AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **54** (2012), Pages 133–140

Trees with strong equality between the Roman domination number and the unique response Roman domination number

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Abstract

A Roman dominating function (RDF) on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF f is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. A function $f : V(G) \to \{0, 1, 2\}$ with the ordered partition (V_0, V_1, V_2) of $V(G)$, where $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i = 0, 1, 2$, is a unique response Roman function if $x \in V_0$ implies $|N(x) \cap V_2| \leq 1$ and $x \in V_1 \cup V_2$ implies that $|N(x) \cap V_2| = 0$. A function $f : V(G) \to \{0, 1, 2\}$ is a unique response Roman dominating function (or just URRDF) if it is a unique response Roman function and a Roman dominating function. The Roman domination number $\gamma_R(G)$ (respectively, the unique response Roman domination number $u_R(G)$) is the minimum weight of an RDF (respectively, URRDF) on G. We say that $\gamma_R(G)$ strongly equals $u_R(G)$, denoted by $\gamma_R(G) \equiv u_R(G)$, if every RDF on G of minimum weight is a URRDF. In this paper we provide a constructive characterization of trees T with $\gamma_R(T) \equiv u_R(T)$.

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1 Introduction

Let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* $N(v)$ of a vertex $v \in V$ is $\{u \in V : uv \in E\}$ and the *closed neighborhood* $N[v]$ is $N(v) \cup \{v\}$. If D is a subset of $V(G)$, then the subgraph induced by D in G is denoted by $G[D]$. The *degree* of v, denoted by $\deg(v)$, is the cardinality of its open neighborhood. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support vertex is denoted by $S_{p,q}$. A set $S \subseteq V(G)$ is a 2-*packing* of G if for every pair of vertices $x, y \in S$, $N[x] \cap N[y] = \emptyset$. The *distance* $d(u, v)$ between two vertices u and v in a graph G is the minimum number of edges of a path from u to v. The *diameter* $\text{diam}(G)$ of G, is $\max_{u,v \in V(G)} d(u, v)$.

For a graph G, let $f : V(G) \to \{0, 1, 2\}$ be a function, and let (V_0, V_1, V_2) be the ordered partition of $V = V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$. There is a one-one correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of $V(G)$. So we will write $f = (V_0, V_1, V_2)$ (or $f = (V_0^f, V_1^f, V_2^f)$) to indicate the function f.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex u of $V(G)$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight $f(V(G))$ of an RDF f on G is the value $\sum_{u \in V(G)} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G. A function $f = (V_0, V_1, V_2)$ is called a $\gamma_R(G)$-*function* or γ_R-*function* on G if it is an RDF on G and $f(V(G)) = \gamma_R(G)$. Roman domination has been introduced by Cockayne et al. [4] and has been studied for example in [1, 2, 3, 6, 7, 8, 9].

A function $f : V(G) \longrightarrow \{0, 1, 2\}$ with ordered partition (V_0, V_1, V_2) is a *unique response Roman function* if $x \in V_0$ implies $|N(x) \cap V_2| \leq 1$ and $x \in V_1 \cup V_2$ implies that $|N(x) \cap V_2| = 0$. A function $f : V(G) \to \{0, 1, 2\}$ is a *unique response Roman dominating function* (URRDF) if it is a unique response Roman function and an RDF. The *unique response Roman domination number* of G, denoted by $u_R(G)$, is the minimum weight of a URRDF on G. A function $f = (V_0, V_1, V_2)$ is called an $u_R(G)$-*function* or u_R-*function* on G if it is a URRDF and $f(V(G)) = u_R(G)$. Note that unique response Roman dominating functions exist in a graph G, since $(\emptyset, V(G), \emptyset)$ is always a unique response Roman dominating function. Unique response Roman domination has been introduced by Rubalcaba et al. [10] and studied in [5].

Observe that $\gamma_R(G) \leq u_R(G)$ for every graph G. Clearly, if G is a graph with $\gamma_R(G) = u_R(G)$, then every $u_R(G)$-function is a $\gamma_R(G)$-function. However, not every $\gamma_R(G)$-function is an $u_R(G)$-function even when $\gamma_R(G) = u_R(G)$. For example, the double star $S_{2,3}$ has two $\gamma_R(S_{2,3})$-functions, but only one of them is an $u_R(S_{2,3})$-function. We say that $\gamma_R(G)$ and $u_R(G)$ are *strongly equal*, denoted by $\gamma_R(G) \equiv u_R(G)$, if every $\gamma_R(G)$-function is an $u_R(G)$-function.

