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An extension of the Corrádi-Hajnal Theorem

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Abstract

Corrádi and Hajnal [Acta Math. Acad. Sci. Hungar. 14 (1963), 423–439] showed that if G is a graph of order at least $3k$ with minimum degree at least $2k$ then G contains k disjoint cycles. In this paper, we extend this result to disjoint cycles of length at least 4. We prove that if G is a graph of order at least $4k$ with $k \geq 2$ and the minimum degree of G is at least $2k$ then with three easily recognized exceptions, G contains k disjoint cycles of length at least 4. We propose two conjectures for a graph to contain k disjoint cycles of length at least s for each $s \geq 5$.

1 Introduction

Several graphs are said to be disjoint if no two of them have any common vertex. Corrádi and Hajnal [2] investigated the maximum number of disjoint cycles in a graph. They proved that if G is a graph of order at least $3k$ with minimum degree at least $2k$, then G contains k disjoint cycles. Erdős and Faudree [5] conjectured that if G is a graph of order $4k$ with minimum degree at least $2k$, then G contains k disjoint cycles of length 4. In [8], we confirmed this conjecture. In this paper, we show that if a graph G of order $n \geq 4k$ with $k \geq 2$ has minimum degree at least $2k$ then with three easily recognized exceptions, G contains k disjoint cycles of length at least 4. Motivated by this work, we propose two conjectures. We list these two conjectures before stating our main theorem as follows:

Conjecture 1. *Let d and k be two positive integers with $k \geq 2$. If G is a graph of order at least $(2d+1)k$ and the minimum degree of G is at least $(d+1)k$ then G contains k disjoint cycles of length at least $2d+1$.*

Conjecture 2. *Let d and k be two positive integers with $k \geq 3$ and $d \geq 3$. Let G be a graph of order $n \geq 2dk$ with minimum degree at least dk. Then G contains*

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k disjoint cycles of length at least $2d$, unless k is odd and $n = 2dk + r$ for some $1 \leq r \leq 2d-2$.

El-Zahar [3] conjectured that if G is a graph of order $n = n_1 + n_2 + \cdots + n_k$ with $n_i \geq 3$ $(1 \leq i \leq k)$ and the minimum degree of G is at least $\lceil n_1/2 \rceil + \lceil n_2/2 \rceil + \cdots + \lceil n_k/2 \rceil$, then G contains k disjoint cycles of lengths $n_1, n_2, \ldots, n_k$, respectively. In Conjecture 1, if G has order $(2d+1)k$ then the conjecture reduces to the special case of El-Zahar's conjecture where $n_i = 2d+1$ for all $1 \leq i \leq k$. Similarly, if G has order $2dk$ in Conjecture 2, then the conjecture reduces to the special case of El-Zahar's conjecture where $n_i = 2d$ for all $1 \leq i \leq k$. To see the necessity of $n \neq 2dk + r$ for $1 \leq r \leq 2d-2$ in Conjecture 2, we observe the disjoint union $K_{2dt+d+r_1} \cup K_{2dt+d+r_2}$ with $0 \leq r_1 \leq r_2 \leq d-1$ and $r_2 \neq 0$. Clearly, the minimum degree of this graph is at least $2dt + d + r_1 - 1 = dk + r_1 - 1$ and each of $K_{2dt+d+r_1}$ and $K_{2dt+d+r_2}$ does not contain $t+1$ disjoint cycles of length at least $2d$. Therefore if $k = 2t+1$ then $K_{2dt+d+r_1} \cup K_{2dt+d+r_2}$ does not contain k disjoint cycles of length at least $2d$. If $r_1 \neq 0$ then this graph has minimum degree at least dk. If $r_1 = 0$, we choose a fixed vertex from $K_{2dt+d+r_2}$ and join this vertex to every vertex of K_{2dt+d}. The resulting graph has minimum degree $2dt + d = dk$ and still does not contain k disjoint cycles of length at least $2d$.

To state our result, we define exceptional graphs as follows. First, We say that a cycle is a *feasible* cycle if its order is at least 4. If G is a graph and X and Y are two disjoint subgraphs of G or two subsets of $V(G)$, we use $e(X, Y)$ to denote the number of edges of G between X and Y.

For each odd integer $k \geq 3$, we let Γ_k be a set of graphs such that a graph G belongs to Γ_k if and only if G contains two disjoint complete subgraphs G_1 and G_2 such that $V(G) = V(G_1 \cup G_2)$, $|V(G_1)| = 2k+1$ and $2k \leq |V(G_2)| \leq 2k+1$. Moreover, if $|V(G_2)| = 2k+1$ then $e(G_1, G_2) \leq 1$ and if $|V(G_2)| = 2k$ then $e(G_1, G_2) = 2k$ and G_1 has a vertex x adjacent to every vertex of G_2. Clearly, each of G_1 and G_2 contain at most $(k-1)/2$ disjoint feasible cycles and therefore G does not contain k disjoint feasible cycles.

For each integer $k \geq 2$ and each odd integer $n \geq 4k+1$, let $\Sigma_{k,n}$ be a set of graphs such that a graph G belongs to $\Sigma_{k,n}$ if and only if G has order n and there exists a partition $V(G) = V_1 \cup V_2$ such that $|V_1| = 2k-1$, $|V_2| = n - 2k + 1$, $e(V_1, V_2) = (2k-1)(n-2k+1)$ and the subgraph induced by V_2 consists of $(n-2k+1)/2$ independent edges. Clearly, each feasible cycle of G contains at least two vertices of V_1 and therefore G does not contain k disjoint feasible cycles.

Let F_9 be a 4-regular graph of order 9 with $V(F_9) = \{a_1, a_2, a_3, a_4\} \cup \{x_1, x_2, x_3, x_4, x_5\}$ such that $\{x_1a_1, x_1a_2, x_2a_3, x_2a_4, x_4a_2, x_4a_3, x_5a_1, x_5a_4\} \subseteq E(F_9)$ and $a_1a_2a_3a_4a_1$, $x_1x_2x_3x_1$ and $x_3x_4x_5x_3$ are three cycles of F_9.

Main Theorem *Let k and n be two integers with $k \geq 2$ and $n \geq 4k$. If G is a graph of order $n \geq 4k$ and the minimum degree of G is at least $2k$, then G contains k disjoint cycles of length at least 4 if and only if $G \not\cong F_9$ and $G \notin \Gamma_k \cup \Sigma_{k,n}$.*

1.1 Terminology and Notation

Let G be a graph. Let H be a subgraph of G or a subset of $V(G)$ or a sequence of distinct vertices of G. Let $u \in V(G)$. We define $N(u, H)$ to be the set of neighbors of u contained in H, and let $e(u, H) = |N(u, H)|$. Clearly, $N(u, G) = N(u)$ and $e(u, G)$ is the degree of u in G. If X is a subgraph of G or a subset of $V(G)$ or a sequence of distinct vertices of G, we define $N(X, H) = \cup_u N(u, H)$ and $e(X, H) = \sum_u e(u, H)$ where u runs over all the vertices in X. Let each of $X_1, X_2, \ldots, X_r$ be a subgraph of G or a subset of $V(G)$. We use $[X_1, X_2, \ldots, X_r]$ to denote the subgraph of G induced by the set of all the vertices that belong to at least one of $X_1, X_2, \ldots, X_r$. For each integer $i \geq 3$, we use C_i to denote a cycle of length i and $C_{\geq i}$ to denote a cycle of length at leat i. Use P_j to denote a path of order j for all integers $j \geq 1$. For a cycle C of G, a chord of C is an edge of $G - E(C)$ which joins two vertices of C, and we use $\tau(C)$ to denote the number of chords of C in G. The length of C is denoted by $l(C)$. For each integer $k \geq 3$, a k-cycle is a cycle of length k.

If S is a set of subgraphs of G, we write $G \supseteq S$. For an integer $k \geq 1$ and a graph G', we use kG' to denote a set of k disjoint graphs isomorphic to G'. If G_1 and G_2 are two graphs, we use $G_1 \uplus G_2$ to denote a set of two disjoint graphs, one isomorphic to G_1 and the other isomorphic to G_2. For two graphs H_1 and H_2, the union of H_1 and H_2 is still denoted by $H_1 \cup H_2$ as usual, that is, $H_1 \cup H_2 = (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2))$. Let each of Y and Z be a subgraph of G, or a subset of $V(G)$, or a sequence of distinct vertices of G. If Y and Z do not have any common vertices, we define $E(Y, Z)$ to be the set of all the edges of G between Y and Z. Clearly, $e(Y, Z) = |E(Y, Z)|$. If $C = x_1 x_2 \ldots x_r x_1$ is a cycle, then the operations on the subscripts of the x_i's will be taken by modulo r in $\{1, 2, \ldots, r\}$.

If we write a graph G as a sequence $x_1 x_2 \ldots x_l$ of its vertices, it means that $V(G) = \{x_1, x_2, \ldots, x_l\}$ and $E(G) = \{x_i x_{i+1} | 1 \leq i \leq l - 1\}$. Note that the sequence may have repeated vertices. We use F to denote a graph of order 5 such that $F = x_3 x_1 x_2 x_3 x_4 x_5$. We use B_n to denote a graph of order $n \geq 5$ and size $n + 1$ with a hamiltonian path, say $u_1 u_2 \ldots u_n$, such that $\{u_1 u_3, u_{n-2} u_n\} \subseteq E$, i.e., $B_n = u_3 u_1 u_2 \ldots u_{u-2} u_{n-1} u_n u_{n-2}$. If B is a graph isomorphic to B_n for some $n \geq 5$, we use B^* to denote the set of the four vertices of degree 2 in B which are contained in the two triangles of B. We use C_4^+ to denote a graph of order 4 with exactly five edges.

If $\sigma = (L_1, \ldots, L_s)$ is a sequence of cycles of G, we define $V(\sigma) = \cup_{i=1}^s V(L_i)$ and $\tau(\sigma) = \sum_{i=1}^s \tau(L_i)$. Let $\{H, Q_1, \ldots, Q_t\}$ be a set of $t + 1$ disjoint subgraphs of G such that $Q_i \cong C_4$ for $i = 1, \ldots, t$. We say that $\{H, Q_1, \ldots, Q_t\}$ is optimal if $[H, Q_1, \ldots, Q_t]$ does not contain $t + 1$ disjoint subgraphs $H', Q_1', \ldots, Q_t'$ such that $H' \cong H$, $Q_i' \cong C_4 (1 \leq i \leq t)$ and $\sum_{i=1}^t \tau(Q_i') > \sum_{i=1}^t \tau(Q_i)$.

Let Q be a 4-cycle and H a subgraph of order 4 in G. We write $H \geq Q$ if H has a 4-cycle Q' such that $\tau(Q') \geq \tau(Q)$. Moreover, if $\tau(Q') > \tau(Q)$, we write $H > Q$. If $d \in V(Q)$, we use d^* to denote the vertex of Q with $dd^* \notin E(Q)$.

Let Q be a 4-cycle of G and $u \in V(Q)$. Let $x \in V(G) - V(Q)$. We write $x \to (Q, u)$ if $[Q - u + x] \supseteq C_4$. In this case, we say that u is replaceable by x in Q. Moreover, if $[Q - u + x] \geq Q$ then we write $x \Rightarrow (Q, u)$ and if $[Q - u + x] > Q$ then we write $x \xrightarrow{a} (Q, u)$. In addition, if it does not hold that $x \xrightarrow{a} (Q, u)$ then we write $x \xrightarrow{na} (Q, u)$. Clearly, $x \Rightarrow (Q, u)$ when $x \xrightarrow{a} (Q, u)$. If $x \rightarrow (Q, u)$ for all $u \in V(Q)$ then we write $x \rightarrow Q$. Similarly, we define $x \Rightarrow Q$. Note that if $e(x, Q) = 3$ then $x \rightarrow Q$ if and only if $dd^* \in E$ where $d \in V(Q)$ with $xd \notin E$.

Let P be a path of order at least 2 or a sequence of distinct vertices of length at least 2 in $G - V(Q + x)$. Let X be a subset of $V(G) - V(Q + x)$ with $|X| \geq 2$. We write $x \rightarrow (Q, u; P)$ if $x \rightarrow (Q, u)$ and u is adjacent to the two end vertices of P. In this case, if P is a path of order 3, then $[x, Q, P] \supseteq 2C_4$. We write $x \rightarrow (Q, u; X)$ if $x \rightarrow (Q, u; yz)$ for some $\{y, z\} \subseteq X$ with $y \neq z$. We write $x \rightarrow (Q; P)$ if $x \rightarrow (Q, u; P)$ for some $u \in V(Q)$. Similarly, we define $x \rightarrow (Q; X)$.

2 Lemmas

Let $G = (V, E)$ be a graph. We will use the following lemmas. Lemmas 2.7, 2.10 and 2.11 are already proved in [7] which play important role in this paper. As defined in the introduction, a feasible cycle is a cycle of order at least 4.

Lemma 2.1 *The following two statements hold:*

(a) If L is a cycle of order $p \geq 5$ and $v \in V(G) - V(L)$ such that $e(v, L) \geq 2$, then either $[L + v]$ contains a feasible cycle C with $l(C) < p$, or $e(v, L) = 2$ and v is adjacent to two consecutive vertices of L.

(b) If P is a path of order $p \geq 4$ and $u \in V(G) - V(P)$ such that $e(u, P) \geq 3$, then for some endvertex z of P, $[P + u - z]$ contains a feasible cycle C with $l(C) \leq p$. Moreover, if $p \geq 5$ and $[P + u]$ does not contain a feasible cycle of length less than p, then there exists a labelling $P = z_1 z_2 \dots z_p$ such that $N(u, P) = \{z_1, z_2, z_p\}$.

Proof. The statement (a) is an easy observation. We prove (b) as follows. Say $P = z_1 z_2 \dots z_p$. As $p \geq 4$ and $e(u, P) \geq 3$, we readily see that $[P + u - z]$ contains a feasible cycle C with $l(C) \leq p$ for some endvertex z of P. So for the proof of (b), we may assume that $e(u, P) = 3$ with $\{z_1, z_p\} \subseteq N(u)$. Then we readily see that if $p \geq 5$ and $[P + u]$ does not contain a feasible cycle of length less than p, then $e(u, z_2 z_{p-1}) = 1$. So (b) holds. ∎

Lemma 2.2 *Suppose that C is a 4-cycle of G and $x \in V(G) - V(C)$ such that $e(x, C) \geq 3$. Then either $x \rightarrow C$ or there exists $v \in V(C)$ such that $xv \notin E$ and $x \xrightarrow{a} (C, v)$.*

Proof. Say $C = v_1 v_2 v_3 v_4 v_1$ with $e(x, C - v_4) = 3$. If $xv_4 \in E$ or $v_2 v_4 \in E$ then $x \rightarrow C$. Otherwise $xv_4 \notin E$ and $v_2 v_4 \notin E$ and so $x \xrightarrow{a} (C, v_4)$. ∎

Lemma 2.3 *Suppose that x and y are two distinct vertices in G and C is a 4-cycle of $G - \{x, y\}$ with $e(xy, C) \geq 5$. Then there exists a vertex $u \in V(C)$ such that either $x \rightarrow (C, u)$ and $uy \in E$ or $y \Rightarrow (C, u)$ and $ux \in E$.*

Proof. If $e(x,C)=4$ or $e(y,C)=4$, the lemma obviously holds. So assume that $e(x,C)\le 3$ and $e(y,C)\le 3$. Say $C=u_1u_2u_3u_4u_1$. For the proof, we assume that $x\not\rightarrow (C,u_i)$ for each $u_i\in V(C)$ with $u_iy\in E$. Then $x\not\rightarrow C$. Suppose that $e(x,C)=3$. Say without loss of generality $e(x,u_1u_2u_3)=3$. Then $u_2u_4\notin E$ as $x\not\rightarrow C$. Moreover, we see that $e(y,u_2u_4)=0$ since $x\rightarrow (C,u_i)$ for $i=2,4$. Thus $e(y,u_1u_3)=2$ and so $y\Rightarrow (C,u_2)$ with $u_2x\in E$. Hence the lemma holds. If $e(x,C)<3$ then $e(x,C)=2$ and $e(y,C)=3$. Say without loss of generality $e(y,u_1u_2u_3)=3$. Then $x\not\rightarrow (C,u_2)$ and so $e(x,u_1u_3)\le 1$. If $xu_4\in E$ then the lemma holds as $y\Rightarrow (C,u_4)$. So assume $xu_4\notin E$. Thus $e(x,u_1u_3)=1$ and $xu_2\in E$. Say without loss of generality $xu_1\in E$. Then $u_2u_4\notin E$ for otherwise $x\rightarrow (C,u_3)$ with $u_3y\in E$. Thus $y\Rightarrow (C,u_2)$ with $u_2x\in E$. ∎

Lemma 2.4 *Let S be a subgraph of order 4 with $V(S)=\{x_0,x_1,x_2,x_3\}$ and $E(S)=\{x_0x_1,x_0x_2,x_0x_3\}$ and C a 4-cycle in G with $V(C)\cap V(S)=\emptyset$. Suppose that $e(x_1x_2x_3,C)\ge 7$. Then either $[S,C]\supseteq 2C_4$ or there exists $\{i,j\}\subseteq\{1,2,3\}$ with $i\ne j$ and $v\in V(C)$ such that $x_i\Rightarrow (C,v)$ and $vx_j\in E$.*

Proof. Assume that the latter conclusion does not hold. We shall prove that $[S,C]\supseteq 2C_4$. As $e(x_1x_2x_3,C)\ge 7$, we may assume without loss of generality $e(x_1,C)\ge 3$. Say $C=v_1v_2v_3v_4v_1$ with $e(x_1,v_1v_2v_3)=3$. If $e(x_1,C)=4$ or $e(v_4,x_2x_3)\ge 1$, we see that $x_1\Rightarrow (C,v)$ and $vx_j\in E$ for some $v\in V(C)$ and $j\in\{2,3\}$, a contradiction. Hence $e(v_4,x_1x_2x_3)=0$ and so $e(x_2x_3,v_1v_2v_3)\ge 4$. If $v_2v_4\notin E$ then $x_1\Rightarrow (C,v_2)$ and so $e(v_2,x_2x_3)=0$. Thus $e(x_2,v_1v_3)=2$ and consequently, $x_2\Rightarrow (C,v_2)$ and $v_2x_1\in E$, a contradiction. Hence $v_2v_4\in E$. Therefore $x_1\rightarrow (C;x_2x_0x_3)$, i.e., $[S,C]\supseteq 2C_4$. ∎

Lemma 2.5 *Let p and q be two integers with $p\ge q\ge 4$ and $p\ge 5$. Let C and L be two disjoint cycles of G with $l(C)=q$, and $l(L)=p$. If $e(L,C)\ge 2p+1$ then one of the following two statements holds:*

(a) $p=5$ and $q=4$;

(b) $[C,L]$ contains two disjoint feasible cycles C' and L' such that if $q=4$ then $l(C')=4$ and $l(L')<p$, and if $q>4$ then either $l(C')+l(L')=q+p$ with $l(C')<q$ or $l(C')+l(L')<q+p$.

