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On the construction of nested orthogonal arrays

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Abstract

Nested orthogonal arrays are useful in obtaining space-filling designs for an experimental set up consisting of two experiments, the expensive one of higher accuracy to be nested in a larger inexpensive one of lower accuracy. Systematic construction methods of some families of symmetric and asymmetric nested orthogonal arrays were provided recently by Dey [*Discrete Math.* 310 (2010), 2831–2834]. In this paper, we provide some more methods of construction of nested orthogonal arrays.

1 Introduction

An (ordinary) orthogonal array, $\mathrm{OA}(N, k, s_1 \times s_2 \times \cdots \times s_k, g)$, having N rows, k columns, $s_1, \ldots, s_k$ symbols and strength g $(2 \leq g < k)$ is an $N \times k$ matrix with elements in the ith column from a set of $s_i \geq 2$ distinct symbols $(1 \leq i \leq k)$, in which all possible combinations of symbols appear equally often as rows in every $N \times g$ subarray.

In an $\mathrm{OA}(N, k, s_1 \times \cdots \times s_k, g)$, if among $s_1, \ldots, s_k$, there are w_i that equal μ_i $(1 \leq i \leq u)$, where $w_1, \ldots, w_u, \mu_1, \ldots, \mu_u$ are positive integers $(\mu_i \geq 2, 1 \leq i \leq u, w_1 + \cdots + w_u = k)$, then we will use the notation $\mathrm{OA}(N, k, \mu_1^{w_1} \times \cdots \times \mu_u^{w_u}, g)$ for $\mathrm{OA}(N, k, s_1 \times \cdots \times s_k, g)$. In particular, if $s_1 = s_2 = \cdots = s_k = s$, then the array reduces to a *symmetric* orthogonal array, denoted simply by $\mathrm{OA}(N, k, s, g)$. Otherwise, the array is an *asymmetric* orthogonal array. Orthogonal arrays have been studied extensively and for a comprehensive account of the theory and applications of such arrays, a reference may be made to Hedayat et al. [2].

In recent years, considerable attention has been paid to experimental situations consisting of two experiments, the expensive one of higher accuracy being nested in a larger and relatively less expensive one of lower accuracy. The higher accuracy experiment can, for instance, correspond to a smaller physical experiment while the lower accuracy one can be a larger computer experiment. The modeling and analysis of data from such nested experiments have been addressed by several authors (see e.g., Kennedy and O'Hagan [4], Reese et al. [8], Qian et al. [6] and Qian and Wu [7]). Nested orthogonal arrays are useful in designing such nested experiments.

We now recall the definition of a nested orthogonal array.

Definition. *A nested orthogonal array, $NOA((N, M), k, (s_1 \times s_2 \times \cdots \times s_k, r_1 \times r_2 \times \cdots \times r_k), g)$, where $r_i \leq s_i$, with strict inequality for at least one i, $1 \leq i \leq k$, and $M < N$, is an orthogonal array $OA(N, k, s_1 \times \cdots \times s_k, g)$ which contains an $OA(M, k, r_1 \times \cdots \times r_k, g)$ as a subarray.*

If $s_1 = s_2 = \cdots = s_k = s$ and $r_1 = r_2 = \cdots = r_k = r$, then one obtains a *symmetric* nested orthogonal array, denoted by $\mathrm{NOA}((N, M), k, (s, r), g)$, where $M < N$ and $r < s$. Otherwise, the array is an *asymmetric* nested orthogonal array.

As noted by Dey [1], in the context of asymmetric nested orthogonal arrays, the above definition does not preclude the possibility of the existence of an asymmetric nested orthogonal array wherein the smaller orthogonal array is a symmetric orthogonal array, nested within a larger asymmetric orthogonal array.

The question of existence of symmetric nested orthogonal arrays has been examined in detail by Mukerjee et al. [5], where some examples of such arrays can also be found. Methods of construction of several families of symmetric and asymmetric nested orthogonal arrays have been provided recently by Dey [1]. In this paper, some more methods of construction of nested orthogonal arrays are provided.

