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Super (a, d)-edge antimagic total labelings of friendship graphs

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Abstract

An (a, d)-edge-antimagic total labeling of a graph G with p vertices and q edges is a bijection f from the set of all vertices and edges to the set of positive integers $\{1, 2, 3, \ldots, p + q\}$ such that all the edge-weights $w(uv) = f(u) + f(v) + f(uv); uv \in E(G)$, form an arithmetic progression starting from a and having common difference d. An (a, d)-edge-antimagic total labeling is called a super (a, d)-edge-antimagic total labeling ((a, d)-SEAMT labeling) if $f(V(G)) = \{1, 2, 3, \ldots, p\}$. In this paper we investigate the existence of super (a, d)-edge antimagic total labeling for friendship graphs and generalized friendship graphs.

1 Introduction

Throughout this paper $G = (V, E)$ is a finite undirected graph with neither loops nor multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [5].

A labeling of a graph G is a mapping that assigns integers to the vertices or edges or both, subject to certain conditions. The labeling is called a vertex labeling or an edge labeling or a total labeling according to whether the domain of the mapping is $V(G)$ or $E(G)$ or $V(G) \cup E(G)$. If f is a total labeling, then the weight of an edge uv is defined by $w(uv) = f(u) + f(v) + f(uv)$.

An (a, d)-edge-antimagic total labeling of a (p, q)-graph G is bijection $f : V \cup E \to \{1, 2, 3, \ldots, p + q\}$ with the property that the edge-weights form an arithmetic

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progression $a, a+d, \ldots, a+(q-1)d$, where $a > 0$ and $d \geq 0$ are two fixed integers. If such a labeling exists, then G is called an (a,d)-edge-antimagic total graph. If, further, the vertex labels are the integers $\{1,2,3,\ldots,p\}$, then f is called a super (a,d)-edge-antimagic total labeling of G (or (a,d)-SEAMT labeling) and a graph which admits such a labeling is called a super (a,d)-edge-antimagic total graph. This labeling was first introduced by Simanjuntak et al. [8].

The graph F_n consisting of n triangles with a common vertex is called the friendship graph. The generalized friendship graph $\mathcal{F}_{m,p}$ consists of m cycles of orders $n_1 \geq n_2 \geq, \ldots, \geq n_m$, having a common vertex. Hence $p = \left(\sum_{i=1}^{m} n_i\right) - m + 1$. Several authors have investigated the existence of a (a,d)-SEAMT labeling of the friendship graph F_n. Slamin et al. [7] showed that the friendship graph F_n has a $(a,0)$-SEAMT labeling if and only if $n \in \{1,3,4,5,7\}$. Bača et al. [2] showed that for $n \in \{1,3,4,5,7\}$, the friendship graph F_n has $(a,0)$-SEAMT labeling and $(a,2)$-SEAMT labeling. For the friendship graph the following problem is posed in [3] and [1].

Problem 1.1. ([3], page 97, [1]) *For the friendship graph F_n, determine whether there is a $(a,2)$-SEAMT labeling, for $n > 7$.*

In this paper we investigate the above Problem and prove that F_n has no $(a,2)$-SEAMT labeling when n is even and $n \not\equiv 4 \pmod{12}$. We also prove that the generalized friendship graph $\mathcal{F}_{2,p}$ has a $(a,1)$-SEAMT labeling if and only if p is odd.

2 Friendship graphs

Let $f : V \cup E \to \{1,2,3,\ldots,p+q\}$ be an (a,d)-edge-antimagic total labeling of a graph $G = (V,E)$. Then $W = \{w(uv) : w(uv) = f(u) + f(v) + f(uv), uv \in E(G)\} = \{a, a+d, \ldots, a+(q-1)d\}$. In the computation of the edge-weights of G each edge label is used once and each label of vertex $v \in V(G)$ is used $\deg(v)$ times. Thus the following equation holds.

$$\sum_{v \in V(G)} \deg(v) f(v) + \sum_{e \in E(G)} f(e) = \sum_{e \in E(G)} w(e) \qquad (1)$$

This equation was first observed by Bača and Yousef [4], which we repeatedly use. We now proceed to investigate the existence of $(a,2)$-SEAMT labeling for friendship graphs.

Theorem 2.1. *The friendship graph F_n with $n \equiv 0 \pmod{12}$ has no $(a,2)$-SEAMT labeling.*

Proof. The order and size of F_n are given by $p = 2n+1$ and $q = 3n = 3(\frac{p-1}{2})$. Let c denote the central vertex of F_n. Suppose there exists an $(a,2)$-SEAMT labeling

$f : V \cup E \to \{1, 2, 3, \ldots, p + q\}$ for F_n and let $f(c) = i$. Since $\deg v_i = 2$ for every $v_i \neq c$ and $\deg c = 2n$, (1) becomes

$$2\left[\frac{p(p+1)}{2} - i\right] + 2ni + \frac{(p+q)(p+q+1)}{2} - \frac{p(p+1)}{2} = aq + q(q-1).$$