Proof. Say $C=a_1a_2\dots a_qa_1$ and $L=x_1x_2\dots x_px_1$ with $e(x_1,C)\ge e(x_i,C)$ for all $x_i\in V(L)$. As $e(L,C)\ge 2p+1$, $e(x_1,C)\ge 3$. For a contradiction, we may assume that neither of (a) and (b) holds.

First, suppose that $q=4$. Then $p\ge 6$. As $e(x_1,C)\ge 3$, $x_1\rightarrow (C,a_i)$ for some $a_i\in V(C)$. As (b) does not hold, $[L-x_1+a_i]$ does not have a feasible cycle of order $\le p-1$. By Lemma 2.1(b), $e(a_i,L-x_1)\le 2$. If $x_1\rightarrow C$, then we would have $e(a_i,L-x_1)\le 2$ for all $a_i\in V(C)$. Consequently, $e(C,L)\le 12$. But $e(C,L)\ge 2p+1\ge 13$, a contradiction. Hence $x_1\not\rightarrow C$ and so $e(x_1,C)=3$. Say $e(x_1,a_1a_2a_3)=3$. Then $x_1\rightarrow (C,a_i)$ and so $e(a_i,L-x_1)\le 2$ for each $i\in\{2,4\}$. Thus $e(a_1a_3,L-x_1)\ge 13-3-4=6$. Suppose $e(a_1,L)\ge 5$. Then $[L-x_1-x_j+a_1]$ contains a feasible cycle of order $\leq p-1$ for each $j \in \{2,p\}$. Thus $[a_2,a_3,a_4,x_1,x_j] \not\supseteq C_4$ for each $j \in \{2,p\}$ since (b) does not hold. This implies that $e(x_2x_p, a_2a_3a_4) = 0$ and so $e(x_j, C) \leq 1$ for $j \in \{2,p\}$. This argument implies that for each $x_i \in V(L)$, if $e(x_i, C) = 3$ then $e(x_{i+1}, C) \leq 1$ and $e(x_{i-1}, C) \leq 1$. It follows that $e(C,L) \leq 2p$, a contradiction. Hence $e(a_1, L) \leq 4$. Similarly, $e(a_3, L) \leq 4$. It follows that $p = 6$, $e(a_1, L-x_1) = e(a_3, L-x_1) = 3$ and $e(a_2, L-x_1) = e(a_4, L-x_1) = 2$. Suppose that $e(a_1a_3, x_2x_6) \geq 0$. Say without loss of generality $a_1x_2 \in E$. Then $[x_1,x_2,a_1,a_2] \supseteq C_4$. Then $[a_3,a_4,x_3,x_4,x_5,x_6]$ does not contain a feasible cycle of length < 6. This implies that $e(a_3a_4, x_3x_4x_5x_6) \leq 2$ and so $e(a_3a_4, L-x_1) \leq 4$, a contradiction. Hence $e(a_1a_3, x_3x_4x_5) = 6$. As $e(a_2a_4, L-x_1) = 4$, we readily see that $[C,L] \supseteq 2C_4$, a contradiction.

Therefore $q \geq 5$. If either $e(x_1, C) \geq 5$ or $N(x_1)$ does not contain two consecutive vertices of C, then we readily see that $[C-a_i-a_{i+1}, x_1]$ contains a feasible cycle of order at most $\leq q-1$ for all $i \in \{1,\ldots,q\}$. Then $[L-x_1, a_i, a_{i+1}]$ does not contain a feasible cycle of order at most p since (b) does not hold. This implies that $e(a_ia_{i+1}, L-x_1) \leq 2$ for all $i \in \{1,\ldots,q\}$. Consequently, $2e(C, L-x_1) \leq 2q$ and so $e(C,L) \leq 2q < 2p+1$, a contradiction. Therefore $e(x_1, C) \leq 4$ and $N(x_1)$ contains two consecutive vertices. Hence $e(x_i, C) \leq 4$ for all $x_i \in V(L)$. Similarly, we see, by exchanging the roles of C and L in this argument, that $e(a_i, L) \leq 4$ for all $a_i \in V(C)$. Say without loss of generality $\{a_1,a_2\} \subseteq N(x_1)$.

We claim that $e(x_1, C) = 3$. If this is false, then $e(x_1, C) = 4$. If all the four vertices of $N(x_1, C)$ are consecutive on C, say without loss of generality $N(x_1, C) = \{a_1,a_2,a_3,a_4\}$, then each of $[x_1,a_1,a_2,a_3]$, $[a_2,a_3,a_4,x_1]$ and $[C-a_2-a_3, x_1]$ contains a feasible cycle of order $\leq q-1$. Thus each of $[L-x_1, a_4, \ldots, a_q]$, $[L-x_1, a_5, \ldots, a_q, a_1]$ and $[L-x_1, a_2, a_3]$ does not contain a feasible cycle since (b) does not hold. It follows that $e(a_4 \ldots a_q, L-x_1) \leq 2$, $e(a_5 \ldots a_qa_1, L-x_1) \leq 2$ and $e(a_2a_3, L-x_1) \leq 2$. This yields that $e(C, L-x_1) \leq 6$ and so $e(C,L) \leq 10 < 2p+1$, a contradiction. If exactly three vertices of $N(x_1, C)$ are consecutive on C, say $N(x_1, C) = \{a_1,a_2,a_3,a_t\}$ with $5 \leq t \leq q-1$, then each of $[x_1,a_1,a_2,a_3]$, $[x_1,a_3,\ldots,a_t]$ and $[x_1,a_t,\ldots,a_q,a_1]$ contains a feasible cycle of order $\leq q-1$. Thus each of $[L-x_1,a_4,\ldots,a_q]$, $[L-x_1,a_{t+1},\ldots,a_q,a_1,a_2]$ and $[L-x_1,a_2,\ldots,a_{t-1}]$ does not contain a feasible cycle. As above, this yields that $e(C,L) \leq 10 < 2p+1$, a contradiction. Hence $N(x_1, C) = \{a_1,a_2,a_k,a_t\}$ with $4 \leq k < t \leq q-1$. Then each of $[x_1,a_2,\ldots,a_k]$ and $[x_1,a_t,\ldots,a_q,a_1]$ contains a feasible cycle of order $\leq q-1$. As above, we would have that $e(a_{k+1} \ldots a_qa_1, L-x_1) \leq 2$ and $e(a_2 \ldots a_k, L-x_1) \leq 2$ and so $e(C,L) \leq 2+2+4 = 8$, a contradiction.

Therefore $e(x_1, C) = 3$. Note that this argument shows that if $q = p$ then we would also have $e(a_i, L) \leq 3$ for all $a_i \in V(C)$. Say $x_1a_k \in E$ with $3 \leq k \leq q$. Say without loss of generality $k \neq q$. If $k \neq 3$, then each of $[a_1,x_1,a_k,\ldots,a_q]$ and $[x_1,a_2,\ldots,a_k]$ contains a feasible cycle of order $\leq q-1$. As above, we shall have that $e(a_2 \ldots a_{k-1}, L-x_1) \leq 2$ and $e(a_{k+1} \ldots a_qa_1, L-x_1) \leq 2$. It follows that $e(a_k, L) \geq 2p+1-2-2-2 \geq 5$, a contradiction since $e(a_i, L) \leq 4$ for all $a_i \in V(C)$. Therefore $k = 3$. Then $[x_1,a_1,a_2,a_3] \supseteq C_4$. Thus $[L-x_1,a_4,\ldots,a_q]$ does not contain a feasible cycle and so $e(a_4 \ldots a_q, L-x_1) \leq 2$. It follows that $e(a_1a_2a_3, L) \geq 2p+1-2 = 2p-1$.

As $e(a_1a_2a_3, L) \le 12$, we obtain $p \le 6$. Suppose that $p = q$. Then similarly, we have $e(a_i, L) \le 3$ for all $a_i \in V(C)$ and so $2p - 1 \le 9$. Consequently, $p = 5$, $e(a_i, L) = 3$ for $i \in \{1, 2, 3\}$ and $e(a_4a_5, L) = 2$. Similarly, $N(a_i, L)$ must contain three consecutive vertices on L for $i \in \{1, 2, 3\}$. Say without loss of generality $e(a_4, L) \ge 1$. Then there exists a labelling $L = y_1y_2y_3y_4y_5y_1$ such that $\{a_4y_1, a_3y_2\} \subseteq E$. Thus $[a_4, a_3, y_1, y_2] \supseteq C_4$ and so $[a_1, a_2, y_3, y_4, y_5]$ does not contain a feasible cycle. This implies that $e(a_1a_2, y_3y_4y_5) \le 2$. It follows that $e(a_1a_2, y_1y_2) = 4$. Thus $e(y_1, C) = e(y_2, C) = 3$. Hence $a_3y_1 \notin E$. It follows that $e(a_3, y_2y_3y_4) = 3$. Thus $[y_1, a_1, a_5, a_4] \supseteq C_4$ and $[a_3, y_2, y_3, y_4] \supseteq C_4$, a contradiction. Hence $p > q$. Thus $p = 6$ and $q = 5$. Then $e(a_1a_2a_3, L) \ge 11$. As $[C - a_2, x_1] \supseteq C_5$, $[L - x_1, a_2]$ does not contain a feasible cycle of order ≤ 5. By Lemma 2.1(b), $e(a_2, L - x_1) \le 2$ and so $e(a_2, L) \le 3$. It follows that $e(a_1, L) = e(a_3, L) = 4$, $e(a_2, L) = 3$ and $e(a_4a_5, L) = 2$. Label $L = y_1y_2y_3y_4y_5y_6y_1$ such that $\{y_1, y_3\} \subseteq N(a_1)$. Thus $[a_1, y_1, y_2, y_3] \supseteq C_4$ and so $[a_2, a_3, y_4, y_5, y_6]$ does not contain a feasible cycle. Therefore $e(a_2a_3, y_4y_5y_6) \le 2$ and so $e(a_2a_3, y_1y_2y_3) \ge 5$. If $e(a_4a_5, y_4y_5y_6) \ge 1$, we may assume without loss of generality that $e(a_4a_5, y_4y_5) \ge 1$. Then $[a_1, y_3, y_4, y_5, a_4, a_5]$ contains a feasible cycle and $[a_2, a_3, y_1, y_2] \supseteq C_4$, a contradiction. Hence $e(a_4a_5, y_1y_2y_3) = 2$. As $e(y_i, C) \le 3$ for $i = 1, 2, 3$, It follows that $e(a_1, y_1y_2y_3) = 2$ and $e(a_2a_3, y_1y_2y_3) = 5$. Therefore $e(a_2a_3, y_4y_5y_6) = 2$ and $e(a_1, y_4y_5y_6) = 2$. Then $e(a_1, y_4y_6) \ge 1$. Say without loss of generality $a_1y_4 \in E$. Then $[a_1, y_1, y_6, y_5, y_4] \supseteq C_5$ and $[a_2, a_3, y_2, y_3] \supseteq C_4$, a contradiction. ∎

Lemma 2.6 *Let C and B be two disjoint subgraphs of G such that $C \cong C_4$ and $B \cong B_t$ with $t \ge 5$. Suppose that $[C, B]$ does not contain a 4-cycle C' such that $\tau(C') > \tau(C)$ and $[C, B] - V(C') \supseteq B_r$ for some $r \ge 5$. If $e(B^*, C) \ge 8$ and $[C, B] \not\supseteq C_4 \uplus C_{\ge 4}$, then there exist two labellings $C = a_1a_2a_3a_4a_1$ and $B = x_3x_1x_2x_3 \dots x_{t-2}x_{t-1}x_tx_{t-2}$ such that $e(\{x_1, x_2, x_{t-1}, x_t\}, C) = 8$ and one of the following nine statements holds:*

(1^0) *$t = 5$ and $N(x_i, C) = \{a_1, a_3\}$ for all $i \in \{1, 2, 4, 5\}$;*

(2^0) *$t = 5$ and $N(x_i, C) = \{a_1, a_2\}$ for all $i \in \{1, 2, 4, 5\}$;*

(3^0) *$e(x_1x_2, C) = 8$ and $e(x_{t-1}x_t, C) = 0$;*

(4^0) *$N(x_1, C) = \{a_1, a_2, a_3\}$, $N(x_2, C) = N(x_{t-1}, C) = \{a_1\}$, $N(x_t, C) = \{a_1, a_4, a_3\}$, $a_1a_3 \in E$, $a_2a_4 \notin E$;*

(5^0) *$N(x_1, C) = \{a_1, a_2, a_3\}$, $N(x_2, C) = \{a_1, a_3\}$, $e(x_{t-1}, C) = 0$, $N(x_t, C) = \{a_1, a_4, a_3\}$, $a_1a_3 \in E$, $a_2a_4 \notin E$;*

(6^0) *$N(x_1, C) = \{a_1, a_4, a_3\}$, $N(x_2, C) = \{a_1, a_2, a_3\}$, $e(x_{t-1}x_t, a_2a_4) = 0$, $e(a_1, x_{t-1}x_t) = 1$, $e(a_3, x_{t-1}x_t) = 1$, $a_2a_4 \notin E$;*

(7^0) *$N(x_1, C) = \{a_1, a_4\}$, $N(x_2, C) = \{a_2, a_3\}$, $N(x_{t-1}, C) = \{a_1, a_2\}$, $N(x_t, C) = \{a_3, a_4\}$, $\tau(C) = 0$;*

(8^0) *$N(x_1, C) = \{a_1, a_4\}$, $N(x_2, C) = \{a_2, a_3\}$, $e(x_{t-1}, C) = 0$, $e(x_t, C) = 4$, $\tau(C) = 0$;*

(9^0) $e(x_1, C) = 4$, $e(x_2 x_{t-1}, C) = 0$, $e(x_t, C) = 4$.

Moreover, if there exists a vertex $v \in V(G) - V(C) \cup B^* \cup \{x_3, x_{t-2}\}$ *such that* $e(v, C) \geq 2$ *and* $G - V(C) \cup B^*$ *has a path* P *from* v *to a vertex of* $B - B^*$*, then either one of* (1^0) *and* (2^0) *holds, or* $[B, C, P] \supseteq C_4 \uplus C_{\geq 4}$.

Proof. Say $C = a_1a_2a_3a_4a_1$ and $B = x_3x_1x_2x_3 \ldots x_{t-2}x_{t-1}x_tx_{t-2}$. Set $T_1 = x_1x_2x_3x_1$, $T_2 = x_{t-2}x_{t-1}x_tx_{t-2}$ and $H = [C, B]$. For the proof, suppose $H \not\supseteq C_4 \uplus C_{\geq 4}$. This implies that $x_i \not\rightarrow (C; B^* - \{x_i\})$ for all $x_i \in B^*$. We divide the proof into the following three cases.

Case 1. $e(a_i, B^*) = 4$ for some $a_i \in V(C)$.

Say $e(a_1, B^*) = 4$. Then $[a_1, T_1] \supseteq C_4$ and $[a_1, T_2] \supseteq C_4$. Thus $[a_2, a_3, a_4, x_i, x_{i+1}] \not\supseteq C_{\geq 4}$ for $i = 1, t-1$. This implies that $e(a_2a_3a_4, x_ix_{i+1}) \leq 2$ for $i = 1, t-1$. As $e(C, B^*) \geq 8$, it follows that $e(a_2a_3a_4, x_ix_{i+1}) = 2$ for $i = 1, t-1$. If $e(a_i, x_1x_2) = 2$ for some $i \in \{2, 3, 4\}$ then $[a_i, T_1] \supseteq C_4$ and so $t = 5$ for otherwise $H \supseteq 2C_4$. Consequently, $e(a_i, x_{t-1}x_t) = 2$ for otherwise $H \supseteq C_4 \uplus C_{\geq 4}$, and therefore ($1^0$) or ($2^0$) holds. Hence we may assume that $e(a_i, x_1x_2) \leq 1$ and similarly, $e(a_i, x_{t-1}x_t) \leq 1$ for $i = 2, 3, 4$. Since $[a_2, a_3, a_4, x_i, x_{i+1}] \not\supseteq C_{\geq 4}$ for $i = 1, t-1$, this yields that $e(x_i, a_2a_3a_4) = 2$ and $e(x_j, a_2a_3a_4) = 2$ for some $i \in \{1, 2\}$ and $j \in \{t-1, t\}$. Say without loss of generality $i = 1$ and $j = t$. As $x_i \not\rightarrow (C; B^* - \{x_i\})$ for all $x_i \in B^*$, it follows that $e(x_1, a_2a_4) = 1$, $e(x_t, a_2a_4) = 1$ and $e(a_3, x_1x_t) = 2$. Say without loss of generality $x_ta_4 \in E$. If $x_1a_4 \in E$ then $[x_1, a_3, x_t, a_4] \supseteq C_4$ and so $H \supseteq C_4 \uplus C_{\geq 4}$, a contradiction. Hence $x_1a_2 \in E$. As $H \not\supseteq 2C_4$, $x_t \not\rightarrow (C, a_1)$ and so $a_2a_4 \notin E$. Clearly, $[x_1, x_2, a_1, a_2] \supseteq C_4^+$ and $[a_3, a_4, x_t, x_{t-1}, x_{t-2}] \supseteq B_5$. By the condition of the lemma, $\tau(C) \geq 1$. Thus $a_1a_3 \in E$ and so (4^0) holds.

Case 2. $e(a_i, B^*) = 3$ for some $a_i \in V(C)$.

