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On 2-dominating kernels in graphs

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Abstract

In this paper we introduce new kinds of kernels in graphs. Using existing concepts of an independent set and a 2-dominating set, we define in a natural way the concept of 2-dominating kernels in graphs. We characterize some classes of graphs having a 2-dominating kernel, also using the idea of a local 2-dominating kernel.

1 Introduction

In general we use standard terminology and notation of graph theory; see [5], [12]. Consider a finite, connected graph G with vertex set $V(G)$ and edge set $E(G)$. The cardinality of $V(G)$ is the order of G and the cardinality of $E(G)$ is its size. By P_n we mean a graph with the vertex set $V(P_n) = \{t_1, \ldots, t_n\}$ and the edge set $E(P_n) = \{t_i t_{i+1} : i = 1, \ldots, n-1\}$, $n \geq 2$. Moreover P_1 is the graph that contains only one vertex. If $xy \in E(G)$, then we say that x is a neighbour of y. The set of all neighbours of x is called the open neighbourhood of x and it is denoted by $N(x)$. The set $N(x) \cup \{x\}$ is the closed neighbourhood and it is denoted by $N[x]$. For a subset $X \subset V(G)$ we write $N(X)$ and $N[X]$ instead of $\bigcup_{x \in X} N(x)$ and $\bigcup_{x \in X} N[x]$, respectively. By $d_G(x, y)$ we denote the distance between vertices x and y in G. For a vertex $x \in V(G)$ let $\deg_G(x)$ denote its degree.

A graph H is a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If every edge of $E(G)$ with both end-vertices in $V(H)$ is in $E(H)$, we say that H is induced by $X \subset V(G)$, and we write $H = G[X]$ or $H \leq G$ if the set X is known. If H is a subgraph of G, then we say that G is a supergraph of H. A vertex $x \in V(G)$ is a simplicial vertex if $N[x]$ induces a complete subgraph. Then $N[x]$ is named as a simplex of G. If every vertex of G belongs to a simplex, then G is a simplicial graph. In a simplicial graph a vertex which is not simplicial will be named as a nonsimplicial vertex. Let H be an arbitrary graph. For a graph G of order n, $n \geq 3$, H-addition (or ad_H) stands for a local augmentation of G which is the operation

$G \to ad_{H(x,y)}(G)$ of adding a graph H to the graph G by identifying the vertex x in G with the vertex y in H.

A vertex v of a graph G is called a cut vertex of G if $G - v$ has more components than G.

Recall that a vertex of degree 1 is called a pendant vertex (or a leaf). For $x \in V(G)$, denote by $L(x)$ the set of leaves attached to the vertex x. A vertex $x \in V(G)$ such that $L(x) \neq \emptyset$ is a support vertex. The set of support vertices in G we denote by $S(G)$. Moreover if $|L(x)| \geq 2$, then x is named as a strong support vertex. If $L(x) = 1$, then x is a weak support vertex and the unique pendant vertex adjacent to a weak support vertex will be called a single leaf. The set of pendant vertices in a graph G attached to the strong support vertices (respectively, weak support vertices) in G we denote by $L_s(G)$ (respectively, $L_w(G)$) and consequently $L(G) = L_s(G) \cup L_w(G)$ is the set of all pendant vertices in G.

A subset $S \subseteq V(G)$ is an independent set of G if no two vertices of S are adjacent in G. Moreover, a subset containing only one vertex and the empty set also are independent. An independent set S is maximal if there is no independent set of G containing S as a proper subset.

A subset $Q \subseteq V(G)$ is a dominating set of G if each vertex from $V(G) \setminus Q$ has a neighbour in Q. A subset $J \subset V(G)$ which is independent and dominating simultaneously is a kernel.

The literature includes many papers dealing with the theory of independent sets and kernels in graphs.

The classical concept of kernels in a digraph was introduced in [13] by Neumann and Morgenstern in the context of the game theory as independent and dominating sets. Since then the concept has been relevant in graph theory for its relations with a variety of problems: in list colourings and perfectness. Berge was one of the pioneers in this area studying the existence of kernels in digraphs and successfully using kernels to solve problems in others areas of mathematics; see [2].

