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Hurwitz-Radon inspired maximal three-dimensional real orthogonal designs

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Abstract

Higher-dimensional real orthogonal designs were originally introduced thirty years ago by Hammer and Seberry for potential applications in coding and in the construction of higher-dimensional orthogonal functions. However, little work has been done on optimizing the parameters of such designs. This paper focuses on $n \times n \times t$ three-dimensional real orthogonal designs (3D-RODs) whose subplanes in all axis-normal directions are themselves real orthogonal designs (RODs) and furthermore whose $n \times n$ subplanes are RODs of maximum rate. It is well-known that the maximum rate of an $n \times n$ square ROD is $\rho(n)/n$, where $\rho(n)$ is the number theoretic Radon number. Thus, a fundamental question is to determine for each n the maximum t such that there exists an $n \times n \times t$ 3D-ROD on $\rho(n)$ variables. This paper provides a lower bound on the maximum value of t for such designs, showing that for $n = 2^a b$ with b odd, $t \geq b\rho(n)$ through a simple construction technique. Furthermore, this paper shows that this lower bound is also the tight upper bound

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when the maximum rate $\rho(n)/n$ of the $n \times n$ subplanes is strictly greater than 1/2, thereby providing a unified alternative proof of the existence and construction of such matrices for $n = 1$, 2, 4, and 8 and proving the existence and optimal parameters of the previously open case of $n = 16$. The proofs and constructions in this paper depend on Hurwitz-Radon theory from the turn of the century.

1 Three-Dimensional Real Orthogonal Designs

Real orthogonal designs have been defined in a variety of ways since they were originally introduced by Geramita and Seberry [3]. In this paper, we will use the following definition that has proven useful in communications applications [16]: A *real orthogonal design (ROD)* is a $p \times n$ matrix with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$, where the x_l are real variables, such that $R^T R = \sum_{l=1}^{u} x_l^2 I_n$, where T represents the matrix transpose and I_n represents the $n \times n$ identity matrix. When the two-dimensional nature of such an ROD should be emphasized, we may write 2D-ROD. When R is $n \times n$, it is also true that $RR^T = \sum_{l=1}^{u} x_l^2 I_n$. We will not consider generalizations wherein entries can be linear combinations of the variables; this technically precludes generalizations wherein variables appear more than once in a column. To prepare for the generalization of RODs to three-dimensions, we can rewrite the orthogonality constraints for an $n \times n$ ROD $R = [r_{ij}]$ as $\sum_{i=1}^{n} r_{ia} r_{ib} = \sum_{j=1}^{n} r_{aj} r_{bj} = \sum_{l=1}^{u} x_l^2 \delta_{ab}$, where δ is the Kronecker delta function and a and b are fixed integers less than or equal to n.

For $n \times n$ square RODs, Geramita and Seberry [3] showed that the number of variables u satisfies $u \leq \rho(n)$, where $\rho(n)$ is the *Radon number* [18] defined by $\rho(n) = 8c + 2^d$, when $n = 2^a b$, where b is odd and $a = 4c + d$ with $0 \leq d < 4$ and $0 \leq c$. An $n \times n$ ROD on $\rho(n)$ variables is said to be of *maximum rate*, as it obtains the largest possible ratio of variables to number of rows.

Approximately thirty years ago, Hammer and Seberry generalized Shlichta's three-dimensional Hadamard matrices [20] by defining $n \times n \times \cdots \times n$ higher-dimensional real orthogonal designs as higher-dimensional matrices over real variables whose columns and subplanes satisfy various orthogonality constraints [4, 5, 6, 19]. Since this time, most research questions related to these higher-dimensional Hadamard matrices and higher-dimensional real orthogonal designs have remained open. Although some work has been done on higher-dimensional Hadamard matrices (*e.g.*, [8, 12, 15, 17]), including a recently updated tutorial on their applications in telecommunications and information security [21] and a recent generalization to higher-dimensional multilevel Hadamard matrices [1], Horadam notes in her 2010 survey that this area is still "seriously under-developed" [7]. With respect to higher-dimensional real orthogonal designs, Hammer and Seberry have discussed a variety of similar potential applications, including in codes and in constructing higher-dimensional Walsh and Haar functions [5], while several constructions for higher-dimensional real orthogonal designs have been proposed, concentrating on cubic designs of size $n \times n \times \cdots \times n$ (see for example [2, 4, 5, 6, 11, 13, 14, 19]).