Say $e(a_1, x_1x_2x_t) = 3$. By Case 1, we may assume that $e(a_i, B^*) \leq 3$ for all $i \in \{2, 3, 4\}$. As $[a_1, T_1] \supseteq C_4$, $[a_2, a_3, a_4, x_{t-1}, x_t] \not\supseteq C_{\geq 4}$ and so $e(a_2a_3a_4, x_{t-1}x_t) \leq 2$. Thus $e(x_1x_2, a_2a_3a_4) \geq 8 - 3 - 2 = 3$. If $e(a_i, x_{t-1}x_t) = 2$ for some $i \in \{2, 3, 4\}$ then $[a_i, T_2] \supseteq C_4$, $e(x_1x_2, C - a_i) \geq 3$ and so $[C - a_i, x_1x_2] \supseteq C_{\geq 4}$, a contradiction. Hence $e(a_i, x_{t-1}x_t) \leq 1$ for all $i \in \{2, 3, 4\}$.

First, suppose that $e(x_{t-1}x_t, a_2a_3a_4) = 2$. Because $[a_2, a_3, a_4, x_{t-1}, x_t] \not\supseteq C_{\geq 4}$, $e(x_j, a_2a_3a_4) = 2$ for some $j \in \{t-1, t\}$. As $x_j \not\rightarrow (C, a_1; B^* - \{x_j\})$, $e(x_j, a_2a_4) \leq 1$ and $a_2a_4 \notin E$. Say without loss of generality $e(x_j, a_3a_4) = 2$. As $x_i \not\rightarrow (C, a_1; B^* - \{x_i\})$ for $i \in \{1, 2\}$, $e(x_i, a_2a_4) \leq 1$ for $i \in \{1, 2\}$. Thus $e(a_3, x_1x_2) \geq 1$. Say without loss of generality $x_1a_3 \in E$. Then $a_4x_2 \notin E$ as $x_1 \not\rightarrow (C, a_4; B^* - \{x_1\})$. Assume that $x_1a_4 \in E$. Then $x_2a_3 \notin E$ as $x_2 \not\rightarrow (C, a_4; B^* - \{x_2\})$, and consequently, $x_2a_2 \in E$. Then $[x_1, x_2, a_1, a_2] \supseteq C_4^+$ and $[a_3, a_4, T_2] \supseteq B_5$. Thus $H \supseteq C_4^+ \uplus B_5$ and so $\tau(C) \geq 1$. Thus $a_1a_3 \in E$ and so $x_2 \rightarrow (C, a_4; B^* - \{x_2\})$, a contradiction. Hence $x_1a_4 \notin E$. Thus $e(x_1x_2, a_2a_3) \geq 3$ and so $e(a_2, x_1x_2) \geq 1$. Then $[x_1, x_2, a_1, a_2] \supseteq C_4^+$. As above, $\tau(C) \geq 1$ and so $a_1a_3 \in E$. Thus $e(a_2, x_1x_2) \leq 1$ as $x_j \not\rightarrow (C, a_2; B^* - \{x_j\})$. Hence $x_2a_3 \in E$ and $e(a_2, x_1x_2) = 1$. Say without loss of generality $a_2x_1 \in E$. Then

$[x_1, x_2, a_2, a_3] \supseteq C_4$. Thus $[a_1, a_4, x_{t-1}, x_t] \not\supseteq C_4$. This yields that $j = t$ and so (5^0) holds.

Next, suppose that $e(x_{t-1}x_t, a_2a_3a_4) \leq 1$. Then $e(x_1x_2, a_2a_3a_4) \geq 8 - 3 - 1 = 4$. As $x_i \not\rightarrow (C; B^* - \{x_i\})$ for $i \in \{1, 2\}$, $e(x_i, a_2a_4) \leq 1$ for $i \in \{1, 2\}$. It follows that $e(x_i, a_2a_3a_4) = 2$ for $i \in \{1, 2\}$ with $e(x_1, a_2a_4) = 1$, $e(x_2, a_2a_4) = 1$, $e(a_3, x_1x_2) = 2$ and $e(a_2a_3a_4, x_{t-1}x_t) = 1$. Say without loss of generality $x_1a_4 \in E$. As $x_2 \not\rightarrow (C; B^* - \{x_2\})$, $e(a_4, x_{t-1}x_t) = 0$ and $a_2a_4 \notin E$. Thus $e(a_2a_3, x_{t-1}x_t) = 1$. If $x_2a_2 \in E$ then similarly, $e(a_2, x_{t-1}x_t) = 0$ and so (6^0) holds. So assume that $x_2a_4 \in E$. Then $[a_4, T_1] \supseteq C_4$ and so $[a_1, a_2, a_3, x_{t-1}, x_t] \not\supseteq C_{\geq 4}$. This yields that $e(a_3, x_{t-1}x_t) = 0$ and $a_2x_{t-1} \notin E$. Thus $a_2x_t \in E$. Consequently, $[x_1, x_2, a_3, a_4] \supseteq K_4$ and $[a_1, a_2, x_t, x_{t-1}, x_{t-2}] \supseteq B_5$. Thus $H \supseteq B_5 \uplus K_4$ and so $\tau(C) = 2$, a contradiction.

Case 3. $e(a_i, B^*) = 2$ for all $a_i \in V(C)$.

First, suppose that $e(a_i, x_1x_2) = 2$ or $e(a_i, x_{t-1}x_t) = 2$ for some $a_i \in V(C)$. Say without loss of generality $e(a_1, x_1x_2) = 2$. Then $[a_1, T_1] \supseteq C_4$. Thus $[a_2, a_3, a_4, x_{t-1}, x_t] \not\supseteq C_{\geq 4}$ and so $e(a_2a_3a_4, x_{t-1}x_t) \leq 2$. Therefore $e(x_1x_2, a_2a_3a_4) \geq 8 - 2 - 2 = 4$ and so $e(a_r, x_1x_2) = 2$ for some $r \in \{2, 3, 4\}$. Then $[x_1, x_2, a_1, a_r] \supseteq C_4$. As $H \not\supseteq 2C_4$, we obtain that $e(a_j, x_{t-1}x_t) \leq 1$ for each $j \in \{2, 3, 4\}$ with $j \neq r$. If $r = 3$ then $e(a_i, x_{t-1}x_t) = 0$ for each $i \in \{2, 4\}$ with $e(a_i, x_1x_2) \geq 1$ because $x_p \not\rightarrow (C; B^* - \{x_p\})$ for each $p \in \{1, 2\}$, and it follows that (3^0) holds. Hence we may assume that $e(a_3, x_1x_2) \leq 1$ and $e(a_2, x_1x_2) = 2$. Then $e(a_1a_2, x_{t-1}x_t) = 0$ and $[a_1, a_2, x_1, x_2] \cong K_4$. Thus $[a_3, a_4, T_2] \not\supseteq C_4$ and so $e(x_{t-1}x_t, a_3a_4) \leq 2$. If $e(x_{t-1}x_t, a_3a_4) \leq 1$ then $e(x_1x_2, C) \geq 7$, and since $x_i \not\rightarrow (C; B^* - \{x_i\})$ for $i \in \{1, 2\}$, it follows that (3^0) holds. Hence $e(x_{t-1}x_t, a_3a_4) = 2$. As $[a_3, a_4, T_2] \not\supseteq C_4$, this implies that $e(x_q, a_3a_4) = 2$ for some $q \in \{t-1, t\}$. Thus $[a_3, a_4, T_2] \supseteq B_5$. By the condition of the lemma, $\tau(C) = 2$. Thus $x_q \rightarrow (C, a_1; x_1x_2)$, a contradiction.

Therefore $e(a_i, x_1x_2) = 1$ and $e(a_i, x_{t-1}x_t) = 1$ for all $a_i \in V(C)$. Assume that $e(x_p, a_ia_{i+2}) = 2$ for some $x_p \in B^*$ and $i \in \{1, 2\}$. Say without loss of generality that $e(x_t, a_1a_3) = 2$. As $x_t \not\rightarrow (C, a_j; B^* - \{x_t\})$ for $j \in \{2, 4\}$, we must have that $e(x_t, a_2a_4) = 2$. Since $x_1 \not\rightarrow (C; x_2x_t)$, $x_2 \not\rightarrow (C; x_1x_t)$ and $e(x_1x_2, C) = 4$, we see that either (8^0) or (9^0) holds. Therefore we may assume that $e(x_p, a_ia_{i+2}) \neq 2$ for all $x_p \in B^*$ and $i \in \{1, 2\}$. Thus $e(x_p, C) = 2$ for all $x_p \in B^*$. As $x_p \not\rightarrow (C; B^* - \{x_p\})$ for all $x_p \in B^*$, it follows that (7^0) holds. This proves that one of (1^0) to (9^0) holds. To see the last statement of the lemma, we notice that $[B, C, P] \supseteq C_4 \uplus C_{\geq 4}$ as one of (3^0) to (9^0) holds. ∎

Lemma 2.7 (*Lemma* 2.8, [7]) *Let Q and R be two disjoint cycles in G such that $Q \cong C_4$, $R \cong C_5$, $e(Q, R) \geq 11$, and $\{Q, R\}$ is optimal. Suppose $[Q \cup R] \not\supseteq 2C_4$. Then there exist two labellings $Q = a_1a_2a_3a_4a_1$ and $R = x_1x_2x_3x_4x_5x_1$ such that $e(x_4x_5, Q) = 0$, $\{a_1, a_2, a_3\} \subseteq N(x_i)$ for each $i \in \{1, 2, 3\}$, and $a_2a_4 \in E$. Moreover, if $e(x_2, Q) = 4$ then $a_1a_3 \in E$.*

Lemma 2.8 (*Lemma* 2.6, [6]) *Let C be a 4-cycle and let P and R be two paths in G with $l(P) = l(R) = 1$. Suppose that C, P and R are disjoint and $e(P \cup R, C) \geq 9$. Then $[C, P, R] \supseteq C_4 \uplus P_4$.*

Lemma 2.9 *Let C be a 4-cycle and P a path of order 5 in G such that C and P are disjoint, $\{C, P\}$ is optimal, $e(C, P) \geq 11$ and $[C, P] \not\supseteq 2C_4$. Let u and v be the two distinct endvertices of P. If $e(uv, C) \geq 1$, then $[C, P]$ contains a 4-cycle C' such that $[C, P] - V(C')$ contains at least five edges.*

Proof. Say $C = a_1a_2a_3a_4a_1$, $P = x_1x_2x_3x_4x_5$ and $H = [C, P]$. Suppose that H does not contain a 4-cycle C' such that $[C, P] - V(C')$ contains at least five edges. We shall prove that $e(x_1x_5, C) = 0$. We divide the proof into the following two cases.

Case 1. $[x_i, x_{i+1}, a_j, a_{j+1}] \supseteq C_4$ for some $i \in \{1, 4\}$ and $j \in \{1, 2, 3, 4\}$.

Say $[x_1, x_2, a_1, a_2] \supseteq C_4$. Then $e([a_3, a_4, x_3, x_4, x_5]) \leq 4$. Thus $e(a_3a_4, x_3x_4x_5) \leq 1$ and so $e(x_3x_4x_5, C) \leq 7$. If we also had that $[x_1, x_2, a_3, a_4] \supseteq C_4$, then we would have that $e(x_3x_4x_5, a_1a_2) \leq 1$ and so $e(P, C) \leq 10$, a contradiction. Hence $[x_1, x_2, a_3, a_4] \not\supseteq C_4$ and so $e(x_1x_2, a_3a_4) \leq 2$. Thus $e(x_1x_2, C) \leq 6$. Therefore

$$e(x_3x_4x_5, a_1a_2) \geq 11 - e(x_3x_4x_5, a_3a_4) - e(x_1x_2, C) \geq 4.$$

Suppose that $N(x_3) \cap N(x_5) \cap \{a_1, a_2\} \neq \emptyset$. Say without loss of generality $\{a_1x_3, a_1x_5\} \subseteq E$. Then $a_1x_3x_4x_5a_1$ is a 4-cycle in H and so we must have that $e(x_1x_2, a_2a_3a_4) \leq 1$, and so $e(x_1x_2, C) \leq 3$. Thus $e(P, C) \leq 3 + 7 = 10$, a contradiction.

Therefore $N(x_3) \cap N(x_5) \cap \{a_1, a_2\} = \emptyset$, and so $e(x_3x_5, a_1a_2) \leq 2$. Thus $e(x_3x_4x_5, C) \leq 5$. As $e(x_1x_2, C) \leq 6$, it follows that $e(x_3x_4x_5, a_3a_4) = 1$, $e(x_3x_5, a_1a_2) = 2$, $e(x_4, a_1a_2) = 2$, $e(x_1x_2, a_1a_2) = 4$ and $e(x_1x_2, a_3a_4) = 2$. Then we see that $e(x_1, a_3a_4) = 0$ for otherwise we readily see that $H - x_5 \supseteq 2C_4$. Thus $e(x_2, C) = 4$. Then $[x_1, x_2, a_1, a_4] \supseteq C_4$ and $[x_1, x_2, a_2, a_3] \supseteq C_4$. Consequently, $e(a_2a_3, x_3x_4x_5) \leq 1$ and $e(a_1a_4, x_3x_4x_5) \leq 1$. Thus $e(P, C) \leq 10$, a contradiction.

Case 2. $[x_i, x_{i+1}, a_j, a_{j+1}] \not\supseteq C_4$ for all $i \in \{1, 4\}$ and $j \in \{1, 2, 3, 4\}$.

This implies that $e(x_1x_2, C) \leq 4$ and $e(x_4x_5, C) \leq 4$. As $e(P, C) \geq 11$, $e(x_3, C) \geq 3$. First, suppose that there exists $i \in \{1, 5\}$, say $i = 1$, such that $N(x_1, C) \cap N(x_3, C) \neq \emptyset$. Say $\{a_1x_1, a_1x_3\} \subseteq E$. Then $e(x_4x_5, a_2a_3a_4) \leq 1$, and so $e(x_4x_5, C) \leq 3$. As $e(P, C) \geq 11$, we obtain that $e(x_3, C) = 4$, $e(x_1x_2, C) = 4$ and $e(x_4x_5, C) = 3$. Thus we also have that $\{a_1x_3, a_1x_5\} \subseteq E$. By the symmetry, we see that $e(x_1x_2, C) = 3$, a contradiction.

Therefore $N(x_i, C) \cap N(x_3, C) = \emptyset$ for each $i \in \{1, 5\}$. Then $e(x_1, C) \leq 1$ and $e(x_5, C) \leq 1$ as $e(x_3, C) \geq 3$. If $e(x_1, C) = 1$, then $e(x_3, C) = 3$ and $e(x_2, C) \leq 2$ as $[x_1, x_2, a_i, a_{i+1}] \not\supseteq C_4$ for all $i \in \{1, 2, 3, 4\}$. Consequently, $e(x_4x_5, C) \geq 11 - 6 = 5$, a contradiction. Hence $e(x_1, C) = 0$, and similarly, $e(x_5, C) = 0$. ∎

Lemma 2.10 (*Lemma* 2.12, [7]) *Let Q and Z be two disjoint subgraphs in G such that $Q \cong C_4$ and $Z \cong F$. Let u be the vertex of Z with degree 3. Suppose that $e(Q, Z - u) \geq 9$, $\{Q, Z\}$ is optimal, and $[Q \cup Z]$ contains none of $2C_4$, $C_4 \uplus C_5$ and $C_4 \uplus B_5$. Then there exist two labellings $Q = a_1a_2a_3a_4a_1$ and $Z = x_3x_1x_2x_3x_4x_5$ such that $N(\{x_4, x_5\}, Q) \subseteq \{a_1, a_3\}$, $N(x_1, Q) \subseteq \{a_1, a_4, a_3\}$, $N(x_2, Q) \subseteq \{a_1, a_2, a_3\}$, $a_2a_4 \notin E$ and $e(x_3, Q) = 0$.*

Lemma 2.11 (*Lemma* 2.13, [7]) *Let Q_1, Q_2 and Z be disjoint subgraphs in G such that $Q_1 \cong C_4$, $Q_2 \cong C_4$, $Z \cong F$ and $\{Q_1, Q_2, Z\}$ is optimal. Let $Q_1 = a_1a_2a_3a_4a_1$ and $Z = x_3x_1x_2x_3x_4x_5$ be such that $e(Q_1, Z - x_3) \geq 9$, $N(x_1, Q_1) \subseteq \{a_1, a_4, a_3\}$, $N(x_2, Q_1) \subseteq \{a_1, a_2, a_3\}$, $N(\{x_4, x_5\}, Q_1) \subseteq \{a_1, a_3\}$, and $e(Q_2, Z + a_2 + a_4) \geq 15$. Suppose that $[Q_1 \cup Q_2 \cup Z]$ contains none of $3C_4$, $2C_4 \uplus C_5$ and $2C_4 \uplus B_5$. Then there exists a labelling $Q_2 = b_1b_2b_3b_4b_1$ such that $b_2b_4 \notin E$, $e(b_1, Z + a_2 + a_4) = e(b_3, Z + a_2 + a_4) = 7$, $e(b_4, Z + a_2 + a_4) = 0$, and $N(b_2, Z + a_2 + a_4) = \{a_i\}$ for some $i \in \{2, 4\}$ such that if $i = 2$ then $a_2x_2 \notin E$ and if $i = 4$ then $a_4x_1 \notin E$.*

3 Proof of the Main Theorem

Let G be a graph of order $n \geq 4k$ with $k \geq 2$ and $\delta(G) \geq 2k$. Suppose, for a contradiction, that G does not contain k disjoint feasible cycles and G is not isomorphic to F_9 or any graph in $\Sigma_k \cup \Gamma_k$. By the result of [8, Theorem B], $n \geq 4k+1$. Let r_0 be the largest integer such that $G \supseteq r_0C_4$. Let k_0 be the largest integer such that G contains k_0 disjoint feasible cycles with r_0 of them being 4-cycles. Then $k_0 < k$. A chain of G is a sequence $(L_1, \ldots, L_{k_0})$ of k_0 disjoint feasible cycles with r_0 of them being 4-cycles such that

$$\sum_{i=1}^{k_0} l(L_i) \text{ is minimal.} \tag{1}$$

For two chains $(L_1, \ldots, L_{k_0})$ and $(L'_1, \ldots, L'_{k_0})$ in G, we write $(L_1, \ldots, L_{k_0}) \prec (L'_1, \ldots, L'_{k_0})$ if there exists $j \in \{1, \ldots, k_0\}$ such that $l(L_i) = l(L'_i)$ for $i = 1, \ldots, j$ and $l(L_{j+1}) < l(L'_{j+1})$. We say that $(L_1, \ldots, L_{k_0})$ is a minimal chain if for any chain $(L'_1, \ldots, L'_{k_0})$, $(L'_1, \ldots, L'_{k_0}) \not\prec (L_1, \ldots, L_{k_0})$. Clearly, if $(L_1, \ldots, L_{k_0})$ is a minimal chain then $l(L_i) = 4$ for $i = 1, \ldots, r_0$ and $5 \leq l(L_{r+1}) \leq \cdots \leq l(L_{k_0})$. We shall prove the following three lemmas.