In the literature there are some variants and generalizations of kernels, for example (k, l)-kernels which generalize kernels in distance sense or kernels by monochromatic paths which generalize kernels with respect to edge-colouring of a graph. Recently there are some interesting results concerning (k, l)-kernels and kernels by monochromatic paths in digraphs; see for instance papers [6, 7, 8, 9, 10, 11] and [15], [17, 18].

In an undirected graph every maximal independent set is a kernel. The problem is more complicated if we add additional restrictions, studying for example special kernels named in the literature as efficient dominating sets, see [1]. In this paper we study kernels in this direction.

A subset $Q \subseteq V(G)$ is a 2-dominating set of G if each vertex from $V(G) \setminus Q$ has at least two neighbours in Q. The concept of 2-dominating sets has been studied in the literature; see for example [3, 4]. In this paper we formally define a new kind of kernel in graphs using the concept of independence and 2-dominating sets of G.

A subset $J \subset V(G)$ is a 2-dominating kernel of G if J is independent and 2-dominating. Clearly a 2-dominating kernel of G is a kernel of G. From the definition of J it follows that if G is connected, then G has order at least 3 and the set J has at least two vertices. Note that if G is totally disconnected, then $V(G)$ is a 2-dominating kernel of G.

Moreover the definition of the set J implies that every pendant vertex of a graph G belongs to a 2-dominating kernel. The definition of the set J gives the following observation.

Fact 1.1 *Let $L(G) \subset V(G)$ be the set of pendant vertices of a graph G. If J is a 2-dominating kernel of G, then J is a maximal independent set of G such that $L \subseteq J$.*

Fact 1.2 *Let G be a graph of order n, $n \geq 5$. If G has a 2-dominating kernel, then $G \setminus N[L_s(G)]$ has a 2-dominating kernel.*

Maximal independent sets including the set of pendant vertices were introduced by Włoch in [16] and next they were studied in special graphs such as trees and unicyclic graphs, in particular with respect to the number $NMI_L(G)$ defined as the number of maximal independent sets including the set of pendant vertices as a subset. For trees the following has been proved.

Theorem 1.3 *[16] Let T be an n-vertex tree with $n \geq 3$. Then $NMI_L(T) \leq Pv(n-3)$, where $Pv(n)$ is the Padovan number defined by $Pv(0) = Pv(1) = Pv(2) = 1$ and $Pv(n) = Pv(n-2) + Pv(n-3)$, $n \geq 3$.*

Evidently every graph has at least one maximal independent set including the set of pendant vertices as a subset, but an arbitrary graph G does not always have a 2-dominating kernel; for example, it is easy to see that a graph P_4 does not possess a 2-dominating kernel.

In this paper we give a characterization of special classes of graphs and their products with a 2-dominating kernel.

2 Graphs with 2-dominating kernels

We first consider paths and cycles and we establish necessary and sufficient conditions for the existence of 2-dominating kernels in these graphs. These results follow by simple observation, so we omit the proofs.

Theorem 2.1 *A graph P_n has a 2-dominating kernel J if and only if $n = 2p + 1$, $p \geq 1$ and this kernel is unique. If it holds, then $|J| = p + 1$.*

Theorem 2.2 *A graph C_n has a 2-dominating kernel J if and only if $n = 2p$, $p \geq 2$ and there are exactly two disjoint 2-dominating kernels. If it holds, then $|J| = p$.*

Theorem 2.3 *If G is a graph of order n, $n \geq 3$ in which every vertex is pendant or it is a strong support vertex, then G has a 2-dominating kernel $J = L(G)$.*

Theorem 2.4 *Let $G = G(V_1, V_2)$ be a bipartite graph. If for each two pendant vertices $x, y \in V(G)$, $d_G(x, y) \equiv 0 \pmod 2$, then G has a 2-dominating kernel.*

PROOF: Assume that $G = G(V_1, V_2)$ is a bipartite graph. Let $L(G)$ be the set of pendant vertices. Because for each two pendant vertices $x, y \in V(G)$, $d_G(x, y) \equiv 0 \pmod 2$, so $L(G) \subseteq V_1$. Moreover every vertex $u \in V_2$ has degree at least two, or otherwise it would be in V_1. This immediately gives that $J = V_1$ is a 2-dominating kernel of a bipartite graph. Thus the theorem is proved. □

Next we give some necessary conditions for graphs having 2-dominating kernels.