Solving for a, we obtain

$$a = \frac{11p^2 + 32p - 27 + 8i(p-3)}{12(p-1)} \tag{2}$$

Since $a \geq p + 4$, we get $i \geq \frac{p+7}{8}$. Further $i = f(c) \leq p$ and hence $i = \frac{p+7+8k}{8}$, where $0 \leq k \leq \frac{7(p-1)}{8}$. Substituting the value of i in (2), we get

$$a = p + 4 + \frac{2k(p-3)}{3(p-1)}, \quad 0 \leq k \leq \frac{7(p-1)}{8}.$$

Since $n \equiv 0 \pmod{12}$, we have $p = 2n + 1 \equiv 1 \pmod{24}$. Let $p = 24m + 1, m \geq 1$. Hence $a = 24m + 5 + \frac{(12m-1)k}{18m}$, $0 \leq k \leq 21m$. Since $18m$ and $12m - 1$ are relatively prime, it follows that $18m$ divides k. Hence $k = 0$ or $18m$. Hence $i = \frac{p+7}{8} = 3m + 1$ or $i = \frac{7p+1}{8} = 21m + 1$. Therefore, the centre vertex label is either $\frac{p+7}{8} = 3m + 1$ or $\frac{7p+1}{8} = 21m + 1$.

Case (i) $i = 3m + 1$.

Then $a = 24m + 5$. Obviously any edge incident with the central vertex i cannot have weight $p + 4 = 24m + 5$ or $p + 6 = 24m + 7$. Hence for the edge $e_1 = u_1v_1$ with $w(e_1) = 24m + 5$, we have $f(e_1) = p + 1$, $f(u_1) = 1$ and $f(v_1) = 2$. Now the edge $e_2 = u_2v_2$ with $w(e_2) = p + 6$ is nonadjacent to the edge e_1 and clearly there is no such edge with weight $p + 6$, a condradiction.

Case (ii) $i = 21m + 1$.

Then $a = 36m + 4$ and the maximum weight of an edge is $a + 2(q-1) = 108m + 2$. Obviously any edge incident with the central vertex i cannot have weight $108m + 2$ or $108m$. Now for the edge $e_1 = u_1v_1$ with $w(e_1) = 108m + 2$, we have $f(u_1) = 24m + 1$, $f(v_1) = 24m$ and $f(e_1) = 60m + 1$. Now, since the edge with weight $108m$ is nonadjacent with e_1, it follows that there is no edge with weight $108m$, a contradiction. Hence there is no $(a, 2)$-SEAMT labeling of F_n. □

Theorem 2.2. *The friendship graph F_n with $n \equiv 8 \pmod{12}$ has no $(a, 2)$-SEAMT labeling.*

Proof. Suppose there exists an $(a, 2)$-SEAMT labeling $f : V \cup E \to \{1, 2, 3, \ldots, p+q\}$ for F_n and let $f(c) = i$. Then proceeding as in Theorem 2.1 we have $a = p+4+\frac{2k(p-3)}{3(p-1)}$, where $0 \leq k \leq \frac{7(p-1)}{8}$. Since $n \equiv 8 \pmod{12}$, we have $p = 2n + 1 \equiv 17 \pmod{24}$. Let $p = 24m + 17, m \geq 1$. Then $a = 24m + 21 + \frac{(12m+7)k}{(3m+2)6}$, $0 \leq k \leq 7(3m + 2)$.

Since $6(3m+2)$ and $12m+7$ are relatively prime, it follows that $6(3m+2)$ divides k. Hence $k = 0$ or $18m+12$. Hence $i = \frac{p+7}{8} = 3m+3$ or $i = \frac{7p+1}{8} = 21m+15$.

Case (i) $i = 3m+3$.

Then $a = 24m+21$. Obviously any edge incident with the central vertex i cannot have weight $p+4 = 24m+21$ or $p+6 = 24m+23$. Hence for the edge $e_1 = u_1v_1$ with $w(e_1) = 24m+21$, we have $f(e_1) = p+1$, $f(u_1) = 1$ and $f(v_1) = 2$. Now the edge $e_2 = u_2v_2$ with $w(e_2) = p+6$ is nonadjacent to the edge e_1 and clearly there is no such edge with weight $p+6$, a condradiction.

Case (ii) $i = 21m+15$.

Then $a = 36m+28$ and the maximum weight of an edge is $a+2(q-1) = \frac{9p-5}{2} = 108m+74$. Obviously any edge incident with the central vertex i cannot have weight $108m+74$ or $108m+72$. Now for the edge $e_1 = u_1v_1$ with $w(e_1) = 108m+74$, we have $f(u_1) = 24m+17$, $f(v_1) = 24m+16$ and $f(e_1) = 60m+41$ and there is no edge with weight $108m+72$, a contradiction. Hence there is no $(a,2)$-SEAMT labeling of F_n. □

Theorem 2.3. *The friendship graph F_n with $n \equiv 2 \pmod 4$ has no $(a,2)$-SEAMT labeling.*