In this paper, we will consider the three-dimensional definition with the most stringent orthogonality requirement, though we allow for more general sizes than previously considered. We define a *three-dimensional real orthogonal design (3D-ROD)* as a $p \times n \times t$ ($p \geq n \geq t$) matrix $G = [g_{ijk}]$ with entries from the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$, where the x_l are real variables, such that all subplanes in any axis-normal direction are themselves RODs on the same set of variables. In other words, when holding k constant, the resulting t subplanes of size $p \times n$ are RODs; when holding j constant, the resulting n subplanes of size $p \times t$ are RODs; and when holding i constant, the resulting p subplanes of size $n \times t$ are RODs. Such a 3D-ROD satisfies the following equations: $\sum_{i=1}^{p} g_{iar} g_{ibr} = \sum_{l=1}^{u} x_l^2 \delta_{ab}$; $\sum_{i=1}^{p} g_{iqa} g_{iqb} = \sum_{l=1}^{u} x_l^2 \delta_{ab}$; and $\sum_{k=1}^{n} g_{sak} g_{sbk} = \sum_{l=1}^{u} x_l^2 \delta_{ab}$, where r is any given value of k, q is any given value of j, and s is any given value of i. This stringent definition parallels what have been referred to in the literature as *proper* higher-dimensional Hadamard matrices and real orthogonal designs. Although it is more difficult to satisfy than other definitions with weaker orthogonality requirements, it is reasonable to expect that it will be more useful for potential applications and more interesting from a combinatorial perspective.

Our motivation to consider non-cubic 3D-RODs is as follows. The original definition of cubic $n \times n \times n$ 3D-RODs did not permit zero entries, so well-known existence results for 2D-RODs implied that such 3D-RODs can exist only for $n = 1$, 2, 4, and 8, and such 3D-RODs have been constructed for these sizes (see [19, 4, 5, 13], respectively). When we permit zero entries, the goal is still to maximize the number of variables, so we should require that the $n \times n$ subplanes are 2D-RODs on the maximum number of variables, namely $\rho(n)$. However, when constraining the size to $n \times n \times n$, our results in this paper imply that under these conditions, it is not even possible for such a 3D-ROD to exist for $n = 16$. Thus, in order to broaden the possible scope of work and to find more examples of 3D-RODs whose subplanes are on the maximum number of variables, we must allow for more general sizes of 3D-RODs, as in our proposed definition.

Thus, we are motivated in particular to study $n \times n \times t$ 3D-RODs whose subplanes in all axis-normal orientations are RODs and furthermore whose $n \times n$ subplanes are RODs on the maximum number $\rho(n)$ of variables. We focus our attention on the fundamental question of maximizing the parameter t in such designs. This is equivalent to determining the maximum number of square maximum rate RODs such that the subplanes in the other two axis-normal directions are also RODs. Our results rely on the Hurwitz-Radon theory (reviewed in Section 2), which is over a century old and has been used in the context of two-dimensional real and complex orthogonal designs [16, 10]; we are not aware of its use in the context of higher-dimensional matrices.

2 Constructing 3D-RODs using Hurwitz-Radon Matrices

Hurwitz-Radon families of matrices (defined below) have been applied in many areas of mathematics, including being used to encode the interactions between variables in two-dimensional real and complex orthogonal designs [10, 16]. Jarfarkhani [9] provides a comprehensive treatment of this application of the Hurwitz-Radon theory, including proofs of many of the results reviewed in this section. Below, we utilize Hurwitz-Radon families of matrices to build 3D-RODs and to establish a lower bound on the maximum t such that an $n \times n \times t$ 3D-ROD on $\rho(n)$ variables exists. We begin by recalling the following formal definition:

Definition 1. *A set of $n \times n$ real matrices $\{B_1, B_2, \ldots, B_r\}$ is called a size r* Hurwitz-Radon family of matrices *if for $1 \leq i \leq r$, we have $B_i^T B_i = I_n$ and $B_i^T = -B_i$, and for $1 \leq i < j \leq r$ we have $B_i B_j = -B_j B_i$.*