2 Preliminaries

We first introduce some notation. For a positive integer m, $\mathbf{1}_m$, I_m and $\mathbf{0}_m$ respectively, denote an $m \times 1$ vector with all elements equal to 1, an identity matrix of order m and an $m \times 1$ null vector. Also A' will denote the transpose of a matrix A.

A square matrix H_n of order n with entries ± 1 is called a Hadamard matrix if $H_n H_n' = nI_n$. A positive integer n is called a Hadamard number if H_n exists. A matrix H_n trivially exists for $n = 1, 2$ and a necessary condition for the existence of a Hadamard matrix of order $n > 2$ is that $n \equiv 0 \pmod 4$. Note that if H_n is a Hadamard matrix, then we also have $H_n' H_n = nI_n$. From the definition of a Hadamard matrix, it is seen easily that a Hadamard matrix remains a Hadamard matrix if any of its rows or columns is multiplied by -1. Therefore, without loss of generality, one can write a Hadamard matrix with its first column consisting of only $+1$'s. For more details on Hadamard matrices, see e.g., Horadam [3].

Let λ ($\lambda \geq 1$) and m, n ($m, n \geq 2$) be integers and $\mathcal{G}$ be a finite additive abelian group consisting of m elements. A $\lambda m \times n$ matrix, $D(\lambda m, n; m)$, with elements from $\mathcal{G}$, is called a difference matrix if among the differences of the corresponding elements of every two distinct columns, each element of $\mathcal{G}$ appears λ times.

Finally, an ordinary orthogonal array $\mathrm{OA}(N, k, s_1 \times \cdots \times s_k, g)$ is called *tight* if the number of rows of the array attains Rao's lower bound on the number of rows; for details on Rao's bounds, see e.g., Hedayat et al. [2]. In particular, Rao's bounds for arrays of strength two and three are given respectively, by

$$N \geq 1 + \sum_{i=1}^{k} (s_i - 1), \quad \text{if } g = 2 \tag{1}$$

$$N \geq 1 + \sum_{i=1}^{k} (s_i - 1) + (s^* - 1) \left\{ \sum_{i=1}^{k} (s_i - 1) - (s^* - 1) \right\}, \quad \text{if } g = 3, \tag{2}$$

where $s^* = \max_{1 \leq i \leq k} s_i$.

3 Symmetric nested orthogonal arrays

Dey [1] constructed a family of symmetric nested orthogonal arrays in which *neither* of s nor r, $r < s$, are powers of 2. Apart from this family, an example of a symmetric nested symmetric orthogonal array was presented by Mukerjee et al. [5] with $s = 3$ and $r = 2$. Nothing beyond these appears to be known about symmetric nested arrays in which s and r are not both powers of 2.

A trivial method of constructing symmetric nested orthogonal arrays where both s and r are not necessarily powers of 2 is as follows: Let s $(s \geq 3)$, r $(2 \leq r < s)$ and n $(n \geq 2)$ be integers. Form an $s^n \times n$ array A, whose rows are all possible n-plets involving s symbols. Then clearly, A is a symmetric $\text{NOA}((s^n, r^n), n, (s, r), n)$. However, such nested arrays are often too large in size to be of much use.

An ordinary orthogonal array can be obtained by "developing" a difference matrix; see Hedayat et al. ([2], Chapter 6) for details. Though a general method of construction of symmetric nested orthogonal arrays using difference matrices has not yet been found, we give below some examples of non-trivial symmetric nested arrays which are obtainable by developing suitable difference matrices. In these arrays, s and r are not both powers of 2.

Example 1. Consider a difference matrix $D(6, 3; 3)$ shown below:

$$D(6,3;3) = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

By developing this difference matrix, a $\text{NOA}((18, 4), 3, (3, 2), 2)$ is obtained, which is shown below in transposed form:

$$\begin{bmatrix} 0011 & 0000 & 1111 & 2222 & 22 \\ 0101 & 0122 & 0122 & 0011 & 22 \\ 0110 & 2201 & 2102 & 0102 & 12 \end{bmatrix}'.$$

The first four rows of the above array form an OA$(4, 3, 2, 2)$ while the full array is an OA$(18, 3, 3, 2)$. A nested orthogonal array with the same parameters can also be obtained by invoking Theorem 3 of Dey [1].