In this paper, we present a constructive characterization of trees T with $\gamma_R(T) \equiv$

$u_R(T)$. We make use of the following.

Proposition 1 (Cockayne et al. [4]) *Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. Then*
(1) The subgraph induced by V_1 has maximum degree one.
(2) No edge of G joins V_1 to V_2.

2 The characterization

We begin with the following lemmas.

Lemma 2 *Let G be a connected graph of order $n \geq 3$. Then $\gamma_R(G) \equiv u_R(G)$ if and only if for every $\gamma_R(G)$-function $f = (V_0, V_1, V_2)$, V_1 is independent and V_2 is a 2-packing.*

Proof. ($\Longrightarrow$) Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. Since f is a URRDF, we find that V_2 is a 2-packing. Assume that V_1 is not independent. By Proposition 1, $G[V_1]$ has an edge xy. Since $n \geq 3$, we may assume, without loss of generality, that $\deg(y) > 1$. Let $z \in N(y) - \{x\}$. Then $f(z) = 0$, and so there is a vertex $z_1 \in V_2$ such that $z_1 \in N(z)$. Now g defined on $V(G)$ by $g(y) = 2$, $g(x) = 0$, and $g(u) = f(u)$ if $u \notin \{x, y\}$, is a $\gamma_R(G)$-function. This contradicts $\gamma_R(G) \equiv u_R(G)$, since z is adjacent to y and z_1.

($\Longleftarrow$) By the definition of a URRDF, if there is a $\gamma_R(G)$-function $f = (V_0, V_1, V_2)$ such that it is not a URRDF, then V_2 is not a 2-packing, a contradiction. Hence $\gamma_R(G) \equiv u_R(G)$. ■

Lemma 3 *Let T be a tree with $\gamma_R(T) \equiv u_R(T)$, T not a star, and f a $\gamma_R(T)$-function.*

(1) If v is a support vertex, then $f(v) \neq 1$.

(2) If v is adjacent to at least two leaves, then $f(v) = 2$.

(3) If v is a support vertex of degree at least three, and each neighbor of v except at most one is either a leaf or a support vertex of degree two, then $f(v) = 2$.

Proof. (1) If v is a support vertex and $f(v) = 1$, then f maps every leaf adjacent to v to 1. But V_1^f is not an independent set in this case, contradicting Lemma 2.

(2) Suppose that $f(v) \neq 2$, then $f(v) = 0$ by statement 1, and hence f maps all leaves adjacent to v to 1, and f maps a neighbor w of v to 2. Note that w is not a leaf since T is not a star. So replacing $f(v)$ by 2 and $f(u) = 0$ for all leaves u adjacent to v makes f a $\gamma_R(T)$-function such that V_2^f not a 2-packing, contradicting Lemma 2.

(3) $f(v) \neq 1$ by statement (1), and we may assume that v is adjacent to exactly one leaf w by statement (2). If every neighbor of v except w is a support vertex of degree two, then we can easily see that $\gamma_R(T) = t + 2$, where $t \geq 2$ is the number of support vertices adjacent to v, and $f(v) = 2$. So we may assume that there is a vertex x adjacent to v such that x is not a leaf or a support vertex of degree two. Let v_1 be a support vertex of degree two which is adjacent to v, and v_2 be the leaf adjacent to v_1. If $f(v) \neq 2$, then $f(w) = 1$ and $f(v_1) + f(v_2) \geq 2$, so replacing $(f(v_1), f(v_2))$ by $(2, 0)$ still keeps f a $\gamma_R(G)$-function, and hence $f(x) \neq 2$ due to the strong equality. Define g on $V(T)$ by $g(v) = 2$, $g(w) = g(v_1) = 0$, $g(v_2) = 1$, $g(x) = 0$, and $g(u) = f(u)$ if $u \notin \{w, v, v_1, v_2, x\}$. If $f(x) = 1$, then g is an RDF on T with weight less than $\gamma_R(T)$; if $f(x) = 0$, then g is a $\gamma_R(T)$-function which is not a URRDF since V_2^g is not a 2-packing, both of which are contradictions. ■

For nonnegative integers s and t, an (s,t)-*star* is the graph obtained from identifying s copies of P_2's and t copies of P_3's at one leaf of each graph, which is called the *central vertex* of the (s,t)-star. Note that the $(s,0)$-star is the star $K_{1,s}$. Let $\mathcal{F}$ be the set of (s,t)-stars with $s + t \geq 3$ and $t \geq 1$. By *attaching* a tree T' to a vertex v of a tree T at $u \in V(T')$, we mean that adding a disjoint T' to T and an edge uv.