We now recall the following results of Radon [18] which were more recently summarized by Tarokh, Jafarkhani, and Calderback [10] in their work on orthogonal designs:

Theorem 1 ([18])**.** *Let $n = 2^a b$, where b is odd and $a = 4c + d$ with $0 \leq d < 4$ and $0 \leq c$. Any Hurwitz-Radon family of $n \times n$ matrices contains strictly less than $\rho(n) = 8c + 2^d$ matrices. Furthermore $\rho(n) \leq n$. A Hurwitz-Radon family containing $n - 1$ matrices exists if and only if $n = 2, 4,$ or 8.*

We next recall the result of Geramita and Seberry [3] connecting Hurwitz-Radon families of matrices and integer matrices, which are matrices whose entries are restricted to the set {1,0,-1}.

Theorem 2 ([3])**.** *For any n, there exists a Hurwitz-Radon family of matrices of size $\rho(n) - 1$ whose members are $n \times n$ integer matrices.*

From Theorem 2, we may consider the following size $\rho(n) - 1$ Hurwitz-Radon family of $n \times n$ integer matrices $\{B_1, B_2, \ldots, B_{\rho(n)-1}\}$. Define $A_1 = I_n$ and $A_i = B_{i-1}$ for $i = 2, 3, \ldots, \rho(n)$. Previous work by Geramita and Seberry [3] shows that the Hurtwitz-Radon condition implies

$$A_i^T A_i = A_i A_i^T = I_n \tag{1}$$

for $i = 1, 2, \ldots, \rho(n)$ and

$$A_i A_j^T = -A_j A_i^T, \tag{2}$$

so that also

$$A_i^T A_j = -A_j^T A_i \tag{3}$$

for $1 \leq i < j \leq \rho(n)$.

Below, we will use this family of matrices A_i, $i = 1, \ldots, \rho(n)$ to define $n \times n$ RODs on $\rho(n)$ variables in Theorem 3, to define $n \times n \times \rho(n)$ 3D-RODs on $\rho(n)$ variables in Theorem 4, and when $n = 2^a b$ for b odd to define $n \times n \times b\rho(n)$ 3D-RODs on $\rho(n)$ variables in Theorem 5.

Theorem 3. *Let* $R = \sum_{i=1}^{\rho(n)} x_i A_i$*, where the* A_i*,* $i = 1, 2, \ldots, \rho(n)$*, are the matrices derived from the Hurwitz-Radon family described above. Then* R *is an* $n \times n$ *ROD with* $\rho(n)$ *variables* $x_1, x_2, \ldots, x_{\rho(n)}$*.*

We omit the proof of Theorem 3, as it is a straight-forward application of the well-known properties of Hurwitz-Radon families of matrices. This theorem is important, however, because it gives a simple way to produce $n \times n$ RODs with the maximum possible number of variables, $\rho(n)$, for all n. Other constructions of maximum rate square RODs are reviewed by Liang [16].

We next introduce some notation. Let c_j denote the jth column of the matrix R defined in Theorem 3, and let a_{ki} denote the ith row of the matrix A_k utilized in the same theorem. Note that since R and A_k are $n \times n$ matrices, c_j is a $n \times 1$ vector and a_{ki} is a $1 \times n$ vector. It will be useful to visualize this notation formally as follows:

$$R = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}$$
$$A_k = \begin{bmatrix} a_{k1} \\ a_{k2} \\ \vdots \\ a_{kn} \end{bmatrix}$$

We now provide three lemmas that will be used in the proof of Theorem 4 concerning the existence of $n \times n \times \rho(n)$ 3D-RODs on $\rho(n)$ variables.

Lemma 1. *With notation as defined above,* ${c_s}^T c_s = \sum_{i=1}^{\rho(n)} x_i^2$*.*

The proof is straight-forward and omitted for brevity.

Lemma 2. *With notation as defined above,* $a_{iq} {a_{jq}}^T = \delta_{ij}$*.*

The proof follows from Eqns. (1) and (2); it is omitted for brevity.