Theorem 2.5 *Let G be a graph of order n, $n \geq 3$ and size m, $m \geq 2$. If G has a 2-dominating kernel J, then $|J| \geq n - \frac{m}{2}$. Moreover the equality holds if and only if G is a bipartite graph $G = G(V_1, V_2)$ with $V_1 = J$ and $V_2 = V(G) \setminus J$ such that for each $x \in V_2$, $\deg_G x = 2$.*

PROOF: Let J be a 2-dominating kernel of a graph G. Because J is independent, we have that the set J induces the edgeless subgraph on $|J|$ vertices. Moreover every vertex $v \in V(G) \setminus J$ has at least two edges incident to the set J, so $\deg_G v \geq 2$. Because we have $n - |J|$ vertices outside the set J, so $2(n - |J|) \leq m$. Then by simple calculations the result follows.

To prove the equality it suffices to observe that if in a bipartite graph G, for every vertex $v \in V(G) \setminus J$ we have $\deg_G v = 2$, then $m = 2(n - |J|)$, which concludes the necessity.

Assume now that $|J| = n - \frac{m}{2}$. We shall show that G is bipartite with $V_1 = J$ and $V_2 = V(G) \setminus J$ and for each $x \in V_2$, $\deg_G x = 2$. Clearly $V_1 = J$. It suffices to prove that $V(G) \setminus J$ induces an edgeless subgraph and for each $x \in V_2$, $\deg_G x = 2$. Because $|V_2| = |V(G) \setminus J| = n - |J| = \frac{m}{2}$, hence $m = 2|V_2|$. From the assumption J is 2-dominating hence for each $x \in V_2$, $\deg_G x \geq 2$ and by $m = 2|V_2|$ it immediately follows that $\deg_G x = 2$, for each $x \in V_2$. This means that no two vertices from V_2 are joined by an edge. Consequently V_2 induces the edgeless graph, so $G = G(V_1, V_2)$ is a bipartite graph. Thus the theorem is proved. □

Theorem 2.6 *Let T be an arbitrary tree on n vertices, $n \geq 3$. If T has a 2-dominating kernel J, then $|J| \geq \frac{n+1}{2}$. Moreover the equality holds for $T \simeq P_{2k+1}$, $k \geq 1$.*

PROOF: If $n = 3$, then the theorem is obvious. Let $n \geq 4$. Let J be a 2-dominating kernel of T and suppose that $J = \{u_1, \ldots, u_p\}$, where $p \geq 2$. Then there are $n - p$ vertices $x_1, \ldots, x_{n-p} \in V(T) \setminus J$. Since J is a 2-dominating set of T, then without loss of generality assume that $x_1u_1 \in E(T)$ and $x_1u_2 \in E(T)$, where $u_1 \neq u_2$. Then for $x_2 \notin J$ there is at least one vertex from the set J different from $u_1, u_2 \in J$, say u_3, such that u_3 dominates x_2. Otherwise if u_1 and u_2 dominate x_2 then there is a cycle $u_1 - x_2 - u_2 - x_1 - u_1$ in T, a contradiction that T is a tree. Proving analogously for the remaining vertices from the set $V(T) \setminus J$ we deduce that for $n-p$ vertices from $V(T) \setminus J$ we need at least $n-p+1$ vertices in the set J. Otherwise T contains a cycle, a contradiction that T is a tree. Consequently $n \geq n-p+n-p+1$ and by simple calculations we obtain that $|J| = p \geq \frac{n+1}{2}$. The equality immediately follows. Thus the theorem is proved. □

Theorem 2.7 *Let x be a weak-support vertex of G and u be the single leaf adjacent to x. If G has a 2-dominating kernel, then $(N(x) \setminus \{u\}) \setminus S(G) \neq \emptyset$.*

PROOF: Let J be a 2-dominating kernel of a graph G. Clearly a support vertex cannot be in a 2-dominating kernel because every pendant vertex has to belong to a 2-dominating kernel.