Example 2. Consider a difference matrix $D(12, 4; 4)$ shown below:

$$\begin{bmatrix} 00 & 01 & 10 & 11 \\ 00 & 10 & 11 & 01 \\ 00 & 11 & 01 & 10 \\ 01 & 11 & 01 & 10 \\ 01 & 01 & 10 & 11 \\ 01 & 10 & 11 & 01 \\ 10 & 11 & 10 & 01 \\ 10 & 01 & 11 & 10 \\ 10 & 10 & 01 & 11 \\ 11 & 01 & 01 & 01 \\ 11 & 10 & 10 & 10 \\ 11 & 11 & 11 & 11 \end{bmatrix},$$

where the binary operation $(+)$ of the group associated with the difference matrix is defined as below:

$$\begin{array}{c|cccc} + & 00 & 01 & 10 & 11 \\ \hline 00 & 00 & 01 & 10 & 11 \\ 01 & 01 & 00 & 11 & 10 \\ 10 & 10 & 11 & 00 & 01 \\ 11 & 11 & 10 & 01 & 00 \end{array}.$$

By developing this difference matrix, one gets a NOA$((48, 9), 4, (4, 3), 2)$, displayed below in transposed form, where we have used the following mapping for the symbols: $00 \to 0, 01 \to 1, 10 \to 2, 11 \to 3$.

$$\begin{bmatrix} 000111222 & 000000000 & 111111111 & 222222222 & 333333333 & 333 \\ 012012012 & 001122333 & 001122333 & 001122333 & 000111222 & 333 \\ 012120201 & 330203112 & 031203123 & 013312023 & 122013123 & 003 \\ 012201120 & 123331020 & 021331203 & 030232113 & 313210020 & 123 \end{bmatrix}'.$$

The first nine rows of the above array constitute an OA$(9, 4, 3, 2)$ and the full array is an OA$(48, 4, 4, 2)$. Note that no more columns can be added to the above nested array because OA$(9, 4, 3, 2)$ is a tight orthogonal array.

Example 3. Consider a difference matrix $D(12, 4; 3)$ shown below in transposed form:

$$\begin{bmatrix} 021 & 201 & 121 & 002 \\ 020 & 122 & 102 & 101 \\ 011 & 210 & 202 & 021 \\ 010 & 020 & 221 & 112 \end{bmatrix}'.$$

By developing the above difference matrix, one gets a NOA$((36,8),4,(3,2),2)$ shown below in transposed form:

$$\begin{bmatrix} 01010101 & 00000000 & 11111122 & 112222222222 \\ 00111100 & 00112222 & 00112200 & 220110112222 \\ 01011010 & 22120012 & 21022112 & 200020121102 \\ 00110011 & 21021220 & 02221210 & 102102210102 \end{bmatrix}'.$$

The first eight rows of the above array form an OA$(8,4,2,2)$ while all the 36 rows constitute an OA$(36,4,3,2)$.

4 Asymmetric nested orthogonal arrays

4.1 Use of ordinary orthogonal arrays

In Theorem 4 of Dey [1], a method of construction of asymmetric nested orthogonal arrays was proposed. However, there is a slight error in this result. A correct and modified version of the result is provided below.

Theorem 1.

(i) Let t $(t > 2)$ and m $(2 \le m < t)$ be integers. The existence of an OA$(N,k,2,2u)$ implies the existence of a NOA$((tN,mN),k+1,(t\times 2^k, m\times 2^k),2u)$.

(ii) If t and m are both even integers, then the existence of an OA$(N,k,2,2u)$ implies that of a NOA$((tN,mN),\ k+1,(t\times 2^k, m\times 2^k),2u+1)$.

(iii) Furthermore, if $u = 1$ and the OA$(N,k,2,2)$ is tight (i.e., $k = N-1$), then the derived array in (ii) has the maximum number of 2-symbol columns that such an array can accommodate.