We call a vertex v of a graph G to be a *weakly Roman vertex* if $f(v) \neq 2$ for every $\gamma_R(G)$-function f.

Now we present a constructive characterization of trees T with $\gamma_R(T) \equiv u_R(T)$. For this purpose, we define a family of trees as follows: Let $\mathcal{T}$ be the collection of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k = T$ $(k \geq 1)$ of trees, where T_1 is K_1 or a star $K_{1,t}$ with $t \geq 2$, and T_{i+1} can be obtained recursively from T_i by one of the following operations for $1 \leq i \leq k-1$.

- **Operation** $\mathcal{O}_1$: Attach a star $K_{1,r}$, where $r \geq 2$, to a weakly Roman vertex $v \in V(T_i)$ at a leaf of $K_{1,r}$.
- **Operation** $\mathcal{O}_2$: Attach a tree in $\mathcal{F}$ to a vertex $v \in V(T_i)$ at a neighbor u of the central vertex, where v is weakly Roman if u is a leaf, and $f(v) = 0$ for every $\gamma_R(T_i)$-function f if u is a support vertex but not the central vertex.

Lemma 4 *If T_{i+1} is a tree obtained from a tree T_i by Operation $\mathcal{O}_1$, then $\gamma_R(T_i) \equiv u_R(T_i)$ if and only if $\gamma_R(T_{i+1}) \equiv u_R(T_{i+1})$.*

Proof. Let T_{i+1} be obtained from T_i by attaching $K_{1,r}$ to a vertex $v \in V(T_i)$ at a leaf w of a star $K_{1,r}$, where $r \geq 2$. Let O be the central vertex of $K_{1,r}$. Any $\gamma_R(T_i)$-function can be extended to an RDF on T_{i+1} by assigning 2 to O and zero to every vertex in $N(O)-\{w\}$, and so $\gamma_R(T_{i+1}) \leq \gamma_R(T_i)+2$. Let $a \in N(O)-\{w\}$, and f a $\gamma_R(T_{i+1})$-function. Define g on $V(T_i)$ by $g(v) = \max\{1, f(v)\}$, and $g(u) = f(u)$ if $u \in V(T_i)-\{v\}$. It is clear that g is an RDF on T_i. If $f(w) = 2$, then $f(a)+f(O) \geq 1$, and the weight of g is at most $\gamma_R(T_{i+1}) - 2$, so $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 2$. If $f(w) \neq 2$, then $f(w) + f(O) + f(a) \geq 2$, and $f|_{V(T_i)}$ is an RDF on T_i, implying that $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 2$. Hence, $\gamma_R(T_{i+1}) = \gamma_R(T_i) + 2$.

($\Longrightarrow$) Let $\gamma_R(T_i) \equiv u_R(T_i)$. Note that no vertex in K_2 is weakly Roman, so $|V(T_i)| \neq 2$. Also, the lemma holds if $|V(T_i)| = 1$, so we may assume that $|V(T_i)| \geq 3$. Assume that $\gamma_R(T_{i+1}) \not\equiv u_R(T_{i+1})$. By Lemma 2, there is a $\gamma_R(T_{i+1})$-function $f = (V_0^f, V_1^f, V_2^f)$ such that either V_1^f is not independent or V_2^f is not a 2-packing. Suppose that $f(v) = 2$. Since $\gamma_R(K_{1,r} - \{w\}) \geq 2$, $f|_{V(T_i)}$ is an RDF on T_i with weight at most $\gamma_R(T_{i+1}) - 2$, so $f|_{V(T_i)}$ is a $\gamma_R(T_i)$-function that maps v to 2, contradicting v is weakly Roman. So $f(v) \neq 2$. Since $\gamma_R(K_{1,r}) = 2$ and every $\gamma_R(K_{1,r})$-function maps the central vertex to 2, $f(w) = 2$ only if $f(v) = 0$ and $f(x) \neq 2$ for all $x \in N(v) - \{w\}$. However, if $f(w) = 2$, then g is a $\gamma_R(T_i)$-function such that $V_2^g = V_2^f \cap V(T_i)$, so V_2^g is a 2-packing and V_1^g is an independent set by the strong equality for T_i, and hence V_2^f is a 2-packing and V_1^f is an independent set as $f(x) \neq 2$ for all $x \in N(v) - \{w\}$, a contradiction. So $f(w) \neq 2$. Since $f(v) \neq 2$ and $f|_{V(T_i)}$ is a $\gamma_R(T_i)$-function, V_1^f is an independent set and V_2^f is a 2-packing as no $\gamma_R(T_{i+1})$-function maps w to 1, a contradiction.