Let x be a weak-support vertex and assume on the contrary that $(N(x) \setminus \{u\}) \setminus S(G) = \emptyset$. This means that all neighbours (except the vertex u) of x are support vertices. Consequently for each $v \in (N(x) \setminus \{u\})$, $v \notin J$. This means that x has exactly one neighbour in J, a contradiction that G has a 2-dominating kernel. Thus the theorem is proved. □

Theorem 2.8 *Let G be a graph of order n, $n \geq 3$. Suppose that G contains two subgraphs isomorphic to P_t and P_m with $t, m \geq 2$ such that*

(i) P_t and P_m have an endpoint x in common, and

(ii) the endpoints of P_t and P_m distinct from x are leaves in G.

If G has a 2-dominating kernel, then t and m are either both odd or both even.

PROOF: Let J be a 2-dominating kernel of a graph G and assume on the contrary that t is odd and m is even. Let x be the common vertex of P_t and P_m. Clearly the endpoints of P_t and P_m distinct from x belong to J. Since t is odd, then by Theorem 2.1 also $x \in J$, otherwise the vertex $u \in N(x) \cap V(P_t)$ has exactly one neighbour in J, a contradiction to the fact that a graph G has a 2-dominating kernel. Because m is even the vertex $v \in N(x) \cap V(P_m)$ belongs to J, or otherwise its neighbour y, $y \neq x$, has only one neighbour in J. But then $v, x \in J$ and moreover $vx \in E(G)$, contradicting the independence of J. Thus the theorem is proved. □

The following theorems characterize graphs with 2-dominating kernels in terms of their special subgraphs.

Theorem 2.9 *Let G be a graph of order n, $n \geq 5$. Suppose that G contains p, $p \geq 2$, subgraphs isomorphic to $P_{t_1}, \ldots, P_{t_p}$ with $t_i \geq 2$, $i = 1, \ldots, p$ such that* *(i)* $P_{t_1},\ldots,P_{t_p}$ *have only a vertex* x *in common, and*

(ii) the endpoints of P_{t_i}, $i=1,\ldots,p$ *distinct from* x *are leaves in* G*, and*

(iii) $\deg_G x = p+1$.

Let either t_i, $i=1,\ldots,p$ *be odd or* t_i, $i=1,\ldots,p$ *be even. For all* t_i, $i=1,\ldots,p$ *odd, a graph* G *has a* 2*-dominating kernel if and only if* $G\setminus(\bigcup_{i=1}^{p} P_{t_i}\setminus\{x\})$ *has a* 2*-dominating kernel. For all* t_i, $i=1,\ldots,p$ *even, a graph* G *has a* 2*-dominating kernel if and only if* $G\setminus\bigcup_{i=1}^{p} P_{t_i}$ *has a* 2*-dominating kernel.*

Proof: Assume that G is as in the statement of the theorem and consider the following possibilities:

(1). t_i is odd for all $i=1,\ldots,p$.
Let J be a 2-dominating kernel of G. Since $L(G)\subseteq J$ so all endpoints of P_{t_i}, $i=1,\ldots,p$, distinct from x, belong to J. Let $J\cap V(P_{t_i})=J_i$, $i=1,\ldots,p$. Then it is clear that $J=J'\cup\bigcup_{i=1}^{p} J_i$, where $J'=J\cap V(G\setminus\bigcup_{i=1}^{p} P_{t_i})$. The definition of 2-dominating kernel and t_i odd imply that $x\in J_i$, for each $i=1,\ldots,p$. Because $\deg_G x=p+1$ in the graph $G\setminus(\bigcup_{i=1}^{p} P_{t_i}\setminus\{x\})$, the vertex x has to belong to the 2-dominating kernel J^*. Consequently we deduce that $J^*=J'\cup\{x\}$. Assume now that $G\setminus\bigcup_{i=1}^{p} P_{t_i}$ has a 2-dominating kernel J''. Let $G'=G\setminus\bigcup_{i=1}^{p} P_{t_i}$. Since $\deg_G(x)=p+1$, hence $\deg_{G'} x=1$ and $x\in J''$. Let P_{t_i} be an t_i vertex path, $i=1,\ldots,p$ with pendant vertex y_i. Let J_i be a 2-dominating kernel of the subgraph P_{t_i}, $i=1,\ldots,p$. Clearly $y_i\in J_i$, for all $i=1,\ldots,p$. Adding graphs P_{t_i}, $i=1,\ldots,p$ to the graph G' by identifying the vertex x in G with the vertex y_i in P_{t_i} we augment a graph G' to a graph G such that $J''\cup\bigcup_{i=1}^{p} J_i$ is a 2-dominating kernel of G.