Lemma 3. *With notation as defined above, if* v *is any* $n \times 1$ *column vector then* $v^T A_i^T A_j v = \delta_{ij} v^T v$*.*

Proof. It is clear that $v^T A_i^T A_j v$ is a 1×1 matrix, say $[\alpha]$, where α is a scalar. If $i = j$, then Eqn. (1) implies $v^T A_i^T A_j v = v^T I_n v = v^T v = \delta_{ii} v^T v$. Alternatively, if $i \neq j$, we can utilize Eqn. (3) to show that $[\alpha] = -[\alpha]$; therefore $v^T A_i^T A_j v = [0] = \delta_{ij} v^T v$. □

Theorem 4. *Consider the three-dimensional* $n \times n \times \rho(n)$ *matrix* $G = [g_{ijk}]$*, where* $i = 1, 2, \ldots, n$*,* $j = 1, 2, \ldots, n$ *and* $k = 1, 2, \ldots, \rho(n)$*, defined via its constituent planes by setting* $G_{k=p} = A_p R$*, where* A_p *and* R *are as defined above, for each* $p = 1, 2, \ldots, \rho(n)$*. Then* G *is an* $n \times n \times \rho(n)$ *3D-ROD on* $\rho(n)$ *variables.*

Proof. To show that G is a $n \times n \times \rho(n)$ 3D-ROD we need to verify that all constituent subplanes obtained along each axis-normal direction, $G_{k=p}$, $G_{i=q}$ and $G_{j=s}$, are RODs of size $n \times n$, $n \times \rho(n)$, $n \times \rho(n)$ respectively. We first consider the case of holding k constant, making use of Eqn. (1) and the fact that R is a $n \times n$ ROD: $G_{k=p}G_{k=p}^T = A_pRR^TA_p^T = A_p\sum_{i=1}^{\rho(n)} x_i^2 I_n A_p^T = \sum_{i=1}^{\rho(n)} x_i^2 A_pA_p^T = \sum_{i=1}^{\rho(n)} x_i^2 I_n$. Similarly, we have $G_{k=p}^TG_{k=p} = \sum_{i=1}^{\rho(n)} x_i^2 I_n$. Thus $G_{k=p}G_{k=p}^T = G_{k=p}^TG_{k=p} = \sum_{i=1}^{\rho(n)} x_i^2 I_n$ is a $n \times n$ ROD on $\rho(n)$ variables.

We next consider the subplanes obtained by holding i constant. To begin, we use our previously introduced notation to visualize the $n \times n$ matrix $G_{k=p}$ as follows:

$$\begin{aligned} G_{k=p} &= \begin{bmatrix} a_{k1}R \\ a_{k2}R \\ \vdots \\ a_{kn}R \end{bmatrix} \\ &= \begin{bmatrix} A_pc_1 & A_pc_2 & \dots & A_pc_n \end{bmatrix} \end{aligned}$$

Then, since the columns of the $n \times \rho(n)$ matrix $G_{i=q}$ consist of the qth rows of $G_{k=p}$, where $p = 1, 2, \dots, \rho(n)$, we have:

$$G_{i=q} = \begin{bmatrix} R^T{a_{1q}}^T & R^T{a_{2q}}^T & \dots & R^T{a_{\rho(n)q}}^T \end{bmatrix}$$

Now, writing $G_{i=q}^TG_{i=q} = [q_{ij}]$, and using Lemma 2 we have: $q_{ij} = a_{iq}RR^T{a_{jq}}^T = a_{iq}\left[\sum_{i=1}^{\rho(n)} x_i^2 I_n\right]{a_{jq}}^T = \sum_{r=1}^{\rho(n)} x_r^2 a_{iq}{a_{jq}}^T = \delta_{ij}\sum_{r=1}^{\rho(n)} x_r^2$.

Therefore, all diagonal entries of $G_{i=q}^TG_{i=q}$ are $\sum_{r=1}^{\rho(n)} x_r^2$ while all non-diagonal entries are zero. Thus, $G_{i=q}^TG_{i=q} = \sum_{r=1}^{\rho(n)} x_r^2 I_{\rho(n)}$, and $G_{i=q}$ is an $n \times \rho(n)$ ROD on $\rho(n)$ variables.