($\Longleftarrow$) Let $\gamma_R(T_{i+1}) \equiv u_R(T_{i+1})$. Assume that $\gamma_R(T_i) \not\equiv u_R(T_i)$. By Lemma 2, there is a $\gamma_R(T_i)$-function $f = (V_0^f, V_1^f, V_2^f)$ such that either V_1^f is not independent or V_2^f is not a 2-packing. Then h defined on $V(T_{i+1})$ by $h(O) = 2$, $h(u) = 0$ if $u \in N(O)$, and $h(u) = f(u)$ if $u \in V(T_i)$, is a $\gamma_R(T_{i+1})$-function which is not a URRDF, a contradiction. ■

Lemma 5 *If T_{i+1} is a tree obtained from a tree T_i by Operation $\mathcal{O}_2$, then $\gamma_R(T_i) \equiv u_R(T_i)$ if and only if $\gamma_R(T_{i+1}) \equiv u_R(T_{i+1})$.*

Proof. Let T_{i+1} be obtained from T_i by attaching a tree $F \in \mathcal{F}$ to a vertex $v \in V(T_i)$ at w, where w is a neighbor of the central vertex of F, and v is weakly Roman if w is a leaf, and $f(v) = 0$ for every $\gamma_R(T_i)$-function f if w is a support vertex but not the central vertex. Let O be the central vertex of F. We shall prove that $\gamma_R(T_{i+1}) = \gamma_R(T_i) + l + 2$. Any $\gamma_R(T_i)$-function can be extended to an RDF on T_{i+1} by assigning 2 to O, 0 to every vertex in $N(O)$, and 1 to vertices in $V(F) - N[O]$, and so $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + l + 2$. For every $\gamma_R(T_{i+1})$-function f, it is easy to see that $f(V(F)) = \sum_{z \in V(F)} f(z) \geq l + 2 = \gamma_R(F)$, and we define f' to be a RDF on T_i by $f'(v) = \max\{1, f(v)\}$ and $f'(u) = f(u)$ for every $u \in V(T_i) - \{v\}$. If f is a $\gamma_R(T_{i+1})$-function such that $f(V(F)) > \gamma_R(F)$, then f' is an RDF on T_i with weight at most $\gamma(T_{i+1}) - \gamma_R(F) = \gamma_R(T_{i+1}) - l - 2$, so $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - l - 2$. On the other hand, if f is a $\gamma_R(T_{i+1})$-function such that $f(V(F)) = \gamma_R(F)$, then $f(w) \neq 2$, and hence $f|_{V(T_i)}$ is an RDF on T_i, implying that $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - l - 2$. We deduce that $\gamma_R(T_{i+1}) = \gamma_R(T_i) + l + 2$. In other words, $\gamma_R(T_{i+1}) = \gamma_R(T_i) + \gamma_R(F)$.

($\Longrightarrow$) Assume that $\gamma_R(T_i) \equiv u_R(T_i)$, and let f be a $\gamma_R(T_{i+1})$-function. As in the proof of Lemma 4, we may assume that $|V(T_i)| \geq 3$. Note that $f(V(F)) \leq \gamma_R(F) + 1$ since $\gamma_R(T_i) \leq f'(V(T_i)) \leq f(V(T_i)) + 1$. Suppose that $f(V(F)) = \gamma_R(F)$. Then $f(w) = 0$, so $f|_{V(T_i)}$ is a $\gamma_R(T_i)$-function, and hence $f(v) \neq 2$. Furthermore, as $f(V(F)) = \gamma_R(F)$ and $f(v) \neq 2$, $f(O) = 2$ and $f(x) \neq 2$ for $x \in V(F) - \{O\}$. By the strong equality for T_i, $V_2^f = V_2^{f|_{T_i}} \cup \{O\}$ is a 2-packing, and V_1^f is an independent set. This proves that the strong equality holds for T_{i+1} by Lemma 2. So we may assume that $f(V(F)) = \gamma_R(F) + 1$. In this case, $f|_{V(T_i)}$ is not an RDF on T_i, so $f(w) = 2$, $f(v) = 0$, and $f'(v) = 1$. Since f' is a $\gamma_R(T_i)$-function with $f'(v) = 1$, w is a leaf adjacent to O, but this implies that $f(V(F)) \geq \gamma_R(F) + 2$, a contradiction. Hence, $\gamma_R(T_{i+1}) \equiv u_R(T_{i+1})$.