Lemma 3.1 *If $\sigma' = (J_1, \ldots, J_{k_0})$ is a minimal chain and x and y are two distinct vertices of $G - V(\sigma')$ with $e(xy, G - V(\sigma')) \leq 3$, then for some $i \in \{1, \ldots, k_0\}$, $e(xy, J_i) \geq 5$, $l(J_i) = 4$ and $[J_i, x, y]$ contains a 4-cycle J'_i and a path $x'y'$ of order 2 such that $V(J'_i) \cap \{x', y'\} = \emptyset$ and $|\{x, y\} \cap \{x', y'\}| = 1$.*

Proof. Clearly, $e(xy, \cup_{i=1}^{k_0} J_i) \geq 4k - 3 \geq 4k_0 + 1$. Thus $e(xy, J_i) \geq 5$ for some $i \in \{1, \ldots, k_0\}$. By (1) and Lemma 2.1(a), $l(J_i) = 4$. Then this lemma follows from Lemma 2.3. ∎

Lemma 3.2 *Suppose that G does not contain a minimal chain σ with $G - V(\sigma) \supseteq P_5$. Let $\sigma' = (J_1, \ldots, J_{k_0})$ be a minimal chain in G such that $G - V(\sigma') \supseteq P_2 \uplus P_3 = \{y_1y_2, z_1z_2z_3\}$ and $e(y_1y_2z_1z_3, \cup_{i=1}^{k_0} J_i) \geq 8k_0 + 1$. Then there exists a minimal chain σ'' such that $\tau(\sigma'') > \tau(\sigma')$ and either $G - V(\sigma'') \supseteq P_2 \uplus P_3$ or $G - V(\sigma'') \supseteq P_4$.*

Proof. Say $S = \{y_1, y_2, z_1, z_3\}$. Then $e(S, J_i) \geq 9$ for some $i \in \{1, \ldots, k_0\}$. By (1) and Lemma 2.1(a), we see that $l(J_i) = 4$. Say without loss of generality $J_i =$

$J_1 = b_1b_2b_3b_4b_1$. Let $G' = [V(J_1) \cup \{y_1, y_2, z_1, z_2, z_3\}]$. Then $G' \not\supseteq C_4 \uplus P_5$. By the minimality of k_0, $G' \not\supseteq C_4 \uplus C_{\geq 4}$.

To deduce a contradiction, we first assume that $[y_1, y_2, b_i, b_{i+1}] \supseteq C_4$ for some $i \in \{1, 2, 3, 4\}$. Say without loss of generality $[y_1, y_2, b_1, b_2] \supseteq C_4$. Then $e(b_3b_4, z_1z_3) = 0$ as $G' \not\supseteq C_4 \uplus P_5$. Thus $e(z_1z_3, J_1) \leq 4$ and $e(y_1y_2, J_1) \geq 5$. Say without loss of generality that $e(y_1, J_1) \geq 3$ and $\{b_1, b_3\} \subseteq N(y_1)$. Then $b_2 \notin N(z_1) \cap N(z_3)$ for otherwise $G' \supseteq 2C_4$. Thus $e(z_1z_3, J_1) \leq 3$ and so $e(y_1y_2, J_1) \geq 6$. This implies that there exist two distinct edges b_jb_{j+1} and b_lb_{l+1} of J_1 such that $\{y_1b_j, y_2b_{j+1}, y_1b_l, y_2b_{l+1}\} \subseteq E$. Thus $[y_1, y_2, b_j, b_{j+1}] \supseteq C_4$ and $[y_1, y_2, b_l, b_{l+1}] \supseteq C_4$. Therefore $e(z_1z_3, b_{j+2}b_{j+3}) = e(z_1z_3, b_{l+2}b_{l+3}) = 0$. Hence $e(z_1z_3, J_1) \leq 2$. Consequently, $e(y_1y_2, J_1) \geq 7$. Then $[y_1, y_2, b_i, b_{i+1}] \supseteq C_4$ for all $i \in \{1, 2, 3, 4\}$ and it follows that $G' \supseteq C_4 \uplus P_5$, a contradiction.

Next, assume that $[y_1, y_2, b_i, b_{i+1}] \not\supseteq C_4$ for all $i \in \{1, 2, 3, 4\}$. Then $e(y_1y_2, J_1) \leq 4$, and so $e(z_1z_3, J_1) \geq 5$. Say without loss of generality $b_1 \in N(z_1) \cap N(z_3)$. Then $e(y_1y_2, b_2b_4) = 0$ as $G' \not\supseteq C_4 \uplus P_5$. If $b_i \in N(z_1) \cap N(z_3)$ for some $i \in \{2, 4\}$, then we would also have that $e(y_1y_2, b_1b_3) = 0$ and so $e(S, J_1) \leq 8$, a contradiction. Hence $N(z_1) \cap N(z_3) \cap \{b_2, b_4\} = \emptyset$. It follows that $\{b_2, b_4\} \not\subseteq N(z_i)$ for each $i \in \{1, 3\}$ for otherwise $e(b_1, y_1y_2) = 0$ as $G' \not\supseteq C_4 \uplus P_5$, and consequently $e(z_1z_3, J_1) \geq 7$ and therefore $N(z_1) \cap N(z_3) \cap \{b_2, b_4\} \neq \emptyset$, a contradiction. Therefore we may assume without loss of generality that $N(z_1, J_1) = \{b_1, b_4, b_3\}$ and $N(z_3, J_1) \subseteq \{b_1, b_2, b_3\}$. Then $e(y_1y_2, b_1b_3) \geq 3$ and $e(b_1b_3, S) \geq 7$. Say without loss of generality $e(b_1, y_1z_3) = 2$. If $b_2b_4 \in E$ then $G' \supseteq C_4 \uplus P_5 = \{z_1b_4b_2b_3z_1, z_2z_3b_1y_1y_2\}$, a contradiction. Hence $b_2b_4 \notin E$. If $b_2z_3 \in E$ then $G' \supseteq C_4 \uplus P_2 \uplus P_3 = \{z_1b_1b_4b_3z_1, y_1y_2, z_2z_3b_2\}$ with $\tau(z_1b_1b_4b_3z_1) = \tau(J_1)+1$ and so the lemma holds. So assume $b_2z_3 \notin E$. As $e(S, J_1) \geq 9$, it follows that $e(u, b_1b_3) = 2$ for all $u \in \{y_1, y_2, z_3\}$. Let $J_1' = y_1b_1y_2b_3y_1$ and $P' = b_4z_1z_2z_3$. Then $\sigma'' = \{J_1', J_2, \ldots, J_{k_0}\}$ is a minimal chain and $G - V(\sigma'') \supseteq P'$. Clearly, $\tau(J_1') = \tau(J_1) + 1$ and so the lemma holds. ∎

Lemma 3.3 *If σ is a minimal chain in G then $n - |V(\sigma)| \geq 5$.*

Proof. On the contrary, say $n - |V(\sigma)| \leq 4$. We choose σ among all the minimal chains such that $\tau(\sigma)$ is maximal. Say $\sigma = (L_1, \ldots, L_{k_0})$. As $n \geq 4k + 1$, $l(L_{k_0}) \geq 5$. By the minimality of σ, $[L_{k_0}]$ does not contain a p-cycle with $4 \leq p < l(L_{k_0})$ and so $\tau(L_{k_0}) = 0$. Say $H = \cup_{i=1}^{k_0} L_i$ and $D = G - V(H)$. Let $L_{k_0} = x_1x_2 \ldots x_tx_1$. By the minimality of σ and Lemma 2.1(a), $e(y, L_{k_0}) \leq 2$ for all $y \in V(D)$. Thus $e(L_{k_0}, H - V(L_{k_0})) \geq 2tk - 2t - 2|V(D)| \geq 2t(k-2) + 2$. This implies that $e(L_{k_0}, L_i) \geq 2t + 1$ for some $1 \leq i \leq k_0 - 1$. By (1) and Lemma 2.5, $l(L_i) = 4$ and $t = 5$. Thus $e(L_{k_0}, H - V(L_{k_0})) \geq 10(k - 2) + 2$.

Suppose that $|V(D)| \geq 1$. Then $e(u, D) \leq 2$ for some $u \in V(D)$ since $D \not\supseteq C_{\geq 4}$. Thus $e(u, D \cup L_{k_0}) \leq 4$ and so $e(L_{k_0} + u, H - V(L_{k_0})) \geq 10(k - 2) + 2 + 2k - 4 = 12(k - 2) + 2$. This implies that $e(L_{k_0} + u, L_j) \geq 13$ for some $1 \leq j \leq k_0 - 1$. If $e(u, L_j) \geq 3$ then $l(L_j) = 4$ by Lemma 2.1(a). By the maximality of $\tau(\sigma)$ and Lemma 2.2, $u \to L_j$. Hence $e(v, L_{k_0}) \leq 2$ for all $v \in V(L_j)$ by (1) and Lemma 2.1(a). Thus $e(L_{k_0} + u, L_j) \leq 12 < 13$, a contradiction. Hence $e(u, L_j) \leq 2$ and so $e(L_{k_0}, L_j) \geq 11$.

Again, by (1) and Lemma 2.5, $l(L_j) = 4$. Say without loss of generality $L_j = L_1$. By Lemma 2.7, there exist two labellings $L_1 = a_1a_2a_3a_4a_1$ and $L_{k_0} = x_1x_2x_3x_4x_5x_1$ such that $\{a_1, a_2, a_3\} \subseteq N(x_i)$ for $1 \le i \le 3$, $e(x_4x_5, L_1) = 0$ and $a_2a_4 \in E$. Moreover, if $e(x_2, L_1) = 4$ then $a_1a_3 \in E$. If $e(u, L_1) \ge 2$ then $u \to (L_1, a_i)$ for some $a_i \in N(x_1, L_1) \cap N(x_3, L_1)$ and consequently, $[u, L_1, x_1, x_2, x_3] \supseteq 2C_4$, contradicting the minimality of σ. Hence $e(u, L_1) = 1$, $e(x_1x_2x_3, L_1) = 12$ and $a_1a_3 \in E$. As $e(L_{k_0} + u, H - V(L_{k_0})) \ge 12(k-2) + 2$, it follows that $e(L_{k_0} + u, L_s) \ge 13$ for some L_s in $H - V(L_1 \cup L_{k_0})$. By the above argument, we may assume that $s = 2$ with $l(L_2) = 4$, $e(x_lx_{l+1}x_{l+2}, L_2) = 12$ for some $l \in \{1, 2, 3\}$, $e(u, L_2) = 1$ and $\tau(L_2) = 2$. Say $L_2 = b_1b_2b_3b_4b_1$. Say without loss of generality $\{a_1, b_1\} \subseteq N(u)$. Then $[x_l, a_1, u, b_1] \supseteq C_4$ and it clearly follows that $[u, L_1, L_2, x_1, x_2, x_3] \supseteq 3C_4$, contradicting the minimality of σ.

Therefore $V(D) = \emptyset$. As $n \ge 4k+1$, we see that $l(L_{k_0-1}) = 5$. As in the first paragraph, we have that $e(L_{k_0-1}, H - V(L_{k_0-1})) \ge 10(k-2)+2$. By the minimality of σ and Lemma 2.5, $e(L_{k_0-1}, L_{k_0}) \le 10$. Thus $e(L_{k_0-1} \cup L_{k_0}, H - V(L_{k_0-1} \cup L_{k_0})) \ge 20(k-3)+4$. Then $e(L_{k_0-1} \cup L_{k_0}, L_h) \ge 21$ for some L_h in $H - V(L_{k_0-1} \cup L_{k_0})$. Say without loss of generality $e(L_{k_0}, L_h) \ge 11$. As above, we shall have $l(L_h) = 4$. Say $L_h = L_1$. Then there exist two labellings $L_1 = a_1a_2a_3a_4a_1$ and $L_{k_0} = x_1x_2x_3x_4x_5x_1$ such that $\{a_1, a_2, a_3\} \subseteq N(x_i)$ for $1 \le i \le 3$, $e(x_4x_5, L_1) = 0$ and $a_2a_4 \in E$. Moreover, if $e(x_2, L_1) = 4$ then $a_1a_3 \in E$. As $e(L_{k_0-1}, L_1) \ge 21 - 12 = 9$, $e(u, L_1) \ge 2$ for some $u \in V(L_{k_0-1})$. Thus $u \to (L_1, a_j)$ for some $a_j \in N(x_1, L_1) \cap N(x_3, L_1)$ and so $[u, L_1, x_1, x_2, x_3] \supseteq 2C_4$, contradicting the minimality of σ. ∎

We shall apply the above lemmas to prove the following three claims concerning minimal chains.

Claim 1. There exists a minimal chain σ such that $G - V(\sigma)$ has a path of order at least 5.

Proof of Claim 1. On the contrary, suppose that the claim fails. Let t be the largest integer such that $G - V(\sigma)$ contains a path of order t for a minimal chain σ. Then $t \le 4$. Say $\sigma = (L_1, \ldots, L_{k_0})$.

By Lemma 3.1 and Lemma 3.3, we see that $t \ge 2$ and moreover, if $t = 2$, we get a minimal chain σ' such that $G - V(\sigma') \supseteq 2P_2 = \{xy, uv\}$. Then $e(xu, G - V(\sigma')) = 2$. By Lemma 3.1, there exists a 4-cycle J_i in σ' such that $[J_i, x, u]$ contains a 4-cycle J_i' and a path $x'u'$ of order 2 such that $V(J_i') \cap \{x', u'\} = \emptyset$ and $|\{x', u'\} \cap \{x, u\}| = 1$. Thus $[J_i, xy, uv] \supseteq C_4 \uplus P_3$, a contradiction. Hence $t \ge 3$. This argument allows us to see that if $t = 4$ then for any minimal chain σ', $G - V(\sigma') \not\supseteq 2P_4$. To observe this, say $G - V(\sigma') \supseteq 2P_4 = \{R_1, R_2\}$ for a given minimal chain σ'. As $G - V(\sigma')$ does not have a feasible cycle, there exists an endvertex z_i of R_i such that $e(z_i, G - V(\sigma')) = 1$ for $i = 1, 2$. Thus $e(z_1z_2, G - V(\sigma')) < 3$. By Lemma 3.1, we see that there exists a 4-cycle J_i in σ' such that $[J_i, R_1, R_2] \supseteq C_4 \uplus P_5$, a contradiction. Similarly, if $t = 3$ then $G - V(\sigma') \supseteq 2C_3$ for any minimal chain σ' with $G - V(\sigma') \supseteq 2P_3$. To observe this, say $G - V(\sigma') \supseteq 2P_3 = \{x_1x_2x_3, y_1y_2y_3\}$. As $t = 3$ and by Lemma 3.1, we see that $e(x_iy_j, G - V(\sigma')) \ge 4$ for all $\{i, j\} \subseteq \{1, 3\}$. It follows that $x_1x_3 \in E$ and

$y_1y_3 \in E$, i.e., $G - V(\sigma') \supseteq 2C_3$.

We now let σ be chosen with $\tau(\sigma)$ maximal such that $G - V(\sigma)$ has a path of order t. Say $H = [V(\sigma)]$ and $D = G - V(H)$. Let P be a path of order t in D. We show the following Property A.

Property A.
There exists a minimal chain σ' such that $\tau(\sigma') \geq \tau(\sigma)$ and $G - V(\sigma') \supseteq 2C_3$.

Proof of Property A. We divide the proof into the following two cases: $t = 4$ or $t = 3$.

Case 1. $t = 4$.

Say $P = x_1x_2x_3x_4$. We claim that $e(P, D - V(P)) = 0$. If this is false, say without loss of generality $x_iu \in E$ for some $u \in V(D) - V(P)$ and $i \in \{2, 3\}$. Say without loss of generality $x_2u \in E$. Then $e(x_1, D) = 1$ and $e(u, D) = 1$ since D does not have a path of order at least 5. Thus $e(x_1u, D) = 2 < 3$. By Lemma 3.1, for some 4-cycle L_i in H, $[L_i, x_1, u] \supseteq C_4 \uplus P_2$ and consequently, $[L_i, P, u] \supseteq C_4 \uplus P_5$, a contradiction. Hence $e(P, D - V(P)) = 0$.

As D does not have a feasible cycle, $e(x_i, D) = 1$ for some $i \in \{1, 4\}$ and $e(u, D) \leq 2$ for some $u \in V(D) - V(P)$. Say without loss of generality $e(x_4, D) = 1$. Then $e(x_4u, D) \leq 3$. By Lemma 3.1, there exists a 4-cycle L_i in H, say $L_i = L_1$, such that $e(x_4u, L_1) \geq 5$. By the maximality of P, $u \not\to (L_1, v)$ for all $v \in V(L_1)$ with $vx_4 \in E$. By Lemma 2.3, $x_4 \Rightarrow (L_1, v)$ for some $v \in V(L_1)$ with $vu \in E$. Thus $[D - x_4 + v] \supseteq P_3 \uplus P_2$. Therefore if $\sigma' = (J_1, \ldots, J_{k_0})$ is a minimal chain with $\tau(\sigma')$ maximal such that $G - V(\sigma') \supseteq P_3 \uplus P_2$, then $\tau(\sigma') \geq \tau(\sigma)$. Set $H' = [V(\sigma')]$ and $D' = G - V(H')$.

Suppose that $D' \supseteq P_4 \uplus P_2 = \{P', R\}$ with $P' = z_1z_2z_3z_4$ and $R = y_1y_2$. Then $\tau(\sigma') = \tau(\sigma)$ by the maximality of $\tau(\sigma)$. As above, $e(P', D' - V(P')) = 0$. As stated in the second paragraph above *Property A*, $D' \not\supseteq 2P_4$ and so $D' - V(P') \not\supseteq P_4$. As D' does not contain a feasible cycle, $e(z_1, P') = 1$ or $e(z_4, P') = 1$. Say without loss of generality $e(z_1, P') = 1$. By the maximality of $\tau(\sigma')$ and Lemma 3.2, $e(z_1z_3y_1y_2, H') \leq 8k_0$ and so $e(z_1z_3y_1y_2, D') \geq 8(k - k_0) \geq 8$. Hence $e(y_1y_2, D' - V(P')) \geq 5$. Say without loss of generality $e(y_2, D' - V(P')) \geq 3$. Say y_1, u_1 and u_2 are three distinct neighbors of y_2 in $D' - V(P')$. Then $e(u_1u_2, D') = 2$ as $D' \not\supseteq 2P_4$. By Lemma 3.1, for some 4-cycle J_i in H', $[J_i, u_1, u_2] \supseteq C_4 \uplus P_2$ and consequently, $[J_i, y_1, y_2, u_1, u_2] \supseteq C_4 \uplus P_4$. Thus $[J_i, D'] \supseteq C_4 \uplus 2P_4$, a contradiction. Hence $D' \not\supseteq P_4 \uplus P_2$.