Finally, we consider the subplanes obtained by holding j constant. The columns of the $n \times \rho(n)$ matrix $G_{j=s}$ consist of the sth columns of $G_{k=p}$, where $p = 1, 2, \dots, \rho(n)$. Hence, we have

$$G_{j=s} = \begin{bmatrix} A_1c_s & A_2c_s & \dots & A_{\rho(n)}c_s \end{bmatrix}.$$

Let $G_{j=s}^TG_{j=s} = [s_{ij}]$. We have, using Lemma 3 and Lemma 1, that

$$s_{ij} = {c_s}^TA_i^TA_jc_s = \delta_{ij}{c_s}^Tc_s = \delta_{ij}\sum_{r=1}^{\rho(n)} x_r^2.$$

Therefore, all diagonal entries of $G_{j=s}^TG_{j=s}$ are $\sum_{r=1}^{\rho(n)} x_r^2$ while all non-diagonal entries are zero. Thus, $G_{j=s}^TG_{j=s} = \sum_{r=1}^{\rho(n)} x_r^2 I_{\rho(n)}$, and $G_{j=s}$ is an $n \times \rho(n)$ ROD on $\rho(n)$ variables.

We have shown that all subplanes $G_{i=q}$,$G_{j=s}$ and $G_{k=p}$ are 2D-RODs of size $n \times \rho(n)$, $n \times \rho(n)$, $n \times n$ respectively. This proves that G is a $n \times n \times \rho(n)$ 3D-ROD. □

It follows from Theorem 4 that a lower bound on the maximum t such that an $n \times n \times t$ 3D-ROD on $\rho(n)$ variables exists is $\rho(n)$. We will now show that we can improve this bound in certain cases by using permutation matrices and the Kronecker product.

Definition 2. *A* permutation matrix *is a square* $[0,1]$*-matrix that has exactly one entry 1 in each row and each column and 0's elsewhere. A* 3D-permutation matrix *is a 3D matrix in which each of the subplanes in any axis-normal direction are permutation matrices.*

Definition 3. *Let* P *be the* $n \times n \times n$ *matrix* $P = [p_{ijk}]$ *where* $i = 1, 2, \ldots, n$*,* $j = 1, 2, \ldots, n$ *and* $k = 1, 2, \ldots, n$ *defined by*

$$p_{ijk} = \begin{cases} 1 & \textit{if } i + j + k \equiv 0 \pmod{n}, \\ 0 & \textit{if } i + j + k \not\equiv 0 \pmod{n}. \end{cases}$$

It can be verified that P *is a 3D-permutation matrix.*

Lemma 4. *Let* M *be a* $\mu \times \mu$ *permutation matrix and let* R *be a* $p \times n$ *ROD on* u *variables. Then* $M \otimes R$ *is a* $\mu p \times \mu n$ *ROD on* u *variables.*

The proof of Lemma 4 follows directly from properties of the Kronecker product, and the facts that M is a permutation matrix and R is an ROD; the details are omitted for brevity.

Theorem 5. *Let* $n = 2^a b$ *where* b *is odd. Let* P *be a* $b \times b \times b$ *3D-permutation matrix defined using Definition 3. Let* G *be a 3D-ROD of size* $2^a \times 2^a \times \rho(2^a)$ *on* $\rho(2^a)$ *variables. Then* $P \otimes G$ *is 3D-ROD of size* $n \times n \times b\rho(n)$ *with* $\rho(n)$ *variables.*

Proof. First, note that we can obtain the required $2^a \times 2^a \times \rho(2^a)$ 3D-ROD G via the construction described in Theorem 4. Next, note that the definition of the Radon numbers implies that, since b is odd, $\rho(2^a) = \rho(2^a b) = \rho(n)$. Thus, the size $2^a b \times 2^a b \times \rho(2^a) b = n \times n \times b\rho(n)$ of $P \otimes G$ follows directly from the definition of the Kronecker product and the Radon numbers. The number of variables in $P \otimes G$ is also clearly $\rho(2^a) = \rho(n)$. Each plane of $P \otimes G$ is the Kronecker product of a permutation matrix of size $b \times b$ (since P is a 3D-permutation matrix) with a 2D-ROD (since G is a 3D-ROD). By Lemma 4 we can conclude that every plane of $P \otimes G$ is a 2D-ROD. Hence $P \otimes G$ is a 3D-ROD of size $n \times n \times b\rho(n)$ with $\rho(n)$ variables. □