($\Longleftarrow$) By Lemma 2, $\gamma_R(T_i) \equiv u_R(T_i)$ since every $\gamma_R(T_i)$-function can be extended to a $\gamma_R(T_{i+1})$-function by assugning O to be 2 and assigning u to be 0 for $u \in N(O)$, and assigning u to be 1 for $u \in V(F) - N[O]$. ■

We now are ready to establish our main result.

Theorem 6 *Let T be a tree. Then $\gamma_R(T) \equiv u_R(T)$ if and only if T is K_2 or $T \in \mathcal{T}$.*

Proof. ($\Longleftarrow$) Clearly if $T = K_2$, then $\gamma_R(T) \equiv u_R(T)$. Let $T \in \mathcal{T}$. Then there is a sequence of trees T_1, T_2, …, $T_k = T$ ($k \geq 1$) such that T_1 is K_1 or a star $K_{1,t}$ with $t \geq 2$, and T_{i+1} can be obtained recursively from T_i by Operation $\mathcal{O}_1$ or $\mathcal{O}_2$ for $1 \leq i \leq k-1$. We use the induction on the number of operations performed to construct T. Clearly, the property is true if $k = 1$. This establishes the basis case. Assume now that $k \geq 2$ and that the result holds for all trees in $\mathcal{T}$ that can be constructed from a sequence of length at most $k-1$. By the induction hypothesis, $\gamma_R(T_{k-1}) \equiv u_R(T_{k-1})$. By construction, T is obtained from T_{k-1} by using one of Operations $\mathcal{O}_1$ or $\mathcal{O}_2$. Hence, $\gamma_R(T) \equiv u_R(T)$, by Lemmas 4 and 5.

($\Longrightarrow$) Let T be a tree of order n with $\gamma_R(T) \equiv u_R(T)$. We shall do induction on n to show that $T \in \mathcal{T}$. If $1 \leq n \leq 2$, then clearly $T = K_n$. This establishes the base case. Assume that every tree $T' \neq K_2$ with order less than the order of T and with $\gamma_R(T') \equiv u_R(T')$ is in $\mathcal{T}$. Let T be a tree with $n \geq 3$ and $\gamma_R(T) \equiv u_R(T)$, and let f be a $\gamma_R(T)$-function. If $\text{diam}(T) \leq 2$, then T is a star $K_{1,r} \in \mathcal{T}$ for some $r \geq 2$. If $\text{diam}(T) = 3$, then T is a double star. In this case, let x, y be the two non-leaves of T. If $\deg(x) \geq 3$ and $\deg(y) \geq 3$, then $\gamma_R(T) = 4$, and $(V(T) - \{x, y\}, \emptyset, \{x, y\})$, is a $\gamma_R(T)$-function, contradicting $\gamma_R(T) \equiv u_R(T)$. So one of x and y has degree two, and hence T can be obtained from K_1 by Operation $\mathcal{O}_1$. Thus, we may assume that $\text{diam}(T) \geq 4$. Let $k = \text{diam}(T)$, and let $x_0x_1x_2...x_k$ be a diameterical path of T, where x_0 and x_k are two leaves of T. We root T at x_0.

Assume that there is a $\gamma_R(T)$-function f such that $f(x_{k-1}) = 2$. Let T' be the component of $T - x_{k-2}x_{k-3}$ containing x_{k-3}. By the strong equality, $f(u) = 0$ for $u \in N(x_{k-1})$ since $f(x_{k-1}) = 2$, so replacing $f|_{T'}$ by any $\gamma_R(T')$-function still keeps f a $\gamma_R(T)$-function, and hence no $\gamma_R(T')$-function maps x_{k-3} to 2, again by the strong equality. Thus, x_{k-3} is a weakly Roman vertex of T'. If $\deg(x_{k-2}) = 2$, then T is obtained from T' by operation $\mathcal{O}_1$, so it is done by Lemma 4 and the induction hypothesis. So we may assume that $\deg(x_{k-2}) \geq 3$. By the strong equality, x_{k-2} is adjacent to at most one leaf, and no support vertex other than x_{k-1} is a child of x_{k-2}. In other words, children of x_{k-2} are x_{k-1} and a leaf a. By Lemma 3 (3), $\deg(x_{k-1}) \geq 3$, so that T can be obtained by attaching a tree in $\mathcal{F}$ to T' at vertex x_{k-3}. If there is a $\gamma_R(T')$-function g such that $g(x_{k-3}) = 1$, then the function h defined on $V(T)$ by $h(x_{k-1}) = h(x_{k-2}) = 2$, $h(a) = h(x_{k-3}) = 0$, $h(u) = 0$ for $u \in N(x_{k-1}) - \{x_{k-2}\}$, and $h(u) = g(u)$ for $u \in V(T') - \{x_{k-3}\}$, is a $\gamma_R(T)$-function such that V_2^h is not a 2-packing, a contradiction. Therefore, every $\gamma_R(T')$-function maps x_{k-3} to 0, so the result follows by Lemma 5 and the induction hypothesis.