Proof. Let t and m be as above. Denote the OA$(N,k,2,2u)$ by A and let $\bar{A}$ denote the $N\times k$ matrix obtained by interchanging the two symbols in A. Define the $N\times(k+1)$ matrix B as

$$B = \begin{bmatrix} \mathbf{0}_N' & \mathbf{1}_N' & 2\mathbf{1}_N' & 3\mathbf{1}_N' & \cdots & (t-1)\mathbf{1}_N' \\ A' & \bar{A}' & A' & \bar{A}' & \cdots & U \end{bmatrix}',$$

where

$$U \equiv \begin{cases} A' & \text{if } t \text{ is odd} \\ \bar{A}' & \text{if } t \text{ is even.} \end{cases} \quad (3)$$

The array B is clearly an OA$(tN,k+1,t\times 2^k,2u)$. The array

$$C = \begin{bmatrix} \mathbf{0}_N' & \mathbf{1}_N' & 2\mathbf{1}_N' & 3\mathbf{1}_N' & \cdots & (m-1)\mathbf{1}_N' \\ A' & \bar{A}' & A' & \bar{A}' & \cdots & U \end{bmatrix}',$$

where U is as in (3) with t replaced by m, nested within B is an $\mathrm{OA}(mN, k+1, m \times 2^k, 2u)$. This proves (i). If t and m are both even integers, then both B and C, nested within B, are orthogonal arrays of strength $2u+1$ because $[A' \ \bar{A}']'$ is an $\mathrm{OA}(2N, k, 2, 2u+1)$, proving (ii). Finally, (iii) follows from the fact that by (2), in an $\mathrm{OA}(mN, k+1, m \times 2^k, 3)$, $k \le N-1$. □

The following example illustrates Theorem 1.

Example 4. Let $t = 3, m = 2, u = 1$ and

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}'.$$

Following the construction described above, we have

$$B = \begin{bmatrix} 0000 & 1111 & 2222 \\ 0011 & 1100 & 0011 \\ 0101 & 1010 & 0101 \\ 0110 & 1001 & 0110 \end{bmatrix}'.$$

Clearly, B is an asymmetric nested orthogonal array $\mathrm{NOA}((12, 8), 4, (3 \times 2^3, 2^4), 2)$. where the first 8 rows of B form a symmetric $\mathrm{OA}(8, 4, 2, 2)$ while all the 12 rows form an asymmetric $\mathrm{OA}(12, 4, 3 \times 2^3, 2)$.

Next, let $t = 6, m = 4, u = 1$ and A as exhibited above. The array B shown below in transposed form is a $\mathrm{NOA}((24, 16), 4, (6 \times 2^3, 4 \times 2^3), 3)$:

$$B = \begin{bmatrix} 0011 & 1100 & 0011 & 1100 & 0011 & 1100 \\ 0101 & 1010 & 0101 & 1010 & 0101 & 1010 \\ 0110 & 1001 & 0110 & 1001 & 0110 & 1001 \\ 0000 & 1111 & 2222 & 3333 & 4444 & 5555 \end{bmatrix}'.$$

The first 16 rows of B constitute a tight $\mathrm{OA}(16, 4, 4^1 \times 2^3, 3)$, while all the 24 rows form an $\mathrm{OA}(24, 4, 6^1 \times 2^3, 3)$, which is also tight.

4.2 Use of Hadamard matrices

We make use of Hadamard matrices to obtain a family of asymmetric nested orthogonal arrays of strength two. Let $u \ge 4$ be a Hadamard number and H_u be a Hadamard matrix of order u. Write H_u as $H_u = [\mathbf{1}_u \ A^*]$. Let A be a $u \times (u-1)$ matrix obtained by replacing the -1's in A^* by 0, and $\bar{A}$ be a $u \times (u-1)$ matrix obtained by interchanging the two symbols in A. Then each of A and $\bar{A}$ is a symmetric orthogonal array $\mathrm{OA}(u, u-1, 2, 2)$ of strength two with symbols 0 and 1.

