained by deleting an equal number of rows and columns in $\mathbb{Z}_\mu$-schemes representing Desarguesian elliptic semiplanes, e.g. a 15-regular graph on 462 vertices. A somewhat more sophisticated and efficient deletion technique in incidence matrices is due to Balbuena [5], giving rise to instances of 21- and 22-regular graphs on 964 and 1008 vertices, respectively. The methods based on $\mathbb{Z}_\mu$-schemes presented in [2] succeed in tying up with results of [5]:

Proposition 7.1 *For each prime power q, there exist J_2-free $(0, 1)$-matrices of valency $q - 1$ and orders $q^2 - q - 1$ and $q^2 - q - 2$.*

Proof. Consider the simple $\mathbb{Z}_{q-1}$-scheme $L^{(q-1)}$ (see Construction 4.3) and delete two rows and two columns. In general, this yields a simple $\mathbb{Z}_{q-1}$-scheme of order $q - 1$, which has $q - i$ blank entries, with $i = 1, 2, 3$. For the last two cases, we can choose the following minors $M^{(q-1)}$ and $N^{(q-1)}$ of $L^{(q-1)}$, which are respectively obtained by deleting the first two rows as well as

$$\begin{aligned} \text{the first and the last columns} \quad &\text{if } \ i = 2\,, \\ \text{the last two columns} \quad &\text{if } \ i = 3\,. \end{aligned}$$

Embed the $\mathbb{Z}_{q-1}$-schemes $M^{(q-1)}$ and $N^{(q-1)}$ into the mixed schemes

$$M^{(q-1)}_{mix} := \left(\begin{array}{c|cccccc|c} z_{11} & \emptyset & z_{13} & z_{14} & \cdots & z_{1,q-1} & \mathbf{c}_1^{q-2} \\ z_{21} & z_{22} & \emptyset & z_{24} & \cdots & z_{2,q-1} & \mathbf{c}_2^{q-2} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ z_{q-3,1} & z_{q-3,2} & z_{q-3,3} & \ddots & \emptyset & z_{q-3,q-1} & \mathbf{c}_{q-3}^{q-2} \\ z_{q-2,1} & z_{q-2,2} & z_{q-2,3} & \cdots & z_{q-2,q-2} & \emptyset & \mathbf{c}_{q-2}^{q-2} \\ \hline z_{q-1,1} & z_{q-1,2} & z_{q-1,3} & z_{q-1,4} & \cdots & z_{q-1,q-1} & \emptyset \\ \hline \emptyset & \mathbf{r}_1^{q-2} & \mathbf{r}_2^{q-2} & \mathbf{r}_3^{q-2} & \cdots & \mathbf{r}_{q-2}^{q-2} & \emptyset \end{array}\right)^{(q-1)}$$

and $N^{(q-1)}_{mix} :=$

$$= \left(\begin{array}{cc|cccccc|c} z_{11} & z_{12} & \emptyset & z_{14} & z_{15} & \cdots & z_{1,q-1} & \mathbf{c}_1^{q-3} \\ z_{21} & z_{22} & z_{23} & \emptyset & z_{25} & \cdots & z_{2,q-1} & \mathbf{c}_2^{q-3} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ z_{q-4,1} & z_{q-4,2} & z_{q-4,3} & z_{q-4,4} & \ddots & \emptyset & z_{q-4,q-1} & \mathbf{c}_{q-4}^{q-3} \\ z_{q-3,1} & z_{q-3,2} & z_{q-3,3} & z_{q-3,4} & \cdots & z_{q-3,q-2} & \emptyset & \mathbf{c}_{q-3}^{q-3} \\ \hline z_{q-2,1} & z_{q-2,2} & z_{q-2,3} & z_{q-2,4} & z_{q-2,5} & \cdots & z_{q-2,q-1} & \emptyset \\ z_{q-1,1} & z_{q-1,2} & z_{q-1,3} & z_{q-1,4} & z_{q-1,5} & \cdots & z_{q-1,q-1} & \emptyset \\ \hline \emptyset & \emptyset & \mathbf{r}_1^{q-3} & \mathbf{r}_2^{q-3} & \mathbf{r}_3^{q-3} & \cdots & \mathbf{r}_{q-3}^{q-3} & \emptyset \end{array}\right)^{(q-1)}$$