Consequently, we may assume that every $\gamma_R(T)$-function f satisfies $f(x_{k-1}) \neq 2$, and hence $f(x_{k-2}) = 2$, $f(x_k) = 1$, and $f(u) = 0$ for all $u \in N(x_{k-2})$ by Lemma 3 and the strong equality. In fact, f maps every child of x_{k-3} on any longest path in T to 2 by symmetry. Hence, together with Lemma 3 and the strong equality of γ_R and u_R, each child of x_{k-3} other than x_{k-2} is either a leaf or a support vertex of degree two. Also, since no $\gamma_R(T)$-function maps x_{k-3} to 2, the strong equality implies that x_{k-3} has at most one child other than x_{k-2}, and the child (if exists) must be a leaf. Similarly, every support vertex which is a child of x_{k-2} has degree two by Lemma 3. If $\deg(x_{k-2}) = 2$, then we can replace $f(x_{k-1})$ by 2, $f(x_{k-2})$ and $f(x_k)$ by 0, and $f(x_{k-3})$ by $\max\{1, f(x_{k-3})\}$ to obtain a $\gamma_R(T)$-function with $f(x_{k-1}) = 2$, a contradiction. Thus, $\deg(x_{k-2}) \geq 3$. Let X be the component of $T - x_{k-3}x_{k-4}$ containing x_{k-3}, and Y the other component. It is clear that X is in $\mathcal{F}$, so it is sufficient to show that x_{k-4} is a weakly Roman vertex in Y if x_{k-3} does not have children other than x_{k-2}, and show that every $\gamma_R(Y)$-function maps x_{k-4} to 0 for otherwise, and then the theorem follows from Lemma 5.

Note that no $\gamma_R(Y)$-function maps x_{k-4} to 2 by the strong equality. Assume that x_{k-3} has some children other than x_{k-2}, so the children of x_{k-3} are x_{k-2} and a leaf z. If there is a $\gamma_R(Y)$-function q such that $q(x_{k-4}) = 1$, then the function q_1 defined on $V(T)$ by $q_1(x_{k-3}) = 2$, $q_1(x_{k-4}) = q_1(z) = 0$, $q_1(u) = f(u)$ for $u \in V(X) - \{z, x_{k-3}\}$, and $q_1(u) = q(u)$ for $u \in V(Y) - \{x_{k-4}\}$, is a $\gamma_R(T)$-function such that $V_2^{q_1}$ is not a 2-packing, a contradiction. Thus, every $\gamma_R(Y)$-function maps x_{k-4} to 0 as desired, so we may assume that x_{k-3} does not have children other than x_{k-2}. Let t be the number of support vertices in X. Clearly, $\gamma_R(X) \leq t+2$, so $\gamma_R(Y) \geq \gamma_R(T) - t - 2$. Conversely, since every $\gamma_R(T)$-function maps x_{k-3} to 0, $\gamma_R(Y) \leq \gamma_R(T) - t - 2$. Hence, if x_{k-4} is not a weakly Roman vertex of Y, then there is an RDF c of Y with weight $\gamma_R(Y) = \gamma_R(T) - t - 2$ such that $c(x_{k-4}) = 2$, and then c can be extended to an RDF of T with weight $\gamma_R(T)$ by assigning $c(x_{k-2}) = 2$, $c(u) = 1$ for every $u \in V(X) - N(x_{k-2})$, and $c(u) = 0$ for every $u \in V(X) \cap N(x_{k-2})$. However, V_2^c is not a 2-packing, a contradiction. ■

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(Received 26 Sep 2011; revised 13 July 2012)