As $D' \supseteq P_3 \uplus P_2$, let $z_1z_2z_3$ and y_1y_2 be two disjoint paths in D'. Then $e(z_1z_3, D' - \{z_1, z_2, z_3\}) = 0$. As above, we shall have $e(z_1z_3y_1y_2, D') \geq 8$. If $e(y_i, D' - \{z_1, z_2, z_3\}) \geq 3$ for some $i \in \{1, 2\}$, say without loss of generality $i = 1$, let $\{y_2, y_3, y_4\} \subseteq N(y_1, D' - \{z_1, z_2, z_3\})$ with $|\{y_2, y_3, y_4\}| = 3$. Since $D' \not\supseteq P_4 \uplus P_2$ and $D' \not\supseteq P_5$, $e(y_2y_3y_4, D') = 3$ and so $e(y_2y_3y_4, H') \geq 6k - 3 = 6(k - 1) + 3$. Thus $e(y_2y_3y_4, J_i) \geq 7$ for some cycle in H'. By (1) and Lemma 2.1, we see that $l(J_i) = 4$. As $[y_1, y_2, y_3, y_4, J_i] \not\supseteq 2C_4$ and by Lemma 2.4, $[J_i, y_1, y_2, y_3, y_4] \supseteq J_i' \uplus P'$ such that $J_i' \cong C_4$, $\tau(J_i') \geq \tau(J_i)$ and $P' \cong P_4$. With J_i' replacing J_i in σ', we obtain a minimal chain σ'' such that $\tau(\sigma'') \geq \tau(\sigma')$ and $G - V(\sigma'') \supseteq P_4 \uplus P_2 = \{P', z_1z_2\}$. Then we obtain a contradiction by the argument in the previous paragraph with σ'' in place of σ'. Hence $e(y_i, D' - \{z_1, z_2, z_3\}) \leq 2$ for $i \in \{1, 2\}$. As $e(z_1z_3y_1y_2, D') \geq 8$, it follows that $e(z_1z_3, D') = 4$ and so $z_1z_3 \in E$. Thus $e(z_2, D - \{z_1, z_2, z_3\}) = 0$ as $D' \not\supseteq P_5$ and $D' \not\supseteq P_4 \uplus P_2$. It follows that $e(y_i, D' - \{z_1, z_2, z_3\}) = 2$ for $i = 1, 2$. Say without loss of generality $y_3y_2 \in E$ with $y_3 \in V(D') - \{z_1, z_2, z_3, y_1\}$. Then $y_1y_3 \in E$ as $D' \not\supseteq P_4 \uplus P_2$ and $D' \not\supseteq C_5$. Thus $D' \supseteq 2C_3$.

Case 2. $t = 3$.

Say $P = x_1x_2x_3$. As above, we readily see that $e(P, D - V(P)) = 0$ for otherwise G has a minimal chain σ' such that $G - V(\sigma') \supseteq P_4$. If $D - V(P)$ contains an edge, then by the argument in the paragraph right above Case 2 with σ in place of σ', we obtain that $D \supseteq 2C_3$. Hence assume that $D - V(P)$ does not have an edge. Let $u \in V(D) - V(P)$. Then $e(x_3u, D) \leq 3$. By Lemma 3.1, there exists a 4-cycle L_i in H, say $L_i = L_1$, such that $e(x_3u, L_1) \geq 5$. As $t = 3$, $u \not\to (L_1, v)$ for all $v \in N(x_3, L_1)$. By Lemma 2.3, there exists $v \in N(u, L_1)$ such that $x_3 \Rightarrow (L_1, v)$. Let $[L_1 - v + x_3] \supseteq L_1' \cong C_4$. Then $\sigma' = (L_1', L_2, \ldots, L_{k_0})$ is a minimal chain with $\tau(\sigma') \geq \tau(\sigma)$ such that $G - V(\sigma') \supseteq 2P_2 = \{x_1x_2, uv\}$. As above we may assume that $G - V(\sigma') \not\supseteq P_3 \uplus P_2$ for otherwise $G - V(\sigma') \supseteq 2C_3$. Thus $e(x_1x_2uv, V(\sigma')) \geq 8k - 4 \geq 8k_0 + 4$. Hence $e(x_1x_2uv, C) \geq 9$ for some cycle C in σ'. By Lemma 2.1(a), $l(C) = 4$. By Lemma 2.8, $[C, x_1x_2, uv] \supseteq C_4 \uplus P_4$, a contradiction. Q.E.D.

We now let $\sigma' = (J_1, \ldots, J_{k_0})$ be a minimal chain with $\tau(\sigma')$ maximal such that $G - V(\sigma') \supseteq 2C_3$. Then $\tau(\sigma') \geq \tau(\sigma)$. Say $H' = [V(\sigma')]$ and $D' = G - V(H')$. Let $T_1 = x_1x_2x_3x_1$ and $T_2 = y_1y_2y_3y_1$ be two disjoint triangles of D'. We shall show $G \in \Gamma_k$. First, suppose that $e(y_0, T_1 \cup T_2) \geq 1$ for some $y_0 \in V(D') - V(T_1 \cup T_2)$. As $D' \not\supseteq C_{\geq 4}$ and $D' \not\supseteq P_5$, we see that $e(y_0, T_1 \cup T_2) = 1$. Say without loss of generality $y_0y_1 \in E$. Then $t = 4$. Thus $e(y_0y_2y_3, D' - V(T_2 + y_0)) = 0$. As $D' \not\supseteq 2P_4$ and $D' \not\supseteq P_5$, $e(T_1, D' - V(T_1)) = 0$. Say $S = V(T_1) \cup \{y_0, y_2, y_3\}$. Then $e(S, D') = 11$ and so $e(S, H') \geq 12k - 11 \geq 12k_0 + 1$. Thus $e(S, J_i) \geq 13$ for some J_i in H'. Then $e(u, J_i) \geq 3$ for some $u \in S$. By the minimality of σ' and Lemma 2.1(a), $l(J_i) = 4$. Hence $e(z, S) \geq 4$ for some $z \in V(J_i)$. Thus $[S - \{u\}, y_1, z] \supseteq C_4$ or P_5. Therefore $u \not\to (J_i, z)$. If $u \in V(T_1)$ then $D' - u \supseteq P_4$. As $\tau(\sigma') \geq \tau(\sigma)$ and by the maximality of $\tau(\sigma)$, $u \overset{na}{\to} (J_i, v)$ for all $v \in V(J_i)$. By Lemma 2.2, $u \to J_i$, a contradiction. Hence $u = y_a$ for some $a \in \{0, 2, 3\}$. Say $\{a, b, c\} = \{0, 2, 3\}$. Say $J_i = d_1d_2d_3d_4d_1$ with $e(y_a, d_1d_2d_3) = 3$. As $y_a \not\to J_i$, $d_2d_4 \notin E$ and $y_ad_4 \notin E$. As $y_a \overset{a}{\to} (J_i, d_4)$, $[D' - y_a, d_4] \not\supseteq P_4$ by the maximality of $\tau(\sigma)$. Thus $e(d_4, S) = 0$. As $y_a \Rightarrow (J_i, d_2)$, $[D' - y_a + d_2] \not\supseteq P_5$ by the maximality of t. Moreover, $[D' - y_a + d_2]$ does not contain a feasible cycle. It follows that $e(d_2, S - \{y_a\}) \leq 1$. Hence $e(d_1d_3, S - \{y_a\}) \geq 13 - 3 - 1 = 9$. Say without loss of generality $e(d_1, S - \{y_a\}) = 5$. Then $T_1 + d_1 \supseteq C_4$ and $[J_i - d_1, T_2 + y_0] \supseteq P_5$, a contradiction.

Therefore $e(T_1 \cup T_2, D' - V(T_1 \cup T_2)) = 0$. As $D' \not\supseteq P_5$, $e(T_1, T_2) = 0$. Then $e(T_1 \cup T_2, H') \geq 12k - 12 = 12k_0$. Let $i \in \{1, \ldots, k_0\}$ be such that $e(T_1 \cup T_2, J_i) \geq 12$. We claim that $l(J_i) = 4$, $\tau(J_i) = 2$, $e(T_p, J_i) = 12$ and $e(T_q, J_i) = 0$ for some $\{p, q\} = \{1, 2\}$. Suppose that $l(J_i) \geq 5$. By the minimality of σ' and Lemma 2.1(a), we see that $e(u, J_i) = 2$ and the two vertices of $N(u, J_i)$ are consecutive on J_i for each $u \in V(T_1 \cup T_2)$. Then we readily see that $[T_1, J_i]$ contains a feasible cycle C with $l(C) < l(J_i)$, a contradiction. Therefore $l(J_i) = 4$. Then there exists $d \in V(J_i)$ such that $e(d, T_1 \cup T_2) \geq 3$. Say without loss of generality $J_i = d_1d_2d_3d_4d_1$ with $e(d_1, T_1) \geq 2$. Then $[T_1 + d_1] \supseteq C_4$. Thus $[T_2, d_2, d_3, d_4] \not\supseteq P_5$ and so $e(T_2, d_2d_3d_4) = 0$. Thus $e(T_1, J_i) \geq 9$. Then $e(d_j, T_1) \geq 2$ for some $j \in \{2, 3, 4\}$. Thus $e(d_1, T_2) = 0$. It follows that $e(T_1, J_i) = 12$ and $e(T_2, J_i) = 0$. As $[T_1 \cup J_i] \supseteq K_4 \uplus C_3$, we have $\tau(J_i) = 2$ by the maximality of $\tau(\sigma')$. Therefore the claim holds. As $e(T_1 \cup T_2, H') \geq 12k - 12$, it follows, from this argument, that $k_0 = k - 1$, $l(J_i) = 4$, $\tau(J_i) = 2$ and $e(T_1 \cup T_2, J_i) = 12$ for all $i \in \{1, \ldots, k-1\}$. Let $S_p = \{J_i | e(T_p, J_i) = 12, 1 \leq i \leq k-1\}$ for $p = 1, 2$. Then $e(T_p, J_i) = 0$ for all $J_i \in S_q$ where $\{p, q\} = \{1, 2\}$. As $\delta(G) \geq 2k$, it follows that $|S_1| = |S_2|$. As $t \leq 4$, we see that $e(J_i, J_s) = 0$ for all $J_i \in S_1$ and $J_s \in S_2$. Furthermore, we see that if $D' \neq T_1 \cup T_2$ then $e(w, D') \leq 2$ for some $w \in V(D') - V(T_1 \cup T_2)$ because $D' \not\supseteq C_{\geq 4}$. Then $e(w, H') \geq 2(k-1)$ and consequently, $[H', T_1, T_2, w] \supseteq (k-1)C_4 \uplus P_5$, a contradiction. Hence $D' = T_1 \cup T_2$. As $\delta(G) \geq 2k$, it follows that $[\cup_{J_i \in S_1}, T_1] \cong [\cup_{J_i \in S_2}, T_2] \cong K_{4l+3}$ where $2l = k - 1$. Thus $G \in \Gamma_k$. ∎

Claim 2. There exists a minimal chain σ such that $G - V(\sigma)$ has a subgraph D' of order at least 5 with at least $|V(D')|$ edges.

Proof of Claim 2. Suppose that the claim fails. Let $\sigma = (L_1, \ldots, L_{k_0})$ be a minimal chain such that $\tau(\sigma)$ is maximal with $G - V(\sigma) \supseteq P_5$. Let $H = [V(\sigma)]$ and $D = G - V(H)$. Let $P = x_1x_2 \ldots x_t$ be a longest path of D. Then $t \geq 5$. Let D_0 be the component of D with $D_0 \supseteq P$. Then D_0 is a tree. Say $p = |V(D_0)|$, $P' = x_1x_2x_3x_4x_5$ and $R = V(D_0) - V(P')$. Then $\sum_{x \in V(D_0)} e(x, H) = 2pk - 2(p-1) = 2p(k-1) + 2$. Thus $e(D_0, L_i) \geq 2p+1$ for some L_i in H. By the minimality of σ and Lemma 2.1(a), we see that $l(L_i) = 4$. Say without loss of generality $L_i = L_1$. By the maximality of $\tau(\sigma)$ and Lemma 2.2, we see that $u \to L_1$ for each $u \in R$ with $e(u, L_1) \geq 3$. We may enumerate $R = \{u_1, \ldots, u_{p-5}\}$ such that $e(u_1, D_0) = 1$ and $e(u_j, D_0 - \{u_1, \ldots, u_{j-1}\}) = 1$ for $j = 2, \ldots, p-5$. If $e(R, L_1) \geq 2(p-5)+1$, let l be the smallest integer such that $e(u_l, L_1) \geq 3$. Then $e(D_0 - \{u_1, \ldots, u_{l-1}\}, L_i) \geq 2(p - l + 1) + 1$ and so $e(v, D_0 - \{u_1, \ldots, u_{l-1}\}) \geq 3$ for some $v \in V(L_1)$. Thus $[D_0 - \{u_1, \ldots, u_l\}, v]$ has at least $p - l + 1$ edges and so the claim holds since $u_l \to (L_1, v)$, a contradiction. Therefore $e(R, L_1) \leq 2(p-5)$. Thus $e(P', L_1) \geq 11$. By Lemma 2.9, we may assume that $e(x_1x_5, L_1) = 0$ and $e(x_2x_3x_4, L_1) \geq 11$. Clearly, $x_2 \to (L_1, y; x_3x_4)$ for some $y \in V(L_1)$. Thus if $t \geq 6$ then the claim holds. Hence assume that $t = 5$. Then $e(x_1x_5, H - V(L_1)) \geq 4k - 2 = 4(k-1) + 2$. Thus $e(x_1x_5, L_i) \geq 5$ for some L_i in $H - V(L_1)$. By Lemma 2.1(a), we have $l(L_i) = 4$. By Lemma 2.3, we may assume that L_i has a vertex w such that $x_1 \to (L_i, w)$ and $x_5w \in E$. Clearly, $[y, x_3, x_4, x_5, w]$ has at least five edges and so the claim holds, a contradiction. ∎

By Claim 2, G has a minimal chain σ such that $G - V(\sigma)$ has a subgraph G' of order 5 with $e(G') \geq 5$. As $G - V(\sigma) \not\supseteq C_{\geq 4}$, G' has a triangle. If $G - V(\sigma) \not\supseteq F$, then G' has two distinct vertices x and y such that $e(x, G - V(\sigma)) = 1$, $e(y, G - V(\sigma)) = 1$ and $xy \notin E$. Thus $e(xy, G - V(\sigma)) < 3$. By Lemma 3.1, We see that G has a minimal chain σ' such that $G - V(\sigma') \supseteq F$.

Claim 3. There exists a minimal chain σ such that $G - V(\sigma)$ has a subgraph D' with at least $|V(D')| + 1$ edges.

Proof of Claim 3. On the contrary, suppose that the claim fails. Let σ be a minimal chain with $G - V(\sigma) \supseteq F$ such that $\tau(\sigma)$ is maximal. Let $P = x_1x_2 \ldots x_t$ be a longest path in $G - V(\sigma)$ with $x_1x_3 \in E$. Subject to these properties, we may assume that σ and P are chosen with $l(P)$ maximal. Let D_0 be the component of D with $D_0 \supseteq P$. Say $|D_0| = p$ and $R = V(D_0) - \{x_1, x_2, x_3, x_4, x_5\}$. Since the claim is assumed false, the triangle $x_1x_2x_3x_1$ is the unique cycle of D_0. We may enumerate $R = \{u_1, u_2, \ldots, u_{p-5}\}$ such that $e(u_1, D_0) = 1$ and $e(u_i, D_0 - \{u_1, \ldots, u_{i-1}\}) = 1$ for $i = 2, \ldots, p-5$. Let $S = \{x_1, x_2, x_4, x_5\}$. Then $|S \cup R| = p - 1$ and $e(S \cup R, D_0) = 2p - e(x_3, D_0) \leq 2p - 3$. Thus $e(S \cup R, H) \geq 2(p-1)k - 2p + 3 = 2(p-1)(k-1) + 1$. This implies that $e(S \cup R, L_i) \geq 2(p-1) + 1$ for some L_i in σ. By the minimality of σ and Lemma 2.1(a), $l(L_i) = 4$. Say without loss of generality $L_i = L_1$. Assume $e(u, L_1) \geq 3$ for some $u \in R$. Let l be the smallest integer in $\{1, \ldots, p-5\}$ such that $e(u_l, L_1) \geq 3$. Then $e(v, S \cup R - \{u_1, \ldots, u_{l-1}\}) \geq 3$ for some $v \in V(L_1)$. Thus $[S \cup R - \{u_1, \ldots, u_l\}, v]$ is a subgraph of order $p - l$ with at least $p - l + 1$ edges. By the maximality of $\tau(\sigma)$ and Lemma 2.2, $u_l \to (L_1, v)$ and so the claim holds, a contradiction. Hence $e(u, L_1) \leq 2$ for all $u \in R$ and so $e(R, L_1) \leq 2(p-5)$. Thus $e(S, L_1) \geq 9$. By Lemma 2.10, let $L_1 = a_1a_2a_3a_4a_1$ be such that $N(x_4x_5, L_1) \subseteq \{a_1, a_3\}$, $N(x_1, L_1) \subseteq \{a_1, a_4, a_3\}$, $N(x_2, L_1) \subseteq \{a_1, a_2, a_3\}$, $e(x_3, L_1) = 0$ and $a_2a_4 \notin E$. Thus $e(S, L_1) \leq 10$. Since $e(S \cup R, L_1) \geq 2(p-1) + 1$, it follows that for each $u \in R$, $e(u, L_1) \geq 1$ and if $e(S, L_1) = 9$ then $e(u, L_1) = 2$.

