AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **53** (2012), Pages 285–288

Permanent rank and transversals

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Abstract

We use the polynomial method of Alon to give a sufficient condition for the existence of partial transversals in terms of the permanent rank of a certain matrix.

1 Introduction

Let m and n be integers, $2 \leq m \leq n$. An $m \times n$ array is a table of mn cells, arranged in m rows and n columns, with each cell containing exactly one symbol. A section of an array consists of m cells, one from each row and no two from the same column. A transversal is a section in which no two cells contain the same symbol. A partial transversal is a subset of a transversal.

A conjecture of Snevily [7] asserts that, for any odd n, every $k \times k$ sub-matrix of the Cayley addition table of $\mathbb{Z}_n$ contains a transversal. Putting it differently, for any two subsets A and B with $|A| = |B| = k$ of a cyclic group G of odd order $n \geq k$, there exist numberings $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ of the elements of A and B respectively such that the k sums $a_i + b_i$, $1 \leq i \leq k$, are pairwise different. In fact, this is also conjectured for arbitrary Abelian groups G of odd order in [7]. By using a polynomial method, Alon [1] affirmed the conjecture in the particular case when $|G|$ is a prime number. With a new application of the polynomial method, Dasgupta et al. [3] verified Snevily's conjecture for every cyclic group. By employing group rings as a tool, Gao and Wang [5] proved the conjecture for every Abelian group of odd order in the case $k < \sqrt{p}$, where p is the smallest prime divisor of $|G|$. There is also some extension of these kind of results in [8]. Recently, the Snevily's conjecture for arbitrary Abelian groups of odd order has been proved in [2]. An expository paper by Károlyi [6] collects some combinatorial problems in the additive theory and shows the polynomial method as a powerful tool. We used this technique in [4] as well.

In this paper, using the polynomial method, we give a sufficient condition for the existence of partial transversals in terms of the notion of the permanent rank of some

Vandermonde matrices. The permanent rank of a matrix has been introduced in [9]. Our main result is stated in Theorem 3.

2 Result

We start with the following result which we call the polynomial method. The proof of this result can be found in [1].

Theorem 1. *Let F be an arbitrary field and let $f = f(x_1, \ldots, x_k)$ be a polynomial in $F[x_1, \ldots, x_k]$. Suppose that there is a monomial $\prod_{i=1}^{k} x_i^{t_i}$ such that $\sum_{i=1}^{k} t_i$ equals the degree of f and whose coefficient in f is nonzero. Then, if $S_1, \ldots, S_k$ are subsets of F with $|S_i| > t_i$ then there are $s_1 \in S_1, \ldots, s_k \in S_k$ such that $f(s_1, \ldots, s_k) \neq 0$.*

In [3], the authors work with a cyclic group G and they identify it with a subgroup of the multiplicative group of a suitable field. This reduces the original problem to the study of certain Vandermonde matrices as follows.

Denote by $V(a_1, \ldots, a_k)$ the Vandermonde matrix

$$V(a_1, \ldots, a_k) = \begin{pmatrix} 1 & a_1 & \cdots & a_1^{k-1} \\ 1 & a_2 & \cdots & a_2^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & a_k & \cdots & a_k^{k-1} \end{pmatrix}.$$

For a matrix $M = (m_{ij})_{1 \leq i,j \leq k}$, the permanent of M is

$$\mathrm{per} M = \sum_{\pi \in S_k} m_{1\pi(1)} m_{2\pi(2)} \cdots m_{k\pi(k)}.$$

The following result is proved in [3].

Theorem 2. *Let F be an arbitrary field and suppose that* $\mathrm{per} V(a_1, \ldots, a_k) \neq 0$ *for some elements $a_1, \ldots, a_k \in F$. Then, for any subset $B \subset F$ of cardinality k, there is a numbering $b_1, \ldots, b_k$ of the elements of B such that the products $a_1 b_1, \ldots, a_k b_k$ are pairwise different.*

In the same manner, looking for a partial transversal, instead of using the permanent of the above Vandermonde matrix, we consider the permanent of square submatrices of it. This leads us to the notion of the permanent rank of a matrix, introduced in [9]. Define the perrank of an arbitrary matrix A to be the size of the largest square submatrix of A with nonzero permanent. Now, our result can be stated as the following theorem which is a generalization of Theorem 2.

Theorem 3. *Let F be an arbitrary field. Let $a_1, \ldots, a_k$ be arbitrary nonzero distinct elements. Suppose that the perrank of the Vandermonde matrix $V(a_1, \ldots, a_k)$ is equal to r. Then, for any subset $B = \{b_1, \ldots, b_k\} \subset F$ of cardinality k, the square matrix of order k constructed by $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ in the multiplication table of F contains a partial transversal of order r.*

Proof. Note that $1 \leq r \leq k$. If $r = k$, then we are in the case of Theorem 2 and we are done. So, we suppose that $r \leq k-1$. By the definition of perrank, we have a square submatrix A of order r of $V(a_1, \ldots, a_k)$ with nonzero permanent. Without loss of generality, suppose that such a square submatrix A is formed by the last r rows and the columns of indices $j_1, \ldots, j_r$ of $V(a_1, \ldots, a_k)$ with $1 \leq j_1 < j_2 < \cdots < j_r \leq k$. Put $V' = V(a_1x_1, \ldots, a_{k-r}x_{k-r}, a_{k-r+1}, \ldots, a_k)$ and consider the polynomial $f(x_1, \ldots, x_{k-r}) = \mathrm{per}V'$ in $F[x_1, \ldots, x_{k-r}]$. Evaluating $\mathrm{per}V'$ via the first $k-r$ rows gives

$$f(x_1, \ldots, x_{k-r}) = \sum \mathrm{per}V_1' \cdot \mathrm{per}V_2'$$

where the sum is taken over all submatrices V_1' of order $k-r$ of V' whose rows come from the first $k-r$ rows of V' and V_2' denotes the complementary submatrix of V_1' in V'. Clearly, one of the possibilities for V_2' is the submatrix A. As $\mathrm{per}A \neq 0$ by assumption, the polynomial f contains a monomial with nonzero coefficient. In fact, the same monomial cannot appear again with coefficient $-\mathrm{per}A$, because we work with a Vandermonde matrix and $a_1, \ldots, a_{k-r}$ are not zero. So we get different monomials when submatrix V_1' changes. Hence, there is a monomial $\prod_{i=1}^{k-r} x_i^{t_i}$ such that $\sum_{i=1}^{k-r} t_i$ equals the degree of f and whose coefficient in f is nonzero. Note that $t_i \leq k-1$ for all $1 \leq i \leq k-r$. So, for any subset $S \subset F$ of cardinality k with nonzero elements, we can apply Theorem 1 with $S_i = S$ for $i = 1, \ldots, k-r$ to obtain that there are $s_1, \ldots, s_{k-r} \in S$ such that $f(s_1, \ldots, s_{k-r}) \neq 0$. That is, $\mathrm{per}V(a_1', \ldots, a_k') \neq 0$, where $a_1' = a_1s_1, \ldots, a_{k-r}' = a_{k-r}s_{k-r}$ and $a_i' = a_i$ for $k-r+1 \leq i \leq k$. An application of Theorem 2 with $a_1', \ldots, a_k'$ in place of $a_1, \ldots, a_k$ completes the proof.

Acknowledgments

The author is indebted to the Research Council of Sharif University of Technology for support.

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(Received 19 Nov 2011; revised 21 Feb 2012)