The valency of both $M_{mix}^{(q-1)}$ and $N_{mix}^{(q-1)}$ is $q-1$ and their orders are

$$(q-1)(q-1)+q-i = q^2-q-i+1$$

for $i=2$ and $i=3$, respectively. Then their blow-ups will do. □

8 Krčadinac's Configuration of Type (34_6)

In this Section we present a construction yielding four configurations of type (30_5), which will be used to obtain Krčadinac's configuration of type (34_6) (cf. [22]) and four new configurations of type (35_6). The computer results have been obtained by using the software [21].

Construction 8.1 Start with the elliptic semiplane $\mathcal{S}_{15}^{L}$ and represent it by the $\mathbb{Z}_3$-scheme $L^{(3)}$ of order 5 and valency 4, see Construction 4.3. Compose the following simple $\mathbb{Z}_3$-scheme of order 10 and valency 5

$$T = \left(\begin{array}{ccccc|ccccc} \alpha_1 & & & & & & 0 & 1 & 2 & 0 \\ & \alpha_2 & & & & 0 & & 2 & 1 & 0 \\ & & \alpha_3 & & & 1 & 2 & & 0 & 0 \\ & & & \alpha_4 & & 2 & 1 & 0 & & 0 \\ & & & & \alpha_5 & 0 & 0 & 0 & 0 & \\ \hline & 0 & 2 & 1 & 0 & \beta_1 & & & & \\ 0 & & 1 & 2 & 0 & & \beta_2 & & & \\ 2 & 1 & & 0 & 0 & & & \beta_3 & & \\ 1 & 2 & 0 & & 0 & & & & \beta_4 & \\ 0 & 0 & 0 & 0 & & & & & & \beta_5 \end{array}\right)^{(3)}$$

for suitable $\alpha_i, \beta_i \in \mathbb{Z}_3$. The upper right block is a copy of $L^{(3)}$ and the lower left block is obtained by transposing $L^{(3)}$ and substituting each entry by its opposite element in $(\mathbb{Z}_3, +)$.

Lemma 8.2 *The $\mathbb{Z}_3$-schemes obtained for*

$$\begin{array}{lcl} T_{360} & : & (\alpha_1,\ldots,\alpha_5)=(1,1,1,1,1),\ (\beta_1,\ldots,\beta_5)=(1,1,1,1,1), \\ T_{72} & : & (\alpha_1,\ldots,\alpha_5)=(1,1,1,1,0),\ (\beta_1,\ldots,\beta_5)=(1,1,1,1,0), \\ T_{36} & : & (\alpha_1,\ldots,\alpha_5)=(1,1,1,1,1),\ (\beta_1,\ldots,\beta_5)=(1,1,1,1,0), \\ T_{18} & : & (\alpha_1,\ldots,\alpha_5)=(1,1,1,1,1),\ (\beta_1,\ldots,\beta_5)=(1,1,1,0,0) \end{array}$$

represent four pairwise non-isomorphic configurations $\mathcal{T}_{360}$, $\mathcal{T}_{72}$,$\mathcal{T}_{36}$, and $\mathcal{T}_{18}$ of type (30_5), whose automorphism groups have orders 360, 72, 36, *and* 18, *respectively.* □

Proof. Apply Criterion 2.2 to the $\mathbb{Z}_3$-scheme T: all full 2×2 sub-schemes are of type $\begin{pmatrix} \alpha_i & \lambda_{ij} \\ -\lambda_{ij} & \beta_j \end{pmatrix}^{(3)}$, for $i,j \in \{1,\ldots,5\}$ with $i \neq j$. Thus T meets the condition of the criterion if and only if

$$(*) \quad \alpha_i + \beta_j \not\equiv 0 \pmod 3 \quad \text{for all } i,j=1,\ldots,5 \text{ with } i \neq j\,.$$