First, we show that $t = 5$. If $t > 5$ then $e(x_t, L_1) \geq 1$. As $e(S, L_1) \geq 9$, $e(a_i, x_1x_2) = 2$ and $e(a_j, x_4x_5) = 2$ for some $\{i, j\} = \{1, 3\}$. Thus $e(x_t, a_2a_ja_4) = 0$ for otherwise $[P, L_1] \supseteq C_4 \uplus C_{\geq 4}$. Hence $N(x_t, L_1) = \{a_i\}$. Thus $e(S, L_1) = 10$. This yields $[P, L_1] \supseteq C_4 \uplus C_{\geq 4}$, a contradiction. Hence $t = 5$.

Next, we show that $e(a_2a_4, R) = 0$. On the contrary, say without loss of generality $a_2y \in E$ for some $y \in R$. Let N be a shortest path from y to a vertex z of P in D_0. Assume $a_2x_2 \in E$. Then $e(D_0 - x_5 + a_2) \geq p + 1$. Thus $x_5 \not\to (L_1, a_2)$. Hence $e(x_5, a_1a_3) = 1$ and so $e(a_1a_3, x_1x_2x_4) = 6$. If $z \neq x_4$, then $x_4 \to (L_1, a_2)$ and $e([x_1, x_2, x_3, N, a_2]) \geq |V([x_1, x_2, x_3, N, a_2])| + 1$, a contradiction. Hence $z = x_4$. Thus $x_1 \to (L_1, a_2)$ and $[x_2, x_3, x_4, N, a_2] \supseteq C_{\geq 4}$, a contradiction. Hence $a_2x_2 \notin E$. Then $e(a_1a_3, S) = 8$ and $a_4x_1 \in E$. Thus if $z \in \{x_3, x_4, x_5\}$ then $[a_1, a_4, x_1, x_2] \supseteq C_4$ and $[a_2, a_3, x_3, x_4, x_5, N] \supseteq C_{\geq 4}$. If $z \in \{x_1, x_2\}$ then $[N, x_1, x_2, a_1, a_2] \supseteq C_{\geq 5}$ and $[P - z, a_3] \supseteq C_4$, a contradiction. This shows that $e(a_2a_4, R) = 0$.

Let $D' = D - V(D_0)$ and $R' = N(a_2a_4, D')$. We claim that R' consists of isolated vertices of D. If this is not true, say $yz \in E$ with $\{y, z\} \subseteq R'$ and $e(y, a_2a_4) \geq 1$. Say without loss of generality $a_2y \in E$. If $a_2x_2 \in E$ then $x_1x_3x_2a_2yz$ is a longer path than P with $x_1x_2 \in E$ and $x_i \Rightarrow (L_1, a_2)$ for some $i \in \{4, 5\}$, contradicting the maximality of $l(P)$. If $a_2x_2 \notin E$, then $e(x_1x_2, a_1a_4) = 3$, $\tau(x_1x_2a_4a_1x_1) = 1 \geq \tau(L_1)$ and $x_4x_5a_3a_2yz$ is a path with $x_4a_3 \in E$, contradicting the maximality of $l(P)$. Therefore R' consists of isolated vertices of D. Clearly, $N(a_2, R') \cap N(a_4, R') = \emptyset$ for otherwise $[y, L_1, x_1x_2x_3] \supseteq 2C_4$ for each $y \in N(a_2, R') \cap N(a_4, R')$. If $e(y, L_1) \geq 3$

for some $y \in R'$ then $y \to L_1$ by Lemma 2.2, and consequently, $[y, L_1, P] \supseteq 2C_4$, a contradiction. Hence $e(y, L_1) \le 2$ for all $y \in R'$. Therefore $e(a_2a_4, R') \le |R'|$ and $e(R', L_1) \le 2|R'|$. Say $p' = |R'|$, $G_1 = [L_1, D]$ and $X = \{a_2, a_4\} \cup V(D_0) \cup R'$. Then $|X| = p + p' + 2$ and

$$\begin{aligned} e(X, G_1) &= e(a_2a_4, G_1) + e(R', G_1) + e(S \cup R, L_1) + e(S \cup R \cup \{x_3\}, D) \\ &\le (6 + p') + 2p' + (10 + 2(p - 5)) + 2p = 4p + 3p' + 6. \end{aligned}$$

Then

$$\begin{aligned} e(X, H - V(L_1)) &\ge 2(p + p' + 2)k - (4p + 3p' + 6) \\ &= 2(p + p' + 2)(k - 2) + p' + 2. \end{aligned}$$

This implies that $e(X, L_i) \ge 2(p + p' + 2) + 1$ for some L_i in $H - V(L_1)$. By the minimality of σ and Lemma 2.1(a), we see that $l(L_i) = 4$. Say without loss of generality $L_i = L_2$. We claim that $e(u, L_2) \le 2$ for each $u \in R \cup R'$. On the contrary, say $e(u, L_2) \ge 3$ for some $u \in R \cup R'$. If $u \in R$, we may assume that $u = u_l$ where l is the least integer in $\{1, \ldots, p - 5\}$ such that $e(u_l, L_2) \ge 3$. In this case, we let $X' = X - \{u_1, \ldots, u_{l-1}\}$. For the sake of convenience, if $u \in R'$, let $X' = X$. Clearly, $|X'| \ge 8$ and $e(X', L_2) \ge 2|X'| + 1$ in any case. By the maximality of $\tau(\sigma)$ and Lemma 2.2, $u \to L_2$. Let $y \in V(L_2)$ be such that $e(y, X') \ge e(w, X')$ for all $w \in V(L_2)$. As $e(X' - \{u\}, L_2) \ge 2|X'| + 1 - 4 \ge 13$, $e(y, X' - \{u\}) \ge 4$. If $e(y, a_2a_4) = 2$, then $u \to (L_2, y; a_2a_3a_4)$ and so $[u, L_2, L_1, P] \supseteq 3C_4$, a contradiction. If $e(y, X' - R' \cup \{a_2, a_4, u\}) \ge 2$ then $[y, X' - R' \cup \{a_2, a_4, u\}]$ has at least $|V([y, X' - R' \cup \{a_2, a_4, u\}])| + 1$ edges, a contradiction. Thus $e(y, R' - \{u\}) \ge e(y, X' - \{u\}) - 2 \ge 2$. Consequently, $[y, a_2a_3a_4, R' - \{u\}] \supseteq C_{\ge 4}$, $[a_1, P] \supseteq C_4$ and so $[G_1, L_2] \supseteq 2C_4 \uplus C_{\ge 4}$, a contradiction. Thus $e(u, L_2) \le 2$ for all $u \in R \cup R'$ and so $e(P + a_2 + a_4, L_2) \ge 15$.

By Lemma 2.11, there exists a labelling $L_2 = b_1b_2b_3b_4b_1$ such that $e(b_1, P + a_2 + a_4) = e(b_3, P + a_2 + a_4) = 7$, $e(b_4, P + a_2 + a_4) = 0$ and $b_2b_4 \notin E$. Moreover, we may assume without loss of generality that $N(b_2, P + a_2 + a_4) = \{a_2\}$ and $a_2x_2 \notin E$. Thus $e(S, L_1) = 9$ and $e(P + a_2 + a_4, L_2) = 15$. Let $J_1 = a_1x_4a_3x_5a_1$, $J_2 = b_1x_1x_2x_3b_1$, $M = b_4b_3a_2b_2b_3a_4$ and $G_2 = [P, L_1, L_2]$. Clearly, $\tau(J_1) > \tau(L_1)$ and $\tau(J_2) > \tau(L_2)$. Thus $[M, D - V(P)] \not\supseteq F$ by the maximality of $\tau(\sigma)$. Hence $e(a_4b_4, D - V(P)) = 0$. Then $e(a_4b_4, G_2) \le 9$. Consequently, $e(a_4b_4, H - V(L_1 \cup L_2)) \ge 4k - 9 \ge 4(k-3) + 3$. Thus $e(a_4b_4, L_i) \ge 5$ for some L_i in $H - V(L_1 \cup L_2)$. By Lemma 2.1(a), $l(L_i) = 4$. Say $L_i = L_3$. By Lemma 2.3, there exists a permutation (v, w) of $\{a_4, b_4\}$ such that L_3 has a vertex z such that $zv \in E$ and $L_3 - z + w \supseteq J_3 \cong C_4$. Clearly, $\tau(J_3) \ge \tau(L_3) - 1$. Then $[L_1, L_2, L_3, P] \supseteq 3C_4 \cup F = \{J_1, J_2, J_3, M - w + vz\}$ such that $\sum_{i=1}^{3} \tau(J_i) > \sum_{i=1}^{3} \tau(L_i)$, a contradiction. Thus the claim holds. ∎

By Claim 3, G has a minimal chain σ such that $G - V(\sigma)$ has a component of order s with at least $s + 1$ edges for some $s \ge 5$. As $G - V(\sigma) \not\supseteq C_{\ge 4}$, each cycle of $G - V(\sigma)$ is a triangle and so $G - V(\sigma)$ has two edge-disjoint triangles connected by a path, i.e., $G - V(\sigma) \supseteq B_t$ for some $t \ge 5$. We now choose σ with $\tau(\sigma)$ maximal such that $G - V(\sigma) \supseteq B_t$ for some $t \ge 5$. Subject to this requirement, we choose σ such that $G - V(\sigma) \supseteq B_t$ with t maximal. Let $H = [V(\sigma)]$

and $D = G - V(H)$. Let B be a subgraph of D with $B \cong B_t$. Let $x_1 \dots x_t$ be a path of B with $x_1x_3 \in E$ and $x_{t-2}x_t \in E$. Let D_0 be the component of D with $B \subseteq D_0$. Set $D_0' = D - V(D_0)$ and $S = \{x_1, x_2, x_{t-1}, x_t\}$. Let F_1 and F_2 be the two components of $D_0 - \{x_3, \dots, x_{t-2}\}$ such that $\{x_1, x_2\} \subseteq V(F_1)$ and $\{x_{t-1}, x_t\} \subseteq V(F_2)$. Say $R = V(F_1 \cup F_2) - \{x_1, x_2, x_{t-1}, x_t\}$ and $p = |R|$. As $D_0 \not\supseteq C_{\geq 4}$ and by the maximality of t, we see that both F_1 and F_2 are trees. Then $e(S \cup R, D) = e(S \cup R, S \cup R) + e(S, B - S) = 2(p + 4 - 2) + 4 = 2(p + 4)$. Thus $e(S \cup R, H) \geq 2(p + 4)k - 2(p + 4) = 2(p + 4)(k - 1)$. This implies that $e(S \cup R, L_i) \geq 2(p + 4)$ for some L_i in H. If $l(L_i) \geq 5$ then by Lemma 2.1(a), we see that $e(u, L_i) = 2$ for all $u \in S \cup R$ and so $L_i + x_1 + x_2$ contains a feasible cycle of order less than $l(L_i)$, a contradiction. Hence $l(L_i) = 4$. We may list $R = \{u_1, u_2, \dots, u_p\}$ such that $e(u_1, S \cup R) = 1$ and $e(u_i, S \cup R - \{u_1, \dots, u_{i-1}\}) = 1$ for all $i = 2, \dots, p$. Assume that $e(u, L_i) \geq 3$ for some $u \in R$. Let l be the least integer in $\{1, \dots, p\}$ such that $e(u_l, L_i) \geq 3$. Then $e(S \cup R - \{u_1, \dots, u_l\}, L_i) \geq 2(p+4) - 2(l-1) - 4 \geq 6$. Thus $e(y, S \cup R - \{u_1, \dots, u_l\}) \geq 2$ for some $y \in V(L_i)$. By Lemma 2.2, $u_l \to L_i$. Then $[B, R - \{u_1, \dots, u_l\}, y] \not\supseteq C_{\geq 4}$. This implies that either $e(y, F_1 - \{u_1, \dots, u_l\}) = 0$ or $e(y, F_2 - \{u_1, \dots, u_l\}) = 0$. Say without loss of generality $e(y, F_2 - \{u_1, \dots, u_l\}) = 0$. Thus $e(y, F_1 - \{u_1, \dots, u_l\}) \geq 2$. Clearly, $p > l$. This argument implies that $e(y, F_1 - \{u_1, \dots, u_l\}) = 2$. This argument shows that $e(L_i, S \cup R - \{u_1, \dots, u_l\}) \leq 8$. Moreover, we see that $[y, F_1 - \{u_1, \dots, u_l\}]$ contains a triangle with $e(y, x_1x_2) \leq 1$. Thus $[D_0 - u_l, y] \supseteq B_h$ for some $h > t$. By the maximality of t, we must have $u_l \not\Rightarrow (L_i, y)$. Thus $e(u_l, L_i) = 3$. Then $e(S \cup R - \{u_1, \dots, u_l\}, L_i) \geq 2(p+4) - 2(l-1) - 3 \geq 9$, a contradiction. This shows that $e(u, L_i) \leq 2$ for all $u \in R$. As $[B, L_i] \not\supseteq C_4 \uplus C_{\geq 4}$ and by Lemma 2.6, $e(S, L_i) \leq 8$. Thus $e(S, L_i) = 8$ and $e(u, L_i) = 2$ for all $u \in R$ since $e(S \cup R, L_i) \geq 12p + 8$ and $e(R, L_i) \leq 2p$.

As $e(S \cup R, H) \geq 2(p + 4)(k - 1)$, we conclude that $k_0 = k - 1$ and for all $i \in \{1, \dots, k - 1\}$, $l(L_i) = 4$, $e(S, L_i) = 8$ and $e(u, L_i) = 2$ for all $u \in R$. Moreover, $R = \emptyset$ for otherwise R has a vertex u with $e(u, D) = 1$ and so $e(u, G) < 2k$, a contradiction. If $e(v, L_i) \geq 3$ for some $v \in V(D) - V(B)$ and L_i in H, then $v \to L_i$ by Lemma 2.2 and so $[v, L_i, B] \supseteq C_4 \uplus C_{\geq 4}$, a contradiction. Hence $e(v, L_i) \leq 2$ for all $v \in V(D) - V(B)$ and L_i in H. Thus $e(v, D) \geq 2$ for all $v \in V(D) - V(B)$.

We claim that $D = D_0$. If this is not true, then $\delta(D_0') \geq 2$. As $D_0' \not\supseteq C_{\geq 4}$, each cycle of D_0' is a triangle and so D_0' has a path uvw of order 3 such that $e(u, D_0') = e(v, D_0') = 2$. Thus $e(u, L_i) = e(v, L_i) = 2$ for all L_i in H as $\delta(G) \geq 2k$. By Lemma 2.6, there exist two labellings $L_1 = a_1a_2a_3a_4a_1$ and $B = x_3x_1x_2x_3 \dots x_{t-2}x_{t-1}x_tx_{t-2}$ such that one of (1^0) to (9^0) holds with respect to L_1 and B. If $a_i \in N(u, L_1) \cap N(v, L_1)$ for some $i \in \{1, 2, 3, 4\}$ then $[u, v, w, a_i] \supseteq C_4$ and $[L_1 - a_i, B] \supseteq C_{\geq 4}$, a contradiction. Hence $N(u, L_1) \cap N(v, L_1) = \emptyset$. As $[u, v, L_1, B] \not\supseteq C_4 \uplus C_{\geq 4}$, $u \not\to (L_1; S)$ and $v \not\to (L_1; S)$. It follows that $e(u, a_ia_{i+1}) = 2$ and $e(v, a_{i+2}a_{i+3}) = 2$ for some $i \in \{1, 2, 3, 4\}$. Then we readily see that $[u, v, L_1, B] \supseteq C_4 \uplus C_{\geq 4}$, a contradiction. Hence $D = D_0$.

The following four properties will complete the proof of the main theorem. Property 1 is an important step to show that $G \in \sum_{k,n}$ or $G \in \Gamma_k$ or $G \cong F_9$. Properties 2–4 follow from Property 1.

Property 1. For each L_i in H, one of $(1^0), (2^0)$, (3^0) and (7^0) with $t = 5$ in Lemma 2.6 holds with respect to L_i and B, i.e., (2) or (3) or (4) holds below:

$$t = 5 \text{ and for some } \{u, v\} \subseteq V(L_i),\ N(x_j, L_i) = \{u, v\} \text{ for all } j \in \{1, 2, 4, 5\}; \tag{2}$$

$$t \geq 5 \text{ and for some } \{p, q\} = \{1, t-1\},\ e(x_p x_{p+1}, L_i) = 8 \text{ and } e(x_q x_{q+1}, L_i) = 0; \tag{3}$$

$$t = 5 \text{ and there exists a labelling } L_i = u_1 u_2 u_3 u_4 u_1 \text{ such that } N(x_1, L_i) = \{u_1, u_4\},\ N(x_2, L_i) = \{u_2, u_3\},\ N(x_4, L_i) = \{u_1, u_2\},\ N(x_5, L_i) = \{u_3, u_4\},\ \tau(L_i) = 0. \tag{4}$$

Proof of Property 1. On the contrary, suppose that the property fails for some L_i. For convenience, say $L_i = L_1 = a_1 a_2 a_3 a_4 a_1$. Then one of (4^0) to (9^0) in Lemma 2.6 holds with respect to L_1 and B. That is, we may assume that one of (5) to (10) holds in the following:

$$N(x_1, L_1) = \{a_1, a_2, a_3\}, N(x_2, L_1) = N(x_{t-1}, L_1) = \{a_1\}, N(x_t, L_1) = \{a_1, a_4, a_3\},\ a_1 a_3 \in E, a_2 a_4 \notin E; \tag{5}$$

$$N(x_1, L_1) = \{a_1, a_2, a_3\}, N(x_2, L_1) = \{a_1, a_3\}, e(x_{t-1}, L_1) = 0, N(x_t, L_1) = \{a_1, a_4, a_3\}, a_1 a_3 \in E, a_2 a_4 \notin E; \tag{6}$$

$$N(x_1, L_1) = \{a_1, a_4, a_3\}, N(x_2, L_1) = \{a_1, a_2, a_3\}, e(x_{t-1} x_t, a_2 a_4) = 0, e(a_1, x_{t-1} x_t) = 1, e(a_3, x_{t-1} x_t) = 1, a_2 a_4 \notin E; \tag{7}$$

$$N(x_1, L_1) = \{a_1, a_4\}, N(x_2, L_1) = \{a_2, a_3\}, N(x_{t-1}, L_1) = \{a_1, a_2\}, N(x_t, L_1) = \{a_3, a_4\}, \tau(L_1) = 0, t \geq 6; \tag{8}$$

$$N(x_1, L_1) = \{a_1, a_4\}, N(x_2, L_1) = \{a_2, a_3\}, e(x_{t-1}, L_1) = 0, e(x_t, L_1) = 4, \tau(L_1) = 0; \tag{9}$$

$$e(x_1, L_1) = 4, e(x_2 x_{t-1}, L_1) = 0, e(x_t, L_1) = 4. \tag{10}$$

First, assume that either $t \geq 7$ or $V(D) - V(B) \neq \emptyset$. As $D \not\supseteq C_{\geq 4}$, $e(v, D) \leq 2$ for some $v \in V(D) - \{x_1, x_2, x_3, x_{t-2}, x_{t-1}, x_t\}$. Let P be a path of D from v to a vertex u of $B - S$ with $V(P) \cap V(B) = \{u\}$. Then $e(v, L_j) \geq 2$ for some L_j in H. Then one of (1^0) to (9^0) in Lemma 2.6 holds with respect to L_j and B. By the last statement of Lemma 2.6, $t = 5$ and only one of (1^0) and (2^0) holds with respect to L_j and B. If $e(v, L_j) \geq 3$, then by the maximality of $\tau(\sigma)$ and Lemma 2.2, $v \rightarrow L_j$ and so $[L_j, B, v] \supseteq 2C_4$, a contradiction. Hence $e(v, L_j) = 2$. As $e(v, G) \geq 2k$, this argument implies that one of (1^0) and (2^0) holds with respect to L_j for each L_j in H and so Property 1 holds, a contradiction.