Let $\boldsymbol{c} = (0, 1, \ldots, u-1)'$ and define a $2u \times (u+1)$ matrix B as

$$B = \begin{bmatrix} \boldsymbol{c} & \mathbf{0}_u & A \\ \boldsymbol{c} & \mathbf{1}_u & \bar{A} \end{bmatrix}.$$

For $0 \le i \le u-1$, let $\boldsymbol{a}_i'$ be the ith row of A and $\boldsymbol{b}_i'$ be the ith row of $\bar{A}$. Define the $(u-2)\times 1$ vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as $\boldsymbol{\alpha} = (2, 3, \ldots, u-1)'$ and $\boldsymbol{\beta} = (u, u+1, \ldots, 2u-3)'$ and let

$$C = \begin{bmatrix} \mathbf{0}_{u-2} & \boldsymbol{\alpha} & \mathbf{1}_{u-2} \otimes \boldsymbol{a}_0' \\ \mathbf{1}_{u-2} & \boldsymbol{\alpha} & \mathbf{1}_{u-2} \otimes \boldsymbol{a}_1' \\ 2\mathbf{1}_{u-2} & \boldsymbol{\alpha} & \mathbf{1}_{u-2} \otimes \boldsymbol{a}_2' \\ \vdots & & \\ (u-1)\mathbf{1}_{u-2} & \boldsymbol{\alpha} & \mathbf{1}_{u-2} \otimes \boldsymbol{a}_{u-1}' \\ \mathbf{0}_{u-2} & \boldsymbol{\beta} & \mathbf{1}_{u-2} \otimes \boldsymbol{b}_0' \\ \mathbf{1}_{u-2} & \boldsymbol{\beta} & \mathbf{1}_{u-2} \otimes \boldsymbol{b}_1' \\ 2\mathbf{1}_{u-2} & \boldsymbol{\beta} & \mathbf{1}_{u-2} \otimes \boldsymbol{b}_2' \\ \vdots & & \\ (u-1)\mathbf{1}_{u-2} & \boldsymbol{\beta} & \mathbf{1}_{u-2} \otimes \boldsymbol{b}_{u-1}' \end{bmatrix},$$

where for a pair of matrices $E = (e_{ij})$ and F, of orders $m \times n$ and $u \times v$, respectively, $E \otimes F$ denotes their Kronecker (tensor) product, i.e, $E \otimes F$ is an $mu \times nv$ matrix given by $(e_{ij}F)$. We then have the following result.

Theorem 2. *The matrix* $D = \begin{bmatrix} B \\ C \end{bmatrix}$ *is an asymmetric nested orthogonal array* $NOA((2u^2 - 2u, 2u), u+1, (u \times (2u-2) \times 2^{u-1}, u \times 2^u), 2)$. *Furthermore,* $u+1$ *is the maximum number of columns that such an array can accommodate.*

Proof. First observe that B as above is an asymmetric orthogonal array, $\mathrm{OA}(2u, u+1, u \times 2^u, 2)$ of strength two. Furthermore, this array is *tight* as the lower bound in (1) is attained. In B, the first column has u symbols, $0, 1, \ldots, (u-1)$ and the remaining u columns have two symbols each, 0 and 1. Also, it is easy to see that C is an asymmetric orthogonal array $\mathrm{OA}(2u^2 - 4u, u+1, u \times (2u-4) \times 2^{u-1}, 2)$, where the first column has u symbols, $0, 1, \ldots, (u-1)$, the second column has $(2u-4)$ symbols, $2, 3, \ldots, (2u-3)$ and the remaining columns have two symbols each, 0 and 1. It then follows that D is an asymmetric nested orthogonal array with the stated parameters, where B is the smaller array, nested within D. The claim of the maximum number of columns being $u+1$ follows from the fact that B is a tight array.

□

Example 5. Letting $u = 4$ in Theorem 2, one obtains an asymmetric nested orthogonal array $\mathrm{NOA}((24, 8), 5, (4 \times 6 \times 2^3, 4 \times 2^4), 2)$ displayed below in transposed form:

$$\begin{bmatrix} 0123 & 0123 & 0000 & 1111 & 2222 & 3333 \\ 0000 & 1111 & 2345 & 2345 & 2345 & 2345 \\ 0011 & 1100 & 0011 & 0011 & 1100 & 1100 \\ 0101 & 1010 & 0011 & 1100 & 0011 & 1100 \\ 0110 & 1001 & 0011 & 1100 & 1100 & 0011 \end{bmatrix}'.$$

The first 8 rows of the above array constitute an $\mathrm{OA}(8, 5, 4 \times 2^4, 2)$, while all the 24 rows form an $\mathrm{OA}(24, 5, 4 \times 6 \times 2^3, 2)$.