Theorem 5 is important because it shows that for $n = 2^a b$ with b odd, among all $n \times n \times t$ 3D-RODs such that the $n \times n$ constituent subplanes are of maximum rate $\rho(n)/n$, the maximum value for t is bounded below by $b\rho(n)$. When $b \neq 1$, this is a significant improvement over the results given in our previous Theorem 4. In this case, we have improved by a multiplicative factor of b the lower bound on the maximum number of square maximum rate RODs such that the subplanes in the other two axis-normal directions are also RODs.

3 A Maximal Result

In this section, we show that the lower bound implied by Theorem 4 is also a tight upper bound on the maximum value of t for $n \times n \times t$ 3D-RODs when the constituent $n \times n$ subplanes are RODs of maximum rate $\rho(n)/n > 1/2$. If we evaluate $\rho(n)/n$, the maximum rate of an $n \times n$ ROD, we see that $\rho(n)/n > 1/2$ exactly when $n = 1$, 2, 4, 8, and 16. Hence, our work provides a unified alternative proof and construction for the known cases of $n = 1$, 2, 4, and 8 (wherein the 3D-ROD has no zero entries and t is equal to n; see [19, 4, 5, 13], respectively, for constructions of cubic 3D-RODs of these respective sizes) while closing the previously open case of $n = 16$. We begin with the following lemma:

Lemma 5. *Let R be an $n \times n$ ROD with rate strictly greater than 1/2. Then, for any choice of two rows in R, there exists a column in which the entries in both of the rows are non-zero.*

The proof of Lemma 5 is a straight-forward combinatorial argument; the details are omitted for brevity. We are now prepared to present the main result of this paper.

Theorem 6. *Let G be a 3D-ROD of size $n \times n \times t$, where the constituent $n \times n$ subplanes are RODs of maximum rate $\rho(n)/n > 1/2$. Then $t \leq \rho(n)$.*

Proof. Since $G_{k=p}$ is an $n \times n$ square ROD of maximum rate $\rho(n)/n > 1/2$, we have $G^T_{k=p}G_{k=p} = G_{k=p}G^T_{k=p} = \sum_{i=1}^{\rho(n)} x_i^2 I_n$, and thus $det(G_{k=p}) = \left[\sum_{i=1}^{\rho(n)} x_i^2\right]^{\frac{n}{2}} \neq 0$. Thus, the matrix $G_{k=p}$ is invertible for all $p = 1, 2, \ldots, t$. Now, for $p = 1, 2, \ldots, t$, we define $H_p = G_{k=p}G^{-1}_{k=1}$, and upon verifying that $G^{-1}_{k=1} = \frac{1}{\sum_{i=1}^{\rho(n)} x_i^2} G^T_{k=1}$, we can write $H_p = \left[\frac{1}{\sum_{i=1}^{\rho(n)} x_i^2}\right] G_{k=p}G^T_{k=1}$. In this way, we have formed a family of t matrices of size $n \times n$, $H_1, H_2, \ldots, H_t$ with $H_1 = G_{k=1}G^{-1}_{k=1} = I_n$. We will show that $H_2, \ldots, H_t$ form a Hurwitz-Radon family of $n \times n$ matrices.

Note first that $H_p^T H_p = \left[\frac{1}{\sum_{i=1}^{\rho(n)} x_i^2}\right]^2 G_{k=1}G^T_{k=p}G_{k=p}G^T_{k=1} = \left[\frac{1}{\sum_{i=1}^{\rho(n)} x_i^2}\right] G_{k=1}I_nG^T_{k=1} = I_n$, using the derived definition for H_p and the orthogonality of the RODs $G_{k=p}$ and $G_{k=1}$.