We conclude that $t \leq 6$ and $V(D) = V(B)$. Since $[L_1, B] \not\supseteq C_4 \uplus C_{\geq 4}$, it is easy to see that if one of (8),(9) and (10) holds then $e(x_3, L_1) = 0$ and if one of (5), (6) and (7) holds then $e(x_3, a_2a_3a_4) = 0$. Furthermore, we see that if $x_3a_1 \in E$, then $[a_1, B - x_1 - x_2] \supseteq C_{\geq 4}$ and so $[x_1, x_2, a_2, a_3, a_4] \not\supseteq C_4$. This implies that neither of (6) and (7) holds. Thus $e(x_3, L_1) \leq 1$ and if equality holds then (5) holds with $x_3a_1 \in E$. Similarly but simpler, we see that if $t = 6$ then $e(x_4, L_1) \leq 1$.

We claim that $t = 5$. If this is false, say $t = 6$. If $e(x_3, L_1) = 1$ then (5) holds with $x_3a_1 \in E$. Thus $[a_1, x_1, x_2, x_3] \supseteq K_4$ and $[a_3, a_4, x_6, x_5, x_4] \supseteq B_5$. By the maximality of $\tau(\sigma)$, we must have $\tau(L_1) = 2$, a contradiction. Therefore $e(x_3, L_1) = 0$. Thus $e(x_3x_4, H - V(L_1)) \geq 4k - 7 = 4(k - 2) + 1$. This implies that $e(x_3x_4, L_j) \geq 5$ for some L_j in $H - V(L_1)$. Then one of (3^0) to (9^0) in Lemma 2.6 holds with respect to L_j and B. Similarly, if one of (4^0) to (9^0) holds with respect to L_j and B, then $e(x_3, L_j) \leq 1$ and $e(x_4, L_j) \leq 1$, a contradiction. Hence (3^0) holds with respect to L_j and B. As $e(x_3x_4, L_j) \geq 5$, it follows that $[L_j, B] \supseteq 2C_4$, a contradiction.

Thus $t = 5$. Moreover, we see that $e(x_3, L_1) \leq 1$ with equality only if (5) holds and $x_3a_1 \in E$. We may assume that if (4^0) holds with respect to L_i and B for some L_i in H then L_1 has been chosen such that (4^0) (i.e., (5)) holds with respect to L_1 and B and $e(x_3, L_1) \geq e(x_3, L_i)$ for each L_i in H with (4^0) holding with respect to L_i and B.

By Lemma 2.6, there exist two labellings $L_2 = b_1b_2b_3b_4b_1$ and $B = x_3y_1y_2x_3y_4y_5x_3$ such that $\{x_1x_2, x_4x_5\} = \{y_1y_2, y_4y_5\}$ and one of (11) to (19) holds:

$$N(y_i, L_2) = \{b_1, b_3\}, i = 1, 2, 4, 5; \tag{11}$$

$$N(y_i, L_2) = \{b_1, b_2\}, i = 1, 2, 4, 5; \tag{12}$$

$$e(y_1y_2, L_2) = 8, e(y_4y_5, L_2) = 0; \tag{13}$$

$$\begin{aligned} &N(y_1, L_2) = \{b_1, b_2, b_3\}, N(y_2, L_2) = N(y_4, L_2) = \{b_1\}, \\ &N(y_5, L_2) = \{b_1, b_4, b_3\}, b_1b_3 \in E, b_2b_4 \notin E; \end{aligned} \tag{14}$$

$$\begin{aligned} &N(y_1, L_2) = \{b_1, b_2, b_3\}, N(y_2, L_2) = \{b_1, b_3\}, e(y_4, L_2) = 0, \\ &N(y_5, L_2) = \{b_1, b_4, b_3\}, b_1b_3 \in E, b_2b_4 \notin E; \end{aligned} \tag{15}$$

$$\begin{aligned} &N(y_1, L_2) = \{b_1, b_4, b_3\}, N(y_2, L_2) = \{b_1, b_2, b_3\}, e(y_4y_5, b_2b_4) = 0, \\ &e(b_1, y_4y_5) = 1, e(b_3, y_4y_5) = 1, b_2b_4 \notin E; \end{aligned} \tag{16}$$

$$\begin{aligned} &N(y_1, L_2) = \{b_1, b_4\}, N(y_2, L_2) = \{b_2, b_3\}, N(y_4, L_2) = \{b_1, b_2\}, \\ &N(y_5, L_2) = \{b_3, b_4\}, \tau(L_2) = 0; \end{aligned} \tag{17}$$

$$\begin{aligned} &N(y_1, L_2) = \{b_1, b_4\}, N(y_2, L_2) = \{b_2, b_3\}, e(y_4, L_2) = 0, e(y_5, L_2) = 4, \\ &\tau(L_2) = 0; \end{aligned} \tag{18}$$

$$e(y_1, L_2) = 4, e(y_2y_4, L_2) = 0, e(y_5, L_2) = 4. \tag{19}$$

As above, if one of (14) to (19) holds, then $e(x_3, L_2) \leq 1$ with equality only if (14) holds and $x_3b_1 \in E$. Let $G_1 = [L_1, B]$ and $H_1 = G - V(G_1)$. Note that

$[L_1 - a_q, x_1, x_2] \supseteq C_{\geq 4}$ and $[L_1 - a_q, x_4, x_5] \supseteq C_{\geq 4}$ for each $q \in \{2, 4\}$. To prove Property 2, we eliminate each of (5) to (10) as follows. First, (8) does not occur as $t = 5$.

Case 1. One of (5), (6) and (7) holds.

In this case, $e(x_3a_2a_4, G_1) \leq 11$. Thus $e(x_3a_2a_4, H_1) \geq 6k - 11 = 6(k - 2) + 1$. Without loss of generality, say $e(x_3a_2a_4, L_2) \geq 7$. Assume for the moment that $a_q \to L_2$ for some $q \in \{2, 4\}$. If one of (11), (12) and (14) to (18) holds, then for each of (5), (6) and (7), it easy to see that there exists $i \in \{1, 2, 5\}$ and $b_l \in V(L_2)$ such that $e(x_i, a_1a_3) = 2$ and $e(b_l, S - \{x_i\}) \geq 2$. Thus $x_i \to (L_1, a_q)$, $a_q \to (L_2, b_l; S - \{x_i\})$ and so $[B, L_1, L_2] \supseteq 3C_4$, a contradiction. Hence none of (11), (12) and (14) to (18) holds. If (13) holds then $a_q \to (L_2; y_1y_2)$ and so $[a_q, L_2, y_1, y_2, x_3] \supseteq 2C_4$. As $[L_1 - a_q, y_4, y_5] \supseteq C_{\geq 4}$, $[G_1, L_2] \supseteq 2C_4 \uplus C_{\geq 4}$, a contradiction. Hence (19) holds. Thus $e(x_3, L_2) = 0$ and $e(a_2a_4, L_2) \geq 7$. Without loss of generality, say $b_1 \in N(a_2, L_2) \cap N(a_4, L_2)$. Then $[b_1, a_2, a_3, a_4] \supseteq C_4$. As $e(a_1, S) \geq 3$, $e(a_1, S - \{y_j\}) \geq 2$ for some $j \in \{1, 5\}$. Thus $[a_1, B - y_j] \supseteq C_4$ and $y_j \to (L_2, b_1)$. Hence $[G_1, L_2] \supseteq 3C_4$, a contradiction.

Therefore $a_q \not\to L_2$ for each $q \in \{2, 4\}$. Thus $e(a_2, L_2) \leq 3$ and $e(a_4, L_2) \leq 3$. Hence $e(x_3, L_2) \geq 1$. Assume that one of (14) to (19) holds. As $e(x_3, L_2) \geq 1$, we shall have that (14) holds with $x_3b_1 \in E$ (i.e., (4^0) holds with respect to L_2 and B with $e(x_3, L_2) = 1$). By the assumption on L_1, we have that (5) holds with $x_3a_1 \in E$. As $a_q \not\to L_2$ for each $q \in \{2, 4\}$, it follows that $e(a_2, L_2) = 3$ and $e(a_4, L_2) = 3$ with $e(a_2a_4, b_1b_3) = 4$. If $e(b_4, a_2a_4) \geq 1$, then $[a_2, a_3, a_4, b_3, b_4] \supseteq C_5$, $[b_1, b_2, y_1, y_2] \supseteq C_4$ and $[a_1, x_3, y_4, y_5] \supseteq C_4$, a contradiction. Hence $e(b_4, a_2a_4) = 0$ and so $e(b_2, a_2a_4) = 2$. Thus $[b_2, a_2, a_3, a_4] \supseteq C_4$, $[a_1, x_3, y_4, y_5] \supseteq C_4$ and $[y_1, b_1, b_4, b_3] \supseteq C_4$, a contradiction. Hence one of (11), (12) and (13) holds. We break into the following three sub cases.

Subcase 1.1. (5) holds.

In this subcase, first assume that (11) holds. If $e(a_2a_4, b_2b_4) \geq 1$, say without loss of generality $a_2b_2 \in E$. Then $[a_2, b_2, b_1, x_1] \supseteq C_4$, $[b_3, x_2, x_3, x_4] \supseteq C_4$ and $[x_5, a_1, a_4, a_3] \supseteq C_4$, a contradiction. Hence $e(a_2a_4, b_2b_4) = 0$. Thus $e(x_3, L_2) \geq 3$ and so $e(x_3, b_2b_4) \geq 1$. Without loss of generality, say $x_3b_2 \in E$. Then $[x_3, b_2, b_1, x_2] \supseteq C_4$, $[a_1, a_4, x_4, x_5] \supseteq C_4$ and so $[a_2, a_3, x_1, b_3] \not\supseteq C_4$. Thus $a_2b_3 \notin E$. Similarly, $a_2b_1 \notin E$. Hence $e(a_2, L_2) = 0$ and so $e(x_3a_2a_4, L_2) \leq 6$, a contradiction.

Next, assume that (12) holds. If $e(a_q, L_2) \geq 3$ for some $q \in \{2, 4\}$, then $a_q \to (L_2, b_l)$ for some $l \in \{1, 2\}$. Thus $G_2 = [P, L_1, L_2] \supseteq 3C_4$ since $[b_l, x_2, x_3, x_4] \supseteq C_4$ and $x_1 \to (L_1, a_q)$, a contradiction. Hence $e(a_2, L_2) \leq 2$, $e(a_4, L_2) \leq 2$ and so $e(x_3, L_2) \geq 3$. Thus $e(x_3, b_3b_4) \geq 1$. Say without loss of generality $x_3b_3 \in E$. Then $[x_3, b_3, b_2, x_2] \supseteq C_4$ and $[x_1, a_1, a_2, a_3] \supseteq C_4$. Thus $[a_4, x_5, x_4, b_1, b_4] \not\supseteq C_{\geq 4}$. This yields that $e(a_4, b_1b_4) = 0$. Similarly, $e(a_2, b_1b_4) = 0$. If $x_3b_4 \in E$ then we would also have that $e(a_2a_4, b_2b_3) = 0$ and so $e(x_3, L_2) \geq 7$, a contradiction. Hence $x_3b_4 \notin E$ and it follows that $e(x_3, b_1b_2b_3) = 3$ and $e(a_2a_4, b_2b_3) = 4$. Thus $[a_2, b_2, a_4, b_3] \supseteq C_4$, $[x_1, x_2, a_3, a_1] \supseteq C_4$ and $[b_1, x_3, x_4, x_5] \supseteq C_4$, a contradiction.

Therefore (13) holds. Without loss of generality, say $e(x_1x_2, L_2) = 8$.
If $e(a_q, L_2) \geq 3$ for some $q \in \{2, 4\}$, then $a_q \to (L_2, b_l; x_1x_3x_2)$ for some $b_l \in V(L_2)$

and so $G_2 \supseteq 3C_4$ since $x_5 \to (L_1, a_q)$, a contradiction. Hence $e(a_2, L_2) \leq 2$ and $e(a_4, L_2) \leq 2$. Thus $e(x_3, L_2) \geq 3$. Say $e(x_3, b_1b_2b_3) = 3$. Then $[x_3, b_i, b_{i+1}, x_2] \supseteq C_4$ for all $i \in \{1,2,3,4\}$ and $[x_5, a_1, a_4, a_3] \supseteq C_4$. Thus $[a_2, x_1, b_j, b_{j+1}] \not\supseteq C_4$ for all $j \in \{1,2,3,4\}$ and so $e(a_2, L_2) = 0$. Thus $e(x_3a_2a_4, L_2) \leq 6$, a contradiction.

Subcase 1.2. (6) holds.

In this subcase, first assume that (11) holds. If $e(a_2a_4, b_2b_4) \geq 1$, say $\{q, l\} \subseteq \{2, 4\}$ with $a_qb_l \in E$. Let $\{i, j\} = \{1, 5\}$ be such that $a_qx_i \in E$. Then $[a_q, x_i, b_1, b_l] \supseteq C_4$, $x_j \to (L_1, a_q)$ and $[b_3, x_2, x_3, x_4] \supseteq C_4$, i.e., $G_2 \supseteq 3C_4$, a contradiction. Hence $e(a_2a_4, b_2b_4) = 0$. Thus $e(x_3, L_2) \geq 3$ and so $e(x_3, b_2b_4) \geq 1$. Without loss of generality, say $x_3b_2 \in E$. Then $[x_3, b_2, b_1, x_2] \supseteq C_4$ and $x_1 \to (L_1, a_4)$. Thus $[a_4, x_5, x_4, b_3] \not\supseteq C_4$ and so $a_4b_3 \notin E$. Similarly, $a_4b_1 \notin E$. Thus $e(x_3a_2a_4, L_2) < 7$, a contradiction.

Next, assume that (12) holds. If $e(a_2a_4, b_3b_4) \geq 1$, say $q \in \{2, 4\}$ with $e(a_q, b_3b_4) \geq 1$. Without loss of generality, say $a_qb_3 \in E$. Let $\{i, j\} = \{1, 5\}$ be such that $a_qx_i \in E$. Then $[a_q, b_3, b_2, x_i] \supseteq C_4$, $x_j \to (L_1, a_q)$ and $[b_1, x_2, x_3, x_4] \supseteq C_4$, i.e., $G_2 \supseteq 3C_4$, a contradiction. Hence $e(a_2a_4, b_3b_4) = 0$. Thus $e(x_3, L_2) \geq 3$ and so $e(x_3, b_3b_4) \geq 1$. Without loss of generality, say $x_3b_3 \in E$. Then $[x_3, b_3, b_2, x_2] \supseteq C_4$ and $[x_1, a_1, a_2, a_3] \supseteq C_4$. Thus $[a_4, x_5, x_4, b_1] \not\supseteq C_4$ and so $a_4b_1 \notin E$. Similarly, if $x_3b_4 \in E$ then $a_4b_2 \notin E$. It follows that $e(x_3a_2a_4, L_2) < 7$, a contradiction.

Therefore (13) holds. Assume for the moment that $\{y_1, y_2\} = \{x_4, x_5\}$. If $e(a_4, L_2) \geq 1$, say without loss of generality that $a_4b_1 \in E$. Then $[a_4, b_1, x_5, a_3] \supseteq C_4$, $[x_1, x_2, a_1, a_2] \supseteq C_4$ and $[x_4, b_2, b_3, b_4] \supseteq C_4$, a contradiction. Hence $e(a_4, L_2) = 0$. Thus $e(a_2, L_2) \geq 3$. Then $a_2 \to (L_2; x_4x_3x_5)$ and $[x_1, a_1, a_4, a_3] \supseteq C_4$, i.e., $G_2 \supseteq 3C_4$, a contradiction. Therefore $\{y_1, y_2\} = \{x_1, x_2\}$. As $x_5 \to (L_1, a_i)$ for $i \in \{2, 4\}$, we see that $a_i \not\to (L_2; x_1x_3x_2)$ for $i \in \{2, 4\}$ since $G_2 \not\supseteq 3C_4$. This implies that $e(a_i, L_2) \leq 2$ for $i \in \{2, 4\}$ and it follows that $N(a_4, L_2) \cap N(x_3, L_2) \neq \emptyset$. Hence $x_2 \to (L_2; a_4x_5x_3)$. As $x_1 \to (L_1, a_4)$, $G_2 \supseteq 3C_4$, a contradiction.

Subcase 1.3. (7) holds.

In this subcase, we may assume $a_1x_5 \in E$. Assume first that (11) holds. If $e(a_2a_4, b_2b_4) \geq 1$, say without loss of generality that $a_2b_2 \in E$. Then $[a_2, b_2, b_1, x_2] \supseteq C_4$, $[b_3, x_4, x_3, x_5] \supseteq C_4$ and $[x_1, a_1, a_4, a_2] \supseteq C_4$, a contradiction. Hence $e(a_2a_4, b_2b_4) = 0$. Thus $e(x_3, L_2) \geq 3$ and so $e(x_3, b_2b_4) \geq 1$. Without loss of generality, say $x_3b_2 \in E$. Since $e(a_2a_4, b_1b_3) \geq 7 - e(x_3, L_2) \geq 3$, we may assume without loss of generality that $a_2b_1 \in E$. Then $[x_3, b_2, b_3, x_4] \supseteq C_4$, $[a_2, b_1, x_5, a_1] \supseteq C_4$ and $[x_1, x_2, a_3, a_4] \supseteq C_4$, a contradiction.