Similarly, taking $u = 8$, one obtains a $\text{NOA}((112, 16), 9, (8 \times 14 \times 2^7, 8 \times 2^8), 2)$.

4.3 Use of resolvable arrays

Families of asymmetric nested orthogonal arrays of strength two can be obtained via resolvable (ordinary) orthogonal arrays. Let A be an $\text{OA}(N, k, s_1 \times \cdots \times s_k, 2)$, such that its rows can be partitioned into s_1 sets of N/s_1 rows each, say $A_1, A_2, \ldots, A_{s_1}$, and where each A_i $(1 \le i \le s_1)$ is an orthogonal array of strength *unity*. Such an orthogonal array is called resolvable. This means that for $1 \le i \le s_1$, A_i is an $\text{OA}(N/s_1, k, s_1 \times \cdots \times s_k, 1)$ of strength one.

Let $t, m,\ s_1 \le m < t$ be integers such that s_1 divides both t and m. Consider the $tN/s_1 \times (k+1)$ matrix B given by

$$B = \begin{bmatrix} \mathbf{0} & A_1 \\ \mathbf{1} & A_2 \\ \vdots & \\ (s_1-1)\mathbf{1} & A_{s_1} \\ \vdots & \\ (m-s_1)\mathbf{1} & A_1 \\ (m-s_1+1)\mathbf{1} & A_2 \\ \vdots & \\ (m-1)\mathbf{1} & A_{s_1} \\ \vdots & \\ (t-s_1)\mathbf{1} & A_1 \\ (t-s_1+1)\mathbf{1} & A_2 \\ \vdots & \\ (t-1)\mathbf{1} & A_{s_1} \end{bmatrix},$$

where $\mathbf{0}$ and $\mathbf{1}$ are $N/s_1 \times 1$ vectors of all zeros and all ones, respectively. Then, we have the following result.

Theorem 3. *The array B above is a $NOA((tN/s_1, mN/s_1),\ k+1, (t \times s_1 \times \cdots \times s_k, m \times s_1 \times \cdots \times s_k), 2)$.*

Proof. From the resolvability of the array A, it is easy to see that B is an $\text{OA}(tN/s_1, k+1, t \times s_1 \times \cdots \times s_k, 2)$. Also, the first mN/s_1 rows of B form an $\text{OA}(mN/s_1, k+1, m \times s_1 \times \cdots \times s_k, 2)$. □

The following example illustrates Theorem 3.

Example 6. Consider a resolvable $OA(16, 8, 4^2 \times 2^6, 2)$, displayed below in transposed form:

$$\left[\begin{array}{c|c|c|c} 0321 & 3012 & 0312 & 0132 \\ 2103 & 0321 & 0312 & 1023 \\ 0011 & 0011 & 1100 & 1010 \\ 1010 & 1010 & 0110 & 1001 \\ 0110 & 0110 & 0101 & 1100 \\ 1100 & 0011 & 1100 & 0101 \\ 1001 & 1001 & 0101 & 1100 \\ 1010 & 0101 & 0110 & 0110 \end{array}\right]',$$

where each set of four rows forms a resolvable set. Thus, $s_1 = 4$. Following Theorem 3, we have a $\mathrm{NOA}((4t, 4m), 9, (t \times 4^2 \times 2^6, m \times 4^2 \times 2^6), 2)$, where t and m are both multiples of 4 and $4 \le m < t$. For example, taking $t = 8$ and $m = 4$, one gets a $\mathrm{NOA}((32, 16), 9, (8 \times 4^2 \times 2^6, 4^3 \times 2^6), 2)$.