(2). t_i is even for all $i=1,\ldots,p$.
Proving analogously as for the previous case, we obtain that $N(x)\cap\bigcup_{i=1}^{p} V(P_{t_i})\subset J$ and vertex $x\notin J$ is dominated by p, $p\geq 2$, vertices from $N(x)\cap\bigcup_{i=1}^{p} V(P_{t_i})$. Hence in $G''=G\setminus\bigcup_{i=1}^{p} P_{t_i}$ the set $J^*=J\setminus(\bigcup_{i=1}^{p} V(P_{t_i}))$ is a 2-dominating kernel of G''. Similarly as in case (1), using a local augmentation of a graph G', we construct a 2-dominating kernel in a graph G. Thus the theorem is proved. □

Using the same method as above we can prove.

Theorem 2.10 *Let* G *be a graph of order* n, $n\geq 5$*. Suppose that* G *contains a subgraph* P_t, $t\geq 4$*, such that the endpoint* y *of* P_t *is a leaf in* G*. Let* P'_{t-2} *be a subgraph of* P_t *including the leaf* y*. If* G *has a* 2*-dominating kernel, then* $G\setminus P'_{t-2}$ *has a* 2*-dominating kernel.*

The theorems above show that we can reduce the starting graph G to smaller a graph and instead of G we can study this subgraph with respect to the existence of a 2-dominating kernel.

Theorem 2.11 *A simplicial graph G has a 2-dominating kernel if and only if every simplex of G has exactly one simplicial vertex and every nonsimplicial vertex belongs to at least two simplices.*

PROOF: Let J be a 2-dominating kernel of a simplicial graph G. Let $M_1, \ldots, M_k$ be simplices of G. Clearly $V(G) = \bigcup_{i=1}^{k} V(M_i)$. Since every simplex is a complete subgraph, so $|J \cap V(M_i)| \leq 1$, for all $i = 1, \ldots, k$. We shall show that $|J \cap V(M_i)| = 1$, for each $i = 1, \ldots, k$. Assume on the contrary that there is a simplex M_i, $1 \leq i \leq k$, such that $J \cap V(M_i) = \emptyset$. This means that for each $x \in V(M_i)$, $N(x) \nsubseteq M_i$ and we obtain a contradiction that M_i has a simplicial vertex. Consequently $|J \cap V(M_i)| = 1$. We claim that every simplex M_i, $i = 1, \ldots, k$ has exactly one simplicial vertex. Suppose on the contrary that there is $1 \leq i \leq k$ such that M_i has two simplicial vertices $u, v \in V(M_i)$. Because J is a 2-dominating set, we deduce that $u, v \in J$, a contradiction to $|J \cap V(M_i)| = 1$. Since J is 2-dominating, for each $x \in V(M_i)$, if x is a nonsimplicial vertex then $|N(x) \cap J| \geq 2$. From the above considerations we find that every nonsimplicial vertex has to belong to at least two simplices.

Assume now that every simplex of G has exactly one simplicial vertex and every nonsimplicial vertex belongs to at least two simplices. Let $m_i \in M_i$ be a simplicial vertex, for $i = 1, \ldots, k$. We shall show that $J^* = \bigcup_{i=1}^{k} \{m_i\}$ is a 2-dominating kernel of G. Since $m_i \in M_i$, $i = 1, \ldots, k$ is a simplicial vertex, for each simplicial vertex $m_j \in M_j$, $i \neq j$ we have that $m_j \notin N[m_i]$. Consequently $m_i m_j \notin E(G)$ and this immediately gives that J^* is independent. Let $x \in V(G) \setminus J^*$. Then x is a nonsimplicial vertex and by our assumption there are simplices M_t, M_r, $t \neq r$, such that $x \in M_t$ and $x \in M_r$. Moreover, there are simplicial vertices $m_t \in M_t$ and $m_r \in M_r$ such that m_t, $m_r \in J^*$. All this gives $x m_t \in E(G)$ and $x m_r \in E(G)$, so J^* is a 2-dominating set of G. Thus the theorem is proved. □