There are a lot of solutions for $(*)$. A computer search, however, reveals that they lead to only four pairwise non-isomorphic configurations. We can choose the solutions indicated above. □

Construction 8.3 Let $\mathcal{T}$ stand for one of the four configurations $\mathcal{T}_{360}$, $\mathcal{T}_{72}$, $\mathcal{T}_{36}$, or $\mathcal{T}_{18}$ of type (30_5). Rearrange both the rows and columns of the $\mathbb{Z}_3$-scheme T following the order $1, 6, 2, 7, 3, 8, 4, 9, 5, 10$, to obtain an equivalent variant, namely

$$V(T) = \left(\begin{array}{cc|cc|cc|cc|cc}
1 & & & 0 & & 1 & & 2 & & 0 \\
 & 1 & 0 & & 2 & & 1 & & 0 & \\
\hline
 & 0 & 1 & & & 2 & & 1 & & 0 \\
0 & & & 1 & 1 & & 2 & & 0 & \\
\hline
 & 1 & & 2 & 1 & & & 0 & & 0 \\
2 & & 1 & & & 1 & 0 & & 0 & \\
\hline
 & 2 & & 1 & & 0 & 1 & & & 0 \\
1 & & 2 & & 0 & & & \beta_4 & 0 & \\
\hline
 & 0 & & 0 & & 0 & & 0 & \alpha_5 & \\
0 & & 0 & & 0 & & 0 & & & \beta_5
\end{array}\right)^{(3)} .$$

Note that, for $j = 1, 3, 5, 7, 9$, the j^{th} and $j+1^{st}$ rows (and columns) of the scheme $V(T)$ are **non-overlapping**, i.e. the entries in one and the same position of the j^{th} and $j+1^{st}$ rows (and columns) are always one element of $\mathbb{Z}_3$ and one blank entry. Hence, in the blow-up $\overline{V(T)}$ of $V(T)$, the rows (columns) labelled by

$$(\S) \qquad 3(j-1)+1,\ \ 3(j-1)+2,\ \ 3j,\ \ 3j+1,\ \ 3j+2,\ \ 3(j+1)$$

correspond to 6 pairwise parallel points (lines) of $\mathcal{T}$. Denote the sets of these six points and lines by Π_l and Λ_l, respectively, where $l := \frac{1}{2}(j+1)$. The families $\{\Pi_l\}_{l=1,\ldots,5}$ and $\{\Lambda_l\}_{l=1,\ldots,5}$ partition the sets of all points and lines in $\mathcal{T}$. A computer evaluation reveals the following

Lemma 8.4 *The families $\{\Pi_l\}_{l=1,\ldots,5}$ and $\{\Lambda_l\}_{l=1,\ldots,5}$ are invariant under all automorphisms of $\mathcal{T}$.* □

Construction 8.5 Now let $\mathcal{T}$ stand for one of the three configurations $\mathcal{T}_{360}$, $\mathcal{T}_{72}$, and $\mathcal{T}_{36}$ of type (30_5), represented by the schemes $V(T)$ obtained by Construction 2. For $l = 1, \ldots, 4$, add a new "improper" line and point for each set Π_l and Λ_l. Equivalently, add four new rows and columns to the blow-up $\overline{V(T)}$ which, for $j = 1, 3, 5, 7$, have entries 1 in positions (§) and entries 0 else. Simultaneously, substitute the 2×2 sub-scheme $\left(\begin{smallmatrix} \alpha_5 & \\ & \beta_5 \end{smallmatrix}\right)^{(3)}$ of $V(T)$ by $\left(\begin{smallmatrix} \alpha_{5,\eta} & \\ & \beta_{5,\zeta} \end{smallmatrix}\right)^{(3)}$ for some $\eta, \zeta \in \mathbb{Z}_3$. The mixed scheme