A simple method of obtaining a resolvable orthogonal array is as follows: Let $A^* = OA(N, k, s_1 \times s_2 \times \cdots \times s_k, 2)$ denote an orthogonal array of strength two. Clearly, N/s_1 is an integer. Without loss of generality, let the first column of A^* have symbols $0, 1, \ldots, s_1 - 1$. Permute the rows of A^* such that the first N/s_1 rows each have 0 in the first column, the next N/s_1 rows have 1 in the first column, ..., the last N/s_1 rows have the symbol $s_1 - 1$ in the first column. Deleting the first column of (the permuted) A^* leaves a resolvable orthogonal array $\mathrm{OA}(N, k-1, s_2 \times \cdots \times s_k, 2) = A$, say, i.e., $A = [A_1' \ A_2' \ \cdots \ A_{s_1}']'$, where each A_i, as before, is an orthogonal array $\mathrm{OA}(N/s_1, k-1, s_2 \times \cdots \times s_k, 1)$ of strength unity. Using Theorem 3 and the resolvable orthogonal array just constructed, one thus gets the following corollary to Theorem 3.

Corollary. *The existence of an orthogonal array $OA(N, k, s_1 \times s_2 \times \cdots \times s_k, 2)$ implies the existence of a nested orthogonal array $NOA((tN/s_1, mN/s_1), k, (t \times s_2 \times \cdots \times s_k, m \times s_2 \times \cdots \times s_k), 2)$, where t, m are integers and s_1 divides both t and m.*

The following examples illustrate the above corollary.

Example 7. Consider the (ordinary) asymmetric orthogonal array $\mathrm{OA}(12, 5,\ 3 \times 2^4, 2)$, say A, obtained by Wang and Wu [10]. Following the method described above and choosing $s_1 = 2$, we get a resolvable orthogonal array $\mathrm{OA}(12, 4, 3 \times 2^3, 2)$, displayed below in transposed form:

$$\left[\begin{array}{cccccc|cccccc} 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]'.$$

Taking $m = 2$ in Theorem 3, we have a $\mathrm{NOA}(((6t, 12), 5, (t \times 3 \times 2^3, 3 \times 2^4), 2)$, where $t \ge 4$ is an even integer. It was shown by Wang and Wu [10] that in an $\mathrm{OA}(12, k+1, 3 \times 2^k, 2)$, $k \le 4$. In view of this result, one cannot add more 2-symbol columns in the arrays $\mathrm{NOA}((6t, 12), 5,\ (t \times 3 \times 2^3, 3 \times 2^4), 2)$.

For $t = 4, 6$ for example, one obtains a $\mathrm{NOA}((24, 12), 5, (4 \times 3 \times 2^3, 3 \times 2^4), 2)$ and a $\mathrm{NOA}((36, 12), 5, (6 \times 3 \times 2^3, 3 \times 2^4), 2)$, respectively.

Example 8. Next, consider an $\text{OA}(20, 9, 5 \times 2^8, 2)$ given by Wang and Wu [10]. Following the construction described above and again choosing $s_1 = 2$, one obtains a resolvable orthogonal array $\text{OA}(20, 8, 5 \times 2^7, 2)$, displayed below:

$$\left[\begin{array}{ccccc|ccccc} 00 & 11 & 22 & 33 & 44 & 00 & 11 & 22 & 33 & 44 \\ 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 \\ 01 & 10 & 11 & 01 & 00 & 10 & 01 & 00 & 10 & 11 \\ 00 & 01 & 10 & 01 & 11 & 11 & 01 & 10 & 10 & 00 \\ 01 & 00 & 01 & 10 & 11 & 10 & 11 & 01 & 10 & 00 \\ 01 & 01 & 01 & 01 & 10 & 10 & 10 & 10 & 01 & 10 \\ 01 & 10 & 00 & 11 & 10 & 01 & 01 & 11 & 00 & 10 \\ 00 & 10 & 01 & 11 & 01 & 11 & 10 & 10 & 00 & 01 \end{array}\right]'.$$

From Theorem 3 therefore, we get a $\text{NOA}((10t, 20), 9, (t \times 5 \times 2^7, 5 \times 2^8), 2)$, where $t \geq 4$ is an even integer. It is known (Wang and Wu [10]) that in an $\text{OA}(20, k+1, 5 \times 2^k, 2)$, $k \leq 8$ and hence, no further 2-symbol columns can be added to such nested orthogonal arrays.