Now in studying 2-dominating kernels we use the concept of a local 2-dominating kernel which is a useful technical tool in Kernel Theory. A local 2-dominating kernel $J \subseteq V(G)$ is an independent set that is locally 2-dominating, i.e. $N(J) \subseteq V(G) \setminus J$ and every vertex from $N(J)$ has at least two neighbours in J. It is worth mentioning that a 2-dominating kernel is a local 2-dominating kernel, while a local 2-dominating kernel J satisfying the condition $N[J] = V(G)$ is a 2-dominating kernel. A feasible solution for a graph G is a sequence of induced subgraphs and local 2-dominating kernels $< G_i, J_i >_{i=1,\ldots,n}$ such that

(1) $G_1 = G$;

(2) J_i is a local 2-dominating kernel of G_i for all $1 \leq i \leq n-1$;

(3) $G_{i+1} = G_i \setminus N[J_i]$ for all $1 \leq i \leq n-1$;

(4) J_n is a 2-dominating kernel of G_n.

Theorem 2.12 *A graph G has a 2-dominating kernel if and only if G has a feasible solution.*

PROOF: It is obvious that if J is a 2-dominating kernel, then $\langle G_1, J_1\rangle$ is a feasible solution of G.

Assume now that $\langle G_i, J_i\rangle_{i=1,\ldots,n}$ is a feasible solution of graph G. We shall show that $J = \bigcup_{i=1}^{n} J_n$ is a 2-dominating kernel of G. To prove independence of the set J, assume, for a contradiction, that there are $x, y \in J$ such that $xy \in E(G)$. Because every local 2-dominating kernel is independent and the set J is defined by union of local 2-dominating kernels, we assume that $x \in J_i$ and $y \in J_j$, for $i \neq j$. Without lose of generality assume that $i < j$. Then $y \in N[J_i]$, since J_i is a local 2-dominating kernel. Then by the definition of a feasible solution $y \notin V(G_j)$ and consequently $y \notin J_j$, a contradiction to the assumption.

Now we prove that J is 2-dominating. Let $x \in (G \setminus N[J])$ and assume on the contrary that x does not have at least two neighbours in J. This means that $x \notin N[J_i]$ for all $i = 1, \ldots, n$. In this case $x \in V(G_n) \setminus J_n$ and by the condition (4) the set J_n is 2-dominating of G_n, so it follows that x has two neighbours in J_n, a contradiction to the assumption; hence $J = \bigcup_{i=1}^{n} J_i$ also is 2-dominating. Thus the theorem is proved. □

Now we use the concept of local 2-dominating kernels to give sufficient conditions for the existence of a 2-dominating kernel in trees.

For an arbitrary tree T we define the sequence $\eta = \langle T_i, J_i\rangle_{i=1,\ldots,n}$ in a graph T

$T_1 = T$

$J_1 = L_s(T_1)$

$T_2 = T \setminus N[J_1]$

$J_2 = L_s(T_2)$ and for $3 \leq i \leq k$

$T_i = T_{i-1} \setminus N[J_{i-1}]$ for all $i < n$ such that T_n does not possess strong support vertices.

Using the result for feasible solution from Theorem 2.12 we can characterize trees with 2-dominating kernels using the concept of local 2-dominating kernels.

Theorem 2.13 *Let T be an arbitrary tree of order n, $n \geq 3$ and let $\eta = \langle T_i, J_i\rangle_{i=1,\ldots,n}$ be the sequence in T. If for each component T^* of T_n,*

(i) T^* contains only isolated vertices, or

(ii) for each $x, y \in L(T^*)$ we have $d_{T^*}(x, y) \equiv 0 \pmod 2$,

then T has a 2-dominating kernel.