$$V(T)' = \left(\begin{array}{cc|cc|cc|cc|cc|c}
1 & & & 0 & & 1 & & 2 & & 0 & \mathbf{c}_1^4 \\
 & 1 & 0 & & 2 & & 1 & & 0 & & \mathbf{c}_1^4 \\
\hline
 & 0 & 1 & & & 2 & & 1 & & 0 & \mathbf{c}_2^4 \\
0 & & & 1 & 1 & & 2 & & 0 & & \mathbf{c}_2^4 \\
\hline
 & 1 & & 2 & 1 & & & 0 & & 0 & \mathbf{c}_3^4 \\
2 & & 1 & & & 1 & 0 & & 0 & & \mathbf{c}_3^4 \\
\hline
 & 2 & & 1 & & 0 & 1 & & & 0 & \mathbf{c}_4^4 \\
1 & & 2 & & 0 & & & \beta_4 & 0 & & \mathbf{c}_4^4 \\
\hline
 & 0 & & 0 & & 0 & & 0 & \alpha_{5,\eta} & & \\
0 & & 0 & & 0 & & 0 & & & \beta_{5,\zeta} & \\
\hline
\mathbf{r}_1^4 & \mathbf{r}_1^4 & \mathbf{r}_2^4 & \mathbf{r}_2^4 & \mathbf{r}_3^4 & \mathbf{r}_3^4 & \mathbf{r}_4^4 & \mathbf{r}_4^4 & & &
\end{array}\right)^{(3)}$$

suitably represents the result of these modifications. Note that $V(T)'$ has valency 6.

Lemma 8.6 *The $\mathbb{Z}_3$-schemes $V(T_{360})'$, $V(T_{72})'$, and $V(T_{36})'$ turn out to be J_2-free for just one pair (η, ζ) each, namely $(0,0)$, $(1,1)$, and $(0,1)$, respectively. All three solutions lead to one and the same mixed scheme with $\{\alpha_5, \eta\} = \{\beta_5, \zeta\} = \{0,1\}$, whose blow-up represents Krčadinac's configuration of type (34_6) [22]. Its automorphism group has order 72.*

Proof. A straightforward verification shows that the $\mathbb{Z}_3$-scheme $V(T)'$ is J_2-free. The isomorphism with Krčadinac's configuration and the order of its automorphism group have been obtained by computer. □

Remark 8.7 Let $\mathcal{T}$ stand for one of the four configurations $\mathcal{T}_{360}$, $\mathcal{T}_{72}$, $\mathcal{T}_{36}$, or $\mathcal{T}_{18}$. Alternatively, we can also add five new "improper" lines and points for the families $\{\Pi_l\}_{l=1,\ldots,5}$ and $\{\Lambda_l\}_{l=1,\ldots,5}$ in $\mathcal{T}$, respectively. This leads to four configurations of type (35_6), represented by the mixed $\mathbb{Z}_3$-schemes

$$\left(\begin{array}{cc|cc|cc|cc|cc|c}
1 & & & 0 & & 1 & & 2 & & 0 & \mathbf{c}_1^5 \\
& 1 & 0 & & 2 & & 1 & & 0 & & \mathbf{c}_1^5 \\
\hline
& 0 & 1 & & & 2 & & 1 & & 0 & \mathbf{c}_2^5 \\
0 & & & 1 & 1 & & 2 & & 0 & & \mathbf{c}_2^5 \\
\hline
& 1 & & 2 & 1 & & & 0 & & 0 & \mathbf{c}_3^5 \\
2 & & 1 & & & 1 & 0 & & 0 & & \mathbf{c}_3^5 \\
\hline
& 2 & & 1 & & 0 & 1 & & & 0 & \mathbf{c}_4^5 \\
1 & & 2 & & 0 & & & \beta_4 & 0 & & \mathbf{c}_4^5 \\
\hline
& 0 & & 0 & & 0 & & 0 & \alpha_5 & & \mathbf{c}_5^5 \\
0 & & 0 & & 0 & & 0 & & & \beta_5 & \mathbf{c}_5^5 \\
\hline
\mathbf{r}_1^5 & \mathbf{r}_1^5 & \mathbf{r}_2^5 & \mathbf{r}_2^5 & \mathbf{r}_3^5 & \mathbf{r}_3^5 & \mathbf{r}_4^5 & \mathbf{r}_4^5 & \mathbf{r}_5^5 & \mathbf{r}_5^5 &
\end{array}\right)^{(3)}$$