With $t = 4$ for example, one gets a $\text{NOA}((40, 20), 9, (5 \times 4 \times 2^7, 5 \times 2^8), 2)$ with a maximum number of columns.

4.4 Arrays by juxtaposition

A simple method of construction of (ordinary) asymmetric orthogonal arrays, leading to several new asymmetric orthogonal arrays was proposed by Suen [9]. His method can be described as follows: Let $L_1 = \text{OA}(N_1, k+1, u \times s_1 \times \cdots \times s_k, 2)$ and $L_2 = \text{OA}(N_2, k+1, v \times s_1 \times \cdots \times s_k, 2)$ be two orthogonal arrays of strength two each such that $N_1/u = N_2/v$, where the u symbols in the first column of L_1 are $0, 1, \ldots, u-1$, the v symbols in the first column of L_2 are $u, u+1, \ldots, u+v-1$, and for $1 \leq i \leq k$, the s_i symbols in the $(i+1)$st column of both L_1 and L_2 are $0, 1, \ldots, s_i - 1$. Then the array $L = [L_1' \;\; L_2']'$ is an $\text{OA}(N_1 + N_2, (u+v) \times s_1 \times \cdots \times s_k, 2)$.

From the very method of construction, it is easily seen that L in fact is a nested orthogonal array, $\text{NOA}((N_1+N_2, N_1), k+1, ((u+v) \times s_1 \times \cdots \times s_k, u \times s_1 \times \cdots \times s_k), 2)$. The orthogonal array L_1 is nested within the larger orthogonal array L. All the orthogonal arrays in Table 1 of Suen [9] are thus nested asymmetric orthogonal arrays.

Example 9. Let $L_1 = \text{OA}(24, 4 \times 6 \times 2^{11}, 2)$ and $L_2 = \text{OA}(36, 15, 6 \times 6 \times 2^{11}, 2)$, obtained by deleting two 2-symbol columns from an $\text{OA}(36, 20, 6^2 \times 2^{13}, 2)$. Then $L = [L_1' \;\; L_2']'$ is a $\text{NOA}((60, 24), 13, 10 \times 6 \times 2^{11}, 4 \times 6 \times 2^{11}), 2)$. Note that in an $\text{OA}(24, k+2, 4 \times 6 \times 2^k)$, the upper bound on k known so far is 11 and thus, it appears that in the nested array given above, no more than 11 2-symbol columns can be accommodated.

In particular, consider the case $u = v$, which, by virtue of the condition $N_1/u = N_2/v$ implies that $N_1 = N_2$. It follows then that if $L_1 = (N, k+1, u \times s_1 \times \cdots \times s_k, 2)$

and $L_2 = (N, k+1, u \times s_1 \times \cdots \times s_k, 2)$, then $L = [L_1' \ L_2']'$ is a nested asymmetric orthogonal array $\mathrm{NOA}((2N, N), k+1, ((2u) \times s_1 \times \cdots \times s_k, N, k+1, u \times s_1 \times \cdots \times s_k), 2)$.

Example 10. Let $L_1 = \mathrm{OA}(24, 14, 4 \times 3 \times 2^{12}, 2)$, obtained by replacing the 6-symbol column in an $\mathrm{OA}(24, 14, 6 \times 4 \times 2^{11}, 2)$ by two columns containing 3 and 2 symbols, respectively and $L_2 = \mathrm{OA}(24, 14, 4 \times 3 \times 2^{12}, 2)$. Then, by choosing $u = 4$, one obtains a nested array $\mathrm{NOA}((48, 24), 14, (8 \times 3 \times 2^{12}, 4 \times 3 \times 2^{12}), 2)$. Similarly, choosing $u = 3$, one gets a $\mathrm{NOA}((48, 24), 14, (6 \times 4 \times 2^{12}, 3 \times 4 \times 2^{12}), 2)$.

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