PROOF: Using Theorem 2.12 it suffices to show that the sequence $\eta = \langle T_i, J_i\rangle_{1\leq i\leq n}$ is a feasible solution for a graph T. By the definition of the sequence η it suffices to show that the subtree T_n has a 2-dominating kernel. It suffices to consider only connected components of T_n. Let T^* be a connected component of T_n. If T^* has an isolated vertex, say x, then $x \in J$, where J is a 2-dominating kernel of T^*. Assume that for each two pendant vertices in a component T^*, we have $d_{T^*}(x,y) \equiv 0 \pmod 2$. Because T^* is a bipartite graph so by Theorem 2.4 the result immediately follows. Thus the theorem is proved. □

Theorem 2.14 *Let x be a cut vertex of G and let $G_1, \ldots, G_k$, $k \geq 2$ be connected components of a graph $G \setminus \{x\}$. If for each $i = 1, \ldots, k$ a subset $V(G_i) \cup \{x\}$ induces a subgraph R_i with 2-dominating kernel J_i such that $x \in J_i$, then G has a 2-dominating kernel.*

PROOF: Let x be a cut vertex of G and $G_1, \ldots, G_k$, $k \geq 2$ be as in the statement of the theorem. We shall show that $J = \bigcup_{i=1}^{k} J_i$ is a 2-dominating kernel of G. By the assumption we have $x \in J$. So it is obvious that J is a 2-dominating set because of 2-dominating sets J_i, $i = 1, \ldots, k$. Moreover, $|J_i| \geq 2$, for all $i = 1, \ldots, k$. To prove independence of J assume that $u, v \in J$. Let $u \in J_p \subset V(R_p)$, for some p, $1 \leq p \leq k$. If $v \neq x$, then either $v \in V(R_p)$ or $v \in V(R_j)$, $j \neq p$. If $v \in V(R_p)$, then $uv \notin E(G)$ by independence of J_p. If $v \in V(R_j)$, $j \neq p$, then $uv \notin E(G)$, otherwise we obtain a contradiction that x is a cut vertex of a graph G. Thus the theorem is proved. □

Now we describe a local augmentation which preserves the existence of 2-dominating kernels in a graphs.

Theorem 2.15 *Let G be an n-vertex graph $(n \geq 3)$ having a 2-dominating kernel. Then for an arbitrary $m = n + p$, $p \geq 1$, there is a supergraph R of G of order m having a 2-dominating kernel.*

PROOF: Let G be an n-vertex graph, $n \geq 3$, with a 2-dominating kernel J. For an arbitrary $p \geq 1$ we will construct a supergraph R of order $n + p$, $p \geq 1$, with 2-dominating kernel J' such that $J \subset J'$. Let x be an arbitrary vertex from $V(G) \setminus J$. We consider two possibilities:

1. $x \notin J$

Applying the local augmentation for graph $H = K_{1,p}$, $p \geq 1$ by identifying vertices $x \in V(G)$ and the centre of the of the star $K_{1,p}$ we obtain the supergraph $R \supset G$ with order $n + p$. Moreover the set $J' = J \cup L(K_{1,p})$ is a 2-dominating kernel of R.

2. $x \in J$

Then it is clear that there is $y \notin J$ such that $xy \in E(G)$. Proving, as in case 1, by replacing x by y, the result follows. Thus the theorem is proved. □

3 2-dominating kernels in $\sigma(\alpha, G)$

In this section we consider $\sigma(\alpha, G)$ of graphs with respect to the existence of 2-dominating kernels in graphs. To give necessary and sufficient conditions for graphs it is worth working with graphs whose structure can be described using smaller graphs. Clearly such results come from the study of products of graphs. Different operations of graphs allow us to build various families of graphs with 2-dominating kernels.

In this section we present necessary and sufficient conditions for the existence of 2-dominating kernels in G-join $\sigma(\alpha, G)$. Let G be a graph with $V(G) = \{x_1, \ldots, x_n\}$, $n \geq 2$, and $\alpha = (G_i)_{i=1,\ldots,n}$ be a sequence of vertex disjoint graphs on $V(G_i) = \{y_1^i, \ldots, y_{p_i}^i\}$, $p_i \geq 1$, $i = 1, \ldots, n$.
The G-join of the graph G and the sequence α is a graph $\sigma(\alpha, G)$ such that $V(\sigma(\alpha, G)) = \bigcup_{i=1}^{n} (\{x_i\} \times V(G_i))$ and $E(\sigma(\alpha, G)) = \{(x_s, y_j^s)(x_q, y_t^q); s = q \text{ and } y_j^s y_t^s \in E(G_s)$ or $x_s x_q \in E(G)\}$. By G_i^c we mean a copy of the graph G_i in $\sigma(\alpha, G)$. The G-join of graphs is a large-scale graph operation. It is interesting to mention that if $V(G_i) = V$ for $i = 1, \ldots, n$, then from G-join we obtain the generalized lexicographic product $G[G_1, \ldots, G_n]$. If $G_1 = G_2 = \ldots = G_n$, then from the G-join we obtain the composition $G[H]$ of two graphs. If $G = K_2$, then we obtain the join of two graphs $G_1 + G_2$.