whose automorphism groups have still orders 360, 72, 36, and 18 (cf. Lemma 8.4). These configurations are new. Thus far, three configurations of type (35_6) have been exhibited in the literature: In [14] and [26], cyclic configurations are presented in terms of deficient cyclic difference sets, namely

$$\mathcal{C}_G \ : \ \{0,1,8,11,13,17\}^{(35)} \quad \text{and} \quad \mathcal{C}_{MPW} \ : \ \{0,1,3,7,12,20\}^{(35)},$$

respectively, whereas in [12] there is mentioned a configuration $\mathcal{C}_{FLN}$ represented by the following $\mathbb{Z}_7$-scheme:

$$\begin{pmatrix}
0{,}1 & 6 & 2 & 2 & 6 \\
6 & 0{,}1 & 6 & 2 & 2 \\
2 & 6 & 0{,}1 & 6 & 2 \\
2 & 2 & 6 & 0{,}1 & 6 \\
6 & 2 & 2 & 6 & 0{,}1
\end{pmatrix}^{(7)}$$

A computer check reveals that $\mathcal{C}_G$ is isomorphic to $\mathcal{C}_{MPW}$; its automorphism group has order 35, whereas $\mathcal{C}_{FLN}$ has an automorphism group of order 140. It is cyclic as well and isomorphic to the configuration given by the deficient difference set $\{0,1,8,12,14,17\}^{(35)}$. A computer search confirms that there are no further cyclic configurations of type 35_6.

9 Appendix: $(0,1)$-Matrices, Graphs, and Configurations

A *circulant* matrix is a square matrix where each row vector is shifted one element to the right relative to the preceding row vector. Hence a circulant $(0,1)$-matrix is uniquely determined by the positions of the entries 1 in its first row. The transpose of a matrix A is denoted by A^T.

Graph theoretic notations come from [6]. We distinguish *graphs* from *general graphs*, the former having neither loops nor multiple edges. All (general) graphs are supposed to be finite and connected (if not otherwise stated).

Let K be a general graph all of whose edges have been given plus and minus directions. A *cyclic voltage graph* is the pair (K, α) where α is a function from the $+$ directed edges of K into the cyclic group $\mathbb{Z}_\mu$, called a *cyclic voltage assignment.* For slightly different and more general definitions, cf. e.g. [15, 16, 17, 28, 29]. The *derived graph* K^α, also referred to as the *lift* of K in $\mathbb{Z}_\mu$ via α (cf. e.g. [10]) or the (*regular*) *covering graph* (cf. e.g. [28, 31]), is the (not necessarily connected) general graph whose vertex and edges sets are $VK \times \mathbb{Z}_\mu$ and $EK \times \mathbb{Z}_\mu$ and in which (v, a) and (w, b) are incident with (e, a) if EK contains an edge e whose $+$ direction runs from v to w and $a + \alpha(e) = b$. Note that "regular" has a topological meaning (cf. e.g. [16, 17]). The *natural projection* $\pi : K^\alpha \longrightarrow K$ is defined by the rules $(v, a)^\pi = v$ and $(e, a)^\pi = e$.

Lemma 9.1 *The lift of K in $\mathbb{Z}_\mu$ via α is a graph if loops of K don't have image $0 \in \mathbb{Z}_\mu$ and multiple edges do have distinct images under the cyclic voltage assignment.*

Proof. Let e be a v-based loop in K with voltage $a \in \mathbb{Z}_\mu \backslash \{0\}$. If a has order ν, then the loop gives rise to $\frac{\mu}{\nu}$ cycles of length ν in K^α, namely

$$(v, c), (v, c + a), (v, c + 2a), \ldots, (v, c + (\nu - 1)a)$$

for $c = 0, \ldots, \frac{\mu}{\nu} - 1$. Let $e, f \in EK$ be a double edge in K, both running from v to w, with voltages a, b, respectively. This leads to 2μ distinct edges in K^α, no two of which incident with the same pair of vertices, namely

$$(v, c)|(e, a)|(w, c + a) \quad \text{and} \quad (v, c)|(f, b)|(w, c + b)$$

for $c \in \mathbb{Z}_\mu$. □

In the light of this Lemma, we may call a cyclic voltage assignment $\alpha : K \longrightarrow \mathbb{Z}_\mu$ *admissible* if loops of K don't have image $0 \in \mathbb{Z}_\mu$ and multiple edges do have distinct images.