Let d_j denote the jth column of the $n \times n$ matrix $G_{k=1}$ and h_{li} denote the ith row of the matrix H_l. Thus d_j and h_{li} are vectors of size $n \times 1$ and $1 \times n$ respectively. Visually, this notation gives:

$$G_{k=1} = \begin{bmatrix} d_1 & d_2 & \ldots & d_n \end{bmatrix}$$
$$H_l = \begin{bmatrix} h_{l1} \\ h_{l2} \\ \vdots \\ h_{ln} \end{bmatrix}$$

We now consider the 2D-ROD $G_{i=q} = \left[\begin{array}{cccc} G_{k=1}^T {h_{1q}}^T & G_{k=1}^T {h_{2q}}^T & \ldots & G_{k=1}^T {h_{tq}}^T \end{array} \right]$. Let $G_{i=q}^T G_{i=q} = [q_{ij}]$. Since $G_{i=q}$ is an $n \times t$ 2D-ROD, we have $G_{i=q}^T G_{i=q} = \left[\sum_{r=1}^{\rho(n)} x_r^2 I\right]$, and so $q_{ij} = \delta_{ij} \sum_{r=1}^{\rho(n)} x_r^2$. Then,

$$q_{ij} \;=\; h_{iq} G_{k=1} G_{k=1}^T {h_{jq}}^T \;=\; h_{iq} \left[\sum_{i=1}^{\rho(n)} x_i^2 I \right] {h_{jq}}^T \;=\; \sum_{r=1}^{\rho(n)} x_r^2 h_{iq} {h_{jq}}^T.$$

It now follows that $h_{iq} {h_{jq}}^T = \delta_{ij}$.

Now let us consider the $n \times t$ 2D-ROD $G_{j=s}$. The columns of $G_{j=s}$ consist of sth columns of $G_{k=p}$, where $p = 1, 2, \ldots, t$. Hence, we have

$$G_{j=s} \;=\; \left[\begin{array}{cccc} H_1 d_s & H_2 d_s & \ldots & H_t d_s \end{array} \right].$$

Let $G_{j=s}^T G_{j=s} = [s_{ij}]$. Since $G_{j=s}$ is a 2D-ROD, we have

$$\begin{aligned} s_{ij} &= \delta_{ij} \sum_{r=1}^{\rho(n)} x_r^2. \qquad (4) \\ &= {d_s}^T H_i^T H_j d_s \end{aligned}$$

We consider two rows α, β with $\alpha \neq \beta$, $1 \leq \alpha, \beta \leq n$. Because the subplane $G_{k=1}$ is a $n \times n$ ROD with maximum rate $\rho(n)/n > 1/2$, Lemma 5 implies that there exists a column γ such that the elements in positions $[\alpha, \gamma]$ and $[\beta, \gamma]$ are both non-zero. Let those elements be r_α and r_β respectively. We now consider the scalar s_{ij}:

$$\begin{aligned} s_{ij} &= d_\gamma^T H_i^T H_j d_\gamma \\ &= \left[d_\gamma^T H_i^T H_j d_\gamma \right]^T \\ &= d_\gamma^T H_j^T H_i d_\gamma, \text{ and thus,} \\ s_{ij} &= \frac{1}{2} d_\gamma^T \left(H_i^T H_j + H_j^T H_i \right) d_\gamma. \end{aligned}$$

From Eqn. (4) we get for $i \neq j$, $s_{ij} = 0$. The matrix $H_i^T H_j + H_j^T H_i$ is symmetric because $\left[H_i^T H_j + H_j^T H_i\right]^T = \left[H_i^T H_j\right]^T + \left[H_j^T H_i\right]^T = H_j^T H_i + H_i^T H_j = H_i^T H_j + H_j^T H_i$. Let $H_i^T H_j + H_j^T H_i = [\zeta_{ij}]$. By the symmetry of the matrix, $\zeta_{ij} = \zeta_{ji}$ for all $i \neq j$. Since r_α and r_β are both non-zero and since each variable occurs exactly once per column in the ROD, the coefficient of $r_\alpha r_\beta$ in $c_\gamma^T \left(H_i^T H_j + H_j^T H_i \right) c_\gamma$ must be zero. Since for $i \neq j$, $s_{ij} = 0$, the coefficient of $r_\alpha r_\beta$ in $c_\gamma^T \left(H_i^T H_j + H_j^T H_i \right) c_\gamma$ is $\zeta_{\alpha\beta} + \zeta_{\beta\alpha} = 2\zeta_{\alpha\beta} = 2\zeta_{\beta\alpha} = 0$. Hence all non-diagonal entries of the matrix $H_i^T H_j + H_j^T H_i$ are zero. Similarly since the coefficients of r_α^2 and r_β^2 are $\zeta_{\alpha\alpha}$ and $\zeta_{\beta\beta}$ respectively, we can conclude that all diagonal elements of $H_i^T H_j + H_j^T H_i$ are zero. Hence we can conclude that for all $i \neq j$, $H_i^T H_j + H_j^T H_i = 0$. This implies, in particular when taking $i = 1$ and recalling that $H_1 = I_n$, that for all $j = 2, 3, \ldots t$, $H_j = -H_j^T$; it then follows that for $2 \leq i < j \leq t$, $H_j H_i = -H_i H_j$. Thus, combined with the previous demonstration of orthogonality, we can conclude that $\{H_2, H_3, \ldots, H_t\}$ forms a family of $t-1$ Hurwitz-Radon matrices. Hence by Radon's work summarized in Theorem 1, we get the bound $t \leq \rho(n)$. □