Let $X \subseteq V(G)$ and $X = \{x_i; i \in \mathcal{I}\}$. If $G_j = K_1$ for $j \notin \mathcal{I}$ and G_i consists of only two isolated vertices, i.e. $|V(G_i)| = 2$ and $E(G_i) = \emptyset$ for $i \in \mathcal{I}$, then $\sigma(\alpha, G)$ gives the duplication of the subset $X \subseteq V(G)$. In particular if $|\mathcal{I}| = 1$, then we obtain the definition of the duplication of the vertex $x_i \in V(G)$.

To prove the main result of this section, firstly we give the independence lemma.

Lemma 3.1 *[14] A subset $S^* \subset V(\sigma(\alpha, G))$ is an independent set of $\sigma(\alpha, G)$ if and only if $S \subset V(G)$ is an independent set of G such that $S^* = \bigcup_{i \in \mathcal{I}} S_i$ where $\mathcal{I} = \{i : x_i \in S\}$, $S_i \subset V(G_i^c)$ is an independent set of G_i^c, for every $i \in \mathcal{I}$.*

Theorem 3.2 *A graph $\sigma(\alpha, G)$ has a 2-dominating kernel if and only if there is a 2-dominating kernel $J = \{x_i : i \in \mathcal{I}\}$, $\mathcal{I} \subset \{1, \ldots, n\}$, of the graph G such that G_i, $i \in \mathcal{I}$ has a 2-dominating kernel.*

PROOF: Assume that $\sigma(\alpha, G)$ has a 2-dominating kernel, say J^*. We shall prove that the set $J = \{x_i \in V(G) : J^* \cap V(G_i^c) \neq \emptyset\}$ is a 2-dominating kernel of G. It is obvious by Lemma 3.1 and the definition of $\sigma(\alpha, G)$ that $J^* = \bigcup_{i \in \mathcal{I}} J_i$, where $\mathcal{I} \subset \{1, \ldots, n\}$ and $\mathcal{I} = \{i; x_i \in J\}$, where J is an independent set of G. It suffices to show that the set J is 2-dominating. Let $x_t \in (V(G) \setminus J)$. Then from the definition of the set J, for each (x_t, y_p), $1 \leq p \leq p_t$, we have $(x_t, y_p) \notin J^*$. Since J^* is a 2-dominating kernel, there exist two vertices from J^*, say $(x_s, y_r), (x_k, y_l) \in S^*$, where s can be equal to k, such that $(x_t, y_p)(x_s, y_r) \in E(\sigma(\alpha, G))$ and $(x_t, y_p)(x_k, y_l) \in E(\sigma(\alpha, G))$. This means that there are two vertices $x_s, x_k \in J$ such that $x_t x_s \in E(G)$ and $x_t x_k \in E(G)$. Hence there exists a 2-dominating kernel J of G such that if $x_i \in J$, then G_i has a 2-dominating kernel. Let us now suppose that there exists a 2-dominating kernel of G, say $J = \{x_i : i \in \mathcal{I}\}$ where $\mathcal{I} \subset \{1, \ldots, n\}$ such that if $x_i \in J$, then G_i has a 2-dominating kernel.

We shall show that $\sigma(\alpha, G)$ has a 2-dominating kernel. Because G_i has a 2-dominating kernel, so G_i^c also has a 2-dominating kernel in $\sigma(\alpha, G)$. By Lemma 3.1 and the definition of the graph $\sigma(\alpha, G)$ to obtain a 2-dominating kernel of $\sigma(\alpha, G)$ we have to choose a 2-dominating kernel of G_i^c for each $i \in \mathcal{I}$. Evidently the choice of subset $J^* = \bigcup_{i \in \mathcal{I}} J_i$ is a 2-dominating kernel of $\sigma(\alpha, G)$. Thus the theorem is proved. □

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