Suppose that Γ is a graph whose vertex set $V\Gamma$ is the set $\{v_1, \ldots, v_n\}$, and consider the edge set $E\Gamma$ as a set of unordered pairs of elements in $V\Gamma$: then the *adjacency matrix* of Γ is the $n \times n$ matrix $A = A(\Gamma)$ whose entries a_{ij} are given by $a_{ij} := 1$ if $\{v_i, v_j\} \in E\Gamma$, and $a_{ij} := 0$ otherwise. A is a symmetric matrix with entries 0 on the main diagonal. The rows and columns of A correspond to an arbitrary labelling of the vertices of Γ. A permutation π of $V\Gamma$ can be represented by a *permutation matrix* $P_\pi = (p_{ij})$, where $p_{ij} = 1$ if $v_i = v_j^\pi$, and $p_{ij} = 0$ otherwise. Then $P_\pi^{-1} A P_\pi$ becomes the adjacency matrix of Γ with respect to this re-labelling. Thus we focus primarily on the equivalence class $\mathcal{A}$ of $(0, 1)$-matrices represented by A under the equivalence relation

$$A_1 \cong A_2 \quad \text{if} \quad A_2 = P_\pi^{-1} A_1 P_\pi \ \text{ for some permutation matrix } P_\pi \text{ with } \pi \in S_n$$

on the set of symmetric $(0,1)$-matrices with zero diagonal.

A graph is called *k-regular* if every vertex is adjacent to k distinct vertices. A graph is *bipartite* if its vertex set can be partitioned into two parts V_1 and V_2 such that each edge has one vertex in V_1 and one vertex in V_2. If we label the vertices in such a way that those in V_1 come first, then the adjacency matrix of a bipartite graph takes the form

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} .$$

A bipartite graph is *(k,l)-semiregular* if the vertex in V_1 and V_2 are adjacent to k and l vertices, respectively.

An adjacency matrix for the *cycle (graph)* $\mathcal{C}_n$ is the circulant matrix with first row $[0,1,0,\ldots,0,1]$. The *girth* of a graph Γ is the length g of a shortest cycle $\mathcal{C}_g$ which can be embedded into Γ. The following result is generally known.

Lemma 9.2 *Let Γ be a bipartite graph. Then the following are equivalent:*

(i) *Γ has girth ≥ 6;*
(ii) *Γ is $\mathcal{C}_4$-free;*
(iii) *the adjacency matrix $A(\Gamma)$ is J_2-free.*

□

Out of the many ways to introduce configurations, we prefer Levi's definition [23], which best suits our approach to Graph Theory via $(0,1)$-matrices. An *incidence table* or *incidence matrix* C is a J_2-free $(0,1)$-matrix; usually some *regularity* is requested: C is of *type* (m_k, n_l) if C has order (m,n) and if the sums of all entries in the rows and columns have constant values k and l, respectively. The meaning of *points, lines, incidences* etc. are based on the usual interpretation of an incidence table. A *schematic configuration* (m_k, n_l) is an equivalence class $\mathcal{C}$ of incidence tables of type (m_k, n_l) under the equivalence relation

$$C_1 \cong C_2 \quad \text{if} \quad C_2 = PC_1Q \ \text{ for permutation matrices } P \text{ and } Q\,.$$

Other names are *combinatorial configuration* or simply *configuration*, not to be confused with a *geometric configuration* made up by points and lines of the Euclidean plane. If $m = n$ (and hence $k = l$), the symbol (n_k, n_k) will be shortened to (n_k). In the literature, such configurations are called *symmetric*. We avoid this term, since "symmetric" configurations need not admit symmetric incidence tables.