Next, assume that (12) holds. If $e(a_2a_4, b_3b_4) \geq 1$, say without loss of generality $a_2b_3 \in E$. Then $[a_2, b_3, b_2, x_2] \supseteq C_4$, $[b_1, x_4, x_3, x_5] \supseteq C_4$ and $[x_1, a_1, a_4, a_3] \supseteq C_4$, a contradiction. Hence $e(a_2a_4, b_3b_4) = 0$. Thus $e(x_3, L_2) \geq 3$ and so $e(x_3, b_3b_4) \geq 1$. Without loss of generality, say $x_3b_3 \in E$. $[x_3, b_3, b_2, x_4] \supseteq C_4$ and $[x_1, x_2, a_3, a_j] \supseteq C_4$ for $j \in \{2, 4\}$. Hence $[a_j, a_1, x_5, b_1] \not\supseteq C_4$ for $j \in \{2, 4\}$ and so $e(b_1, a_2a_4) = 0$. Thus $e(a_2a_4x_3, L_2) < 7$, a contradiction.

Thus (13) holds. If $\{y_1, y_2\} = \{x_4, x_5\}$, then $x_5 \to L_2$. Clearly, $x_1 \to (L_1, a_2)$ and $x_2 \to (L_1, a_4)$. As $G_2 \not\supseteq 2C_4 \uplus C_5$, we see that $x_5 \not\to (L_2; a_2x_2x_3x_4)$ and $x_5 \not\to$

$(L_2; a_4x_1x_3x_4)$. This implies that $e(a_2a_4, L_2) = 0$, a contradiction. Hence $\{y_1, y_2\} = \{x_1, x_2\}$. Clearly, $[L_1 - a_i, x_4, x_5] \supseteq C_{\geq 4}$ for each $i \in \{2, 4\}$. Thus $a_i \not\to (L_2; x_1x_3x_2)$ and so $e(a_i, L_2) \leq 2$ for each $i \in \{2, 4\}$. As $e(a_2a_4x_3, L_2) \geq 7$, $N(x_3, L_2) \cap N(a_2, L_2) \neq \emptyset$. Say without loss of generality $e(b_1, a_2x_3) = 2$. Then $x_1 \to (L_2, b_1; a_2x_2x_3)$ and so $G_2 \supseteq 2C_4 \uplus C_{\geq 4}$, a contradiction.

Case 2. One of (9) and (10) holds.

By Case 1, each of (5), (6) and (7) does not holds. Then we see, with L_2 in place of L_1, that none of (14), (15) and (16) holds. Therefore one of (11) to (13) or (17) to (19) holds. Then $e(a_2a_4x_3x_4, G_1) = 14$ and so $e(a_2a_4x_3x_4, H_1) \geq 8k - 14 = 8(k-2) + 2$. Thus $e(a_2a_4x_3x_4, L_i) \geq 9$ for some L_i in H_1. Without loss of generality, say $e(a_2a_4x_3x_4, L_2) \geq 9$. First, suppose that one of (17), (18) and (19) holds. Then $e(x_3, L_2) = 0$. Thus $e(a_2a_4x_4, L_2) \geq 9$. It follows that $e(a_2a_4, L_2) \geq 5$ and $e(x_4, L_2) \in \{2, 4\}$. If $e(x_4, L_2) = 4$ then we would have that $x_4 \to (L_2; a_2a_3a_4)$ and $[a_1, x_1, x_3, x_5] \supseteq C_4$, i.e., $G_2 \supseteq 3C_4$, a contradiction. Hence $e(x_4, L_2) \neq 4$. Thus $e(x_4, L_2) = 2$ and $e(a_2a_4, L_2) \geq 7$. Say without loss of generality $e(a_4, L_2) = 4$. Then $a_4 \to (L_2; B - \{x_5\})$ and $x_5 \to (L_1, a_4)$, i.e., $G_2 \supseteq 3C_4$, a contradiction.

Therefore one of (11), (12) and (13) holds. First, assume that (11) holds. Then for each $i \in \{2, 4\}$, we have $N(x_1, L_1) \cap N(b_i, L_1) = \emptyset$ for otherwise $x_5 \to (L_1; x_1b_1b_i)$ and $[b_3, x_2, x_3, x_4] \supseteq C_4$, i.e., $G_2 \supseteq 3C_4$. Similarly, $N(x_2, L_1) \cap N(b_i, L_1) = \emptyset$ for each $i \in \{2, 4\}$. Thus $e(b_2b_4, a_2a_4) = 0$. Hence $e(a_2a_4, b_1b_3) \geq 3$ and $e(x_3, L_2) \geq 3$. As $e(x_3, b_2b_4) \geq 1$, say without loss of generality $x_3b_4 \in E$. As $e(a_2a_4, b_1b_3) \geq 3$, say without loss of generality $a_4b_1 \in E$. Then $[x_3, b_4, b_3, x_4] \supseteq C_4$ and $x_5 \to (L_1, a_4; x_1x_2b_1)$, i.e., $G_2 \supseteq 3C_4$, a contradiction. Next, assume that (12) holds. Similar to the above, we readily see that $e(b_3b_4, a_2a_4) = 0$. Thus $e(a_2a_4, b_1b_2) \geq 3$ and $e(x_3, L_2) \geq 3$. As $e(x_3, b_3b_4) \geq 1$, say without loss of generality $x_3b_3 \in E$. As $e(a_2a_4, b_1b_2) \geq 3$, say without loss of generality $a_4b_1 \in E$. Then $[x_3, b_3, b_2, x_4] \supseteq C_4$ and $x_5 \to (L_1, a_4; x_1x_2b_1)$, a contradiction. Therefore (13) holds.

As $e(a_2a_4x_3x_4, L_2) \geq 9$, either $N(a_2, L_2) \cap N(a_4, L_2) \neq \emptyset$ or $N(x_3, L_2) \cap N(a_i, L_2) \neq \emptyset$ for some $i \in \{2, 4\}$. If $\{y_1, y_2\} = \{x_4, x_5\}$, then either $x_4 \to (L_2; a_2a_3a_4)$ or $x_4 \to (L_2; x_3x_5a_i)$ for some $i \in \{2, 4\}$. Since $[a_1, x_1, x_3, x_5] \supseteq C_4$ and $[x_1, x_2, L_1 - a_i] \supseteq C_4$ for each $i \in \{2, 4\}$, it yields that $G_2 \supseteq 3C_4$, a contradiction. Hence $\{y_1, y_2\} = \{x_1, x_2\}$. Then $e(a_2a_4, L_2) \geq 5$. Thus $x_2 \to (L_2; a_2a_3a_4)$ and $[a_1, x_1, x_3, x_5] \supseteq C_4$, a contradiction. ∎

By Property 1, we partition $\{L_1, L_2, \ldots, L_{k-1}\}$ into four subsets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ and $\mathcal{D}$ as follows. For each $i \in \{1, 2, \ldots, k-1\}$, if (2) holds with respect to L_i and B then $L_i \in \mathcal{A}$; if (3) holds with respect to L_i and B with $e(x_1x_2, L_i) = 8$ then $L_i \in \mathcal{B}$; if (3) holds with respect to L_i and B with $e(x_4x_5, L_i) = 8$ then $L_i \in \mathcal{C}$; and if (4) holds with respect to L_i and B then $L_i \in \mathcal{D}$. For each $L_i \in \mathcal{A}$, let $V(L_i) = \{u_i, v_i, y_i, z_i\}$ be such that $N(x_1, L_i) = \{u_i, v_i\}$.

Property 2. For all $L_i \in \mathcal{A}$ and $L_j \in \mathcal{B} \cup \mathcal{C}$ and $L_r \in \mathcal{D}$, the following statements hold:

(*a*) $e(y_iz_i, L_j) = 0$ and $e(u, L_r) = 0$ for all $u \in V(L_j)$;

(b) $e(y_iz_i, L_r) \leq 2$ and if $e(w, L_r) = 2$ for some $w \in \{y_i, z_i\}$ then $u_iv_i \in E(L_i)$ and $x_b \not\rightarrow (L_i, w)$ for all $b \in \{1, 2, 4, 5\}$.

Proof of Property 2. Let $L_r = c_1c_2c_3c_4c_1$ be such that $N(x_1, L_r) = \{c_1, c_4\}$, $N(x_2, L_r) = \{c_2, c_3\}$, $N(x_4, L_r) = \{c_1, c_2\}$ and $N(x_5, L_r) = \{c_3, c_4\}$.

Say $e(x_lx_{l+1}, L_j) = 8$ with $l \in \{1, 4\}$.

To see (a), first assume that $e(y_iz_i, L_j) \geq 1$. Without loss of generality, say $e(y_i, L_j) \geq 1$ and $y_iu_i \in E$. Then $x_l \rightarrow (L_j; x_{l+1}u_iy_i)$ and $[B - \{x_l, x_{l+1}\}, v_i] \supseteq C_4$, i.e., $[B, L_i, L_j] \supseteq C_4$, a contradiction. Next, assume that $e(u, L_r) \geq 1$ for some $u \in V(L_j)$. Without loss of generality, say $L_j \in \mathcal{C}$ and $L_j = u_1u_2u_3u_4u_1$ with $u_1c_1 \in E$. Then $[B, L_j, L_r] \supseteq 3C_4 = \{c_1u_1x_4c_2c_1, 5_4u_2u_3u_4x_5, x_1c_4c_3x_2x_1\}$, a contradiction. Hence $e(u, L_r) = 0$ and (a) follows.

To see (b), we may assume that either $e(y_i, L_r) \geq 2$ or$e(z_i, L_r) \geq 2$. Without loss of generality, say the former holds. Without loss of generality, say $y_iu_i \in E$. If $e(y_i, c_1c_3) = 2$ or $e(y_i, c_2c_4) = 2$, say without loss of generality $e(y_i, c_1c_3) = 2$, then $y_i \rightarrow (L_r, c_2; x_2x_3x_4)$ and $[x_1, u_i, x_5, v_i] \supseteq C_4$, i.e., $[B, L_i, L_r] \supseteq 3C_4$, a contradiction. Hence $N(y_i, L_r) = \{c_s, c_{s+1}\}$ for some $s \in \{1, 2, 3, 4\}$. Without loss of generality, say $e(y_i, c_1c_2) = 2$. If $x_1 \rightarrow (L_i, y_i)$ then $x_1 \rightarrow (L_i, y_i; c_1x_4c_2)$ and $[c_3, x_2, x_3, x_5] \supseteq C_4$, i.e., $[B, L_i, L_r] \supseteq 3C_4$, a contradiction. Hence $x_1 \not\rightarrow (L_i, y_i)$. Thus $u_iv_i \in E$ and $u_iz_i \notin E$. Hence $x_b \not\rightarrow (L_i, y_i)$ for all $b \in \{1, 2, 4, 5\}$. If we also have $e(z_i, L_r) \geq 1$, then $[y_i, z_i, c_p, c_{p+1}] \supseteq C_4$ for some $p \in \{1, 2, 3, 4\}$. Without loss of generality, say $[y_i, z_i, c_1, c_2] \supseteq C_4$. Then $[x_1, x_2, c_3, c_4] \supseteq C_4$ and $[x_4, x_5, u_i, v_i] \supseteq C_4$, a a contradiction. Hence (b) holds. ∎

Property 3. If $\mathcal{A} \neq \emptyset$ then n is odd and $G \in \Sigma_{k,n}$.

Proof of Property 3. As $\mathcal{A} \neq \emptyset$, $t = 5$. Recall $e(v, D) \geq 2$ for all $v \in V(D) - V(B)$. Since $D \not\supseteq C_{\geq 4}$ and by the maximality of t, we see that $D - V(B)$ consists of independent edges and $N(x_3) \supseteq V(D) - V(B)$. For each edge $xy \in E(D - V(B))$, applying Property 1 with $B' = x_3xyx_3x_4x_5x_3$ in place of B, we see that $N(x, L_i) = N(y, L_i) = \{u_i, v_i\}$ for all $L_i \in \mathcal{A}$. Without loss of generality, say $\mathcal{A} = \{L_1, \ldots, L_p\}$. Say $G' = [D, L_1, \ldots, L_p]$. By Property 2, for each $i \in \{1, \ldots, p\}$ and $w \in \{y_i, z_i\}$, $e(w, G') \geq 2k - e(w, G - V(G')) \geq 2k - 2(k - p - 1) = 2(p + 1)$. We claim that $\tau(L_i) = 2$ and $e(x_3, y_iz_i) = 2$ for all $i \in \{1, \ldots, p\}$. On the contrary, say it fails for L_1. Then $e(y_1z_1, L_1 \cup D) \leq 7$. Then $e(w_1, L_1 \cup D) \leq 3$ for some $w_1 \in \{y_1, z_1\}$. Thus $e(w_1, G' - V(L_1)) \geq 2(p + 1) - 3 = 2(p - 1) + 1$. Thus $e(w_1, L_j) \geq 3$ for some $j \in \{2, \ldots, p\}$. Without loss of generality, say $e(w_1, L_2) \geq 3$. Clearly, $[x_1, x_2, u_1, v_1] \supseteq C_4$, $[u_2, x_3, x_4, x_5] \supseteq C_4$ and $[v_2, x_3, x_4, x_5] \supseteq C_4$. Thus $w_1 \not\rightarrow (L_2, u_2)$ and $w_1 \not\rightarrow (L_2, v_2)$ for otherwise $[L_1, L_2, B] \supseteq 3C_4$. It follows that $u_2v_2 \notin E(L_2)$, $L_2 = u_2y_2v_2z_2u_2$, $e(w_1, L_2) = 3$ and $y_2z_2 \notin E$ with $\{u_2, v_2\} \subseteq N(w_1)$. Let $w_2 \in \{y_2, z_2\}$ be such that $w_1w_2 \notin E$. Thus $w_1 \xrightarrow{a} (L_2, w_2)$ and $e(w_2, L_2 \cup D) \leq 3$. We may define q to be the largest integer such that there exist q distinct 4-cycles L_i in $\{L_1, L_2, \ldots, L_p\}$, say without loss of generality $L_1, \ldots, L_q$, such that the following statements (a) and (b) hold:

(a) For each $i \in \{2, \ldots, q\}$, $L_i = u_iy_iv_iz_iu_i$ and $y_iz_i \notin E$;

(b) There exists $w_i \in \{y_i, z_i\}$ such that $N(w_i, L_{i+1}) = V(L_{i+1}) - \{w_{i+1}\}$ for $i = 1, \ldots, q-1$.

As $e(w_q, L_q \cup D) \le 3$, $e(w_q, L_j) \ge 3$ for some $j \in \{1, \ldots, p\}$ with $j \ne q$. If $j \ge q+1$, we may assume without loss of generality that $L_j = L_{q+1}$. With L_q and L_{q+1} replacing L_1 and L_2 in the above argument, we see that there exists $w_{q+1} \in \{y_{q+1}, z_{q+1}\}$ such that (a) holds with $i = q+1$ and (b) holds with $i = q$, contradicting the maximality of q. Hence $1 \le j \le q-1$. With L_q and L_j replacing L_1 and L_2 in the above argument, we see that $w_q \Rightarrow (L_j, w_j)$. Since $w_i \xrightarrow{a} (L_{i+1}, w_{i+1})$ for $i = j, j+1, \ldots, q-1$, we obtain a contradiction with the maximality of $\tau(\sigma)$. Therefore the claim holds, i.e., $\tau(L_i) = 2$ and $e(x_3, y_i z_i) = 2$ for all $i \in \{1, \ldots, p\}$.

Then we readily see that $e(y_i z_i, y_j z_j) = 0$ for all $1 \le i < j \le p$ for otherwise $G' \supseteq (p+1)C_4$. As $\delta(G) \ge 2k$, it follows that $e(y_i z_i, u_j v_j) = 4$ for all $1 \le i \le j \le p$. Thus $\{x_1x_2, x_4x_5\} \cup \{y_i z_i | 1 \le i \le p\} \cup E(D - V(B))$ is an edge independent set of G' and so $G' \in \Sigma_{p+1,n'}$ where $n' = |V(G')|$. By Property 2, we see that $\mathcal{B} \cup \mathcal{C} \cup \mathcal{D} = 0$ for otherwise $e(y_1 z_1, G) < 4k$. Hence $G' = G \in \Sigma_{k,n}$. ∎

Property 4. If $\mathcal{B} \cup \mathcal{C} \ne \emptyset$, then k is odd and $G \in \Gamma_k$.

Proof of Property 4. By Property 3, $\mathcal{A} = \emptyset$. Clearly, $e(L', L'') = 0$ for all $L' \in \mathcal{B}$ and $L'' \in \mathcal{C}$ for otherwise $[B, L', L''] \supseteq 2C_4 \uplus C_{\ge 4}$. Since $\delta(G) \ge 2$ and by Property 2(a), it follows that $\mathcal{D} = \emptyset$ and $|\mathcal{B}| = |\mathcal{C}| = (k-1)/2$. Thus k is odd. If $t \ge 7$ or $V(D) - V(B) \ne \emptyset$, then $e(v, D) \le 2$ for some $v \in V(D) - \{x_1, x_2, x_3, x_{t-2}, x_{t-1}, x_t\}$ since $D \not\supseteq C_{\ge 4}$. Then $e(v, L_i) \ge 2$ for some L_i in H. By Lemma 2.6, $[D, L_i] \supseteq C_4 \uplus C_{\ge 4}$, a contradiction. Hence $t \le 6$ and $D = B$. As $\delta(G) \ge 2k$, it follows that $[x_1, x_2, x_3, V(\mathcal{B})] \cong [x_{t-2}, x_{t-1}, x_t, V(\mathcal{C})] \cong K_{2k+1}$, i.e., $G \in \Gamma_k$. ∎

By Property 3 and Property 4, if $\mathcal{D} \ne \emptyset$ then $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = \emptyset$. Consequently, $e(x_3, G) = 4$ and so $k = 2$, i.e., $G \cong F_9$. This proves the main theorem.

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