By Lemma 5 and Theorem 6, we have shown that the following $n \times n \times t$ 3D-RODs achieved by our algorithm given in Theorem 4 are the best possible in terms of maximizing t when the constituent $n \times n$ planes are of maximum rate: $1 \times 1 \times 1$ on 1 variable, $2 \times 2 \times 2$ on 2 variables, $4 \times 4 \times 4$ on 4 variables, $8 \times 8 \times 8$ on 8 variables, and $16 \times 16 \times 9$ on 9 variables. While the cases of $n = 1$, 2, 4, and 8 have been treated previously in [19, 4, 5, 13], respectively, the treatment here is unified, uses different methods, and additionally covers the previously open case of $n = 16$. Although they are limited in number, it is reasonable to expect that these $2^a \times 2^a \times \rho(2^a)$ 3D-RODs, for $a = 0, 1, 2, 3, 4$, will be among the most useful for the proposed applications of higher-dimensional RODs related to telecommunications and higher-dimensional orthogonal functions [5] due to their achievement of maximum rate, which is reasonably high at $\rho(2^a)/2^a > 1/2$, and due to their achievement of the maximum number of $2^a \times 2^a$ ROD subplanes given their other parameters.

4 Conclusions

In this paper, we analyzed 3D-RODs of size $n \times n \times t$ such that the $n \times n$ constituent subplanes are RODs of maximum rate and therefore on $\rho(n)$ variables. Our goal was to determine bounds on t. We began by reviewing a simple construction technique for generating maximum rate $n \times n$ RODs on $\rho(n)$ variables. We then used such $n \times n$ RODs to build $n \times n \times \rho(n)$ 3D-RODs on $\rho(n)$ variables, and when $n = 2^a b$ with b odd, to build $n \times n \times b\rho(n)$ 3D-RODs on $\rho(n)$ variables.

We concluded that for $n = 2^a b$ with b odd, $t \geq b\rho(n)$ and when $b = 1$, this bound, simplified to $t \geq \rho(n)$, is also the tight upper bound for designs whose $n \times n$ constituent subplanes are of maximum rate strictly greater than $1/2$. These maximal designs, though limited to $n = 1, 2, 4, 8$, and 16, may be among the most useful for practical applications due to their reasonably high maximum rate and number of $n \times n$ subplanes. Our work leads naturally to the following open research questions: 1) For $n = 2^a$, for any a, is $\rho(2^a)$ the tight upper bound on t for 3D-RODs of size $2^a \times 2^a \times t$ on $\rho(2^a)$ variables? 2) For $n = 2^a b$, for any a and any odd b, is $b\rho(2^a)$ the tight upper bound on t for 3D-RODs of size $n \times n \times t$ on $\rho(n)$ variables? While an affirmative answer to question 2) implies an affirmative answer to question 1), we ask both questions as they may lead to different proof techniques and possibly different answers.

Our work relied on the Hurwitz-Radon theory, and it represents a step in a systematic treatment of optimizing parameters in 3D-RODs. We hope that our results and techniques will lead to more work analyzing higher-dimensional real orthogonal designs.

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