With each (m_k, n_l) configuration $\mathcal{C}$ one associates its *Levi graph* $\Lambda(\mathcal{C})$, see [8]: it is the bipartite graph whose vertices are the points and lines of $\mathcal{C}$; two vertices of $\Lambda(\mathcal{C})$ are adjacent if and only if they make up an incident point-line pair in $\mathcal{C}$. If $\mathcal{C}$ is represented by the incidence table C, then $A := \left(\begin{smallmatrix} 0 & C \\ C^t & 0 \end{smallmatrix}\right)$ is an adjacency matrix for $\Lambda(\mathcal{C})$. Lemma 9.2 implies that Levi graphs of configurations have girth ≥ 6.

For each $n \in \mathbb{N}$ and $1 \leq k \leq \frac{1}{2} + \sqrt{n - \frac{3}{4}}$, a subset $D = \{s_0, \ldots, s_{k-1}\} \subseteq \mathbb{Z}_n$ is called a *deficient cyclic difference set*, denoted by $\{s_0, \ldots, s_{k-1}\}^{(n)}$, if the $k^2 - k$ differences $s_i - s_j \pmod n$ are distinct in pairs for $i, j = 0, \ldots, k-1$ with $i \neq j$, see e.g. [11, 26]. The *deficiency* $d := n - k^2 + k - 1$ counts how many elements in $\mathbb{Z}_n^*$ are not covered by any such difference.

A configuration (n_k) is called *cyclic* if its points can be labelled by the elements of $\mathbb{Z}_n$ such that its lines are given by a *base-line*, i.e. a set $\{z_0, \ldots, z_{k-1}\}$ of k distinct points, and all its *shifts* $\{z_0 + c, \ldots, z_{k-1} + c\}$, numbers taken modulo n, for $c = 1, \ldots, n - 1$.

Lemma 9.3 [14, 24] *A subset $D \subseteq \mathbb{Z}_n$ of cardinality k is the base line of some cyclic configuration (n_k) if and only if D is a deficient cyclic difference set.* □

A finite *elliptic semiplane of order $k - 1$* is an (n_k) configuration satisfying the following axiom of parallels: given a non-incident point line pair (p_1, L_1), there exists at most one line L_2 incident with p_1 and *parallel* to L_1 (i.e. there is no point incident with both L_1 and L_2) and at most one point p_2 incident with L_1 and *parallel* to p_1 (i.e. there is no line incident with both p_1 and p_2), for details, see e.g. [9].

For a survey on the known examples the following notion is useful: a *Baer subset* of a finite projective plane $\mathcal{P}$ is either a Baer subplane $\mathcal{B}$ or, for a distinguished point-line pair (p_0, L_0), the union $\mathcal{B}(p_0, L_0)$ of all lines and points incident with p_0 and L_0, respectively. Trivial examples of elliptic semiplanes are finite projective planes of order n, which are $\big((n^2+n+1)_{n+1}\big)$ configurations. Instances (of *type L*) are obtained from finite projective planes of order n by deleting a Baer subset $\mathcal{B}(p_0, L_0)$ where (p_0, L_0) is a distinguished non-incident point-line pair. The resulting structures are $\big((n^2 - 1)_n\big)$ configurations. Similarly, instances (of *type C*) are obtained from finite projective planes of order n by deleting a Baer subset $\mathcal{B}(p_1, L_1)$ with (p_1, L_1) incident, yielding $\big((n^2)_n\big)$ configurations. Complements $\mathcal{P}\backslash\mathcal{B}$ of Baer subplanes $\mathcal{B}$ make up a third series of instances (of *type D*), furnishing $\big((n^4-n)_{n^2}\big)$ configurations. A sporadic example is the elliptic semiplane (45_7) found by Baker [4]. Elliptic semiplanes of types C, D, and L are said to be *Desarguesian* and denoted by $\mathcal{S}^C$, $\mathcal{S}^D$, and $\mathcal{S}^L$, respectively, if they are derived from $PG(2, q)$.

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