Proof. Suppose there exists an $(a,2)$-SEAMT labeling $f : V \cup E \to \{1,2,3,\ldots,p+q\}$ for F_n and let $f(c) = i$. Then proceeding as in Theorem 2.1 we have $a = p+4+\frac{2k(p-3)}{3(p-1)}$, where $0 \le k \le \frac{7(p-1)}{8}$. Since $n \equiv 2 \pmod 4$, we have $p = 2n+1 \equiv 5 \pmod 8$. Let $p = 8m+5, m \ge 1$. Then $i = m + \frac{3}{2} + k$ and $a = 8m+9+\frac{(4m+1)k}{(2m+1)3}$. Hence $k = j/2$ where j is an odd integer, and it follows that a is not an integer, a contradiction. Hence there is no $(a,2)$-SEAMT labeling of F_n. □

It follows from the above theorems that if n is even and $n \not\equiv 4 \pmod{12}$, then F_n has no $(a,2)$-SEAMT labeling.

3 Generalized friendship graph

Dafik et al. [6] proved that if a (p,q)-graph G is (a,d)-SEAMT labeling then $d \le \frac{2p+q-5}{q-1}$. Hence it follows that if the generalized friendship graph $\mathcal{F}_{2,p}$, where $p \ge 5$, admits an (a,d)-SEAMT labeling then $d < 3$.

Theorem 3.1. *The generalized friendship graph $\mathcal{F}_{2,p}$, where $p \ge 5$ has a $(2p+2,1)$-SEAMT labeling, if and only if p is odd.*

Proof. **Case (i)** p is odd.

Let $C_1 = (u_1u_2 ... u_{n_1}u_1)$ and $C_2 = (v_1 = u_1v_2v_3 \ldots v_{n_2}v_1)$ be the cycles in $\mathcal{F}_{2,p}$. Then $p = n_1+n_2-1$ and $q = p+1$. We define a bijection $f : V \cup E \to \{1,2,3,\ldots,p+q\}$

as follows:

$$
\begin{aligned}
f(v_1) = f(u_1) &= \frac{p+1}{2}\\
f(u_i) &= \frac{p+3}{2} - i, 2 \le i \le \frac{p+1}{2},\\
f\left(u_{\frac{p+1}{2}+i}\right) &= n_1 + 1 - i, 1 \le i \le n_1 - \frac{p+1}{2},\\
f(v_i) &= n_1 - 1 + i,\ 2 \le i \le n_2,\\
f(u_1u_2) &= 2p+1,\\
f(u_iu_{i+1}) &= \frac{3p+1}{2} + i,\ 2 \le i \le \frac{p-1}{2},\\
f\left(u_{\frac{p+1}{2}}u_{\frac{p+1}{2}+1}\right) &= p + n_2,\\
f\left(u_{\frac{p+1}{2}+i}u_{\frac{p+1}{2}+i+1}\right) &= p + n_2 + i,\ 1 \le i \le n_1 - \frac{p+3}{2},\\
f\left(u_1u_{n_1}\right) &= \frac{3p+1}{2},\\
f(u_1v_2) &= p + n_2 - 1,\\
f(v_iv_{i+1}) &= p + n_2 - i,\ 2 \le i \le n_2 \text{ and}\\
f\left(u_1v_{n_2}\right) &= \frac{3p+3}{2}.
\end{aligned}
$$

Then $w(u_1u_2) = 3p+1$, $w(u_iu_{i+1}) = 2p+3+\frac{p+1}{2} - i - 1$, if $2 \le i \le \frac{p-1}{2}$, $w\left(u_{\frac{p+1}{2}}u_{\frac{p+1}{2}+1}\right) = 2p+2$, $w\left(u_{\frac{p+1}{2}+i}u_{\frac{p+1}{2}+i+1}\right) = 2p+2+n_1-i$, $1 \le i \le n_1 - \frac{p+3}{2}$,

$$
w(u_1u_{n_1}) = \begin{cases} 2p+2, & if\ n_1 = \frac{p+1}{2},\\ \frac{5p+5}{2}, & if\ n_1 = p - n_2 + 1, \end{cases}
$$

$w(u_1v_2) = 2p+1+\frac{p+1}{2}$, $w(v_iv_{i+1}) = 2p+n_1+i$, if $2 \le i \le n_2$ and $w(u_1v_{n_2}) = 3p+2$. Hence the set of all edge-weights $W = \{2p+2, 2p+3, \ldots, 2p+q+1\}$. Hence f is a $(2p+2,1)$-SEAMT labeling of $\mathcal{F}_{2,p}$.

Case (ii) p is even.

The order and size of $\mathcal{F}_{2,p}$ are given by $p = n_1 + n_2 - 1$ and $q = n_1 + n_2$. Let c denote the central vertex of $\mathcal{F}_{2,p}$. Suppose there exists an $(a,1)$-SEAMT labeling $f : V \cup E \to \{1,2,3,...,p+q\}$ for $\mathcal{F}_{2,p}$ and let $f(c) = i$. Since $\deg v_i = 2$ for every $v_i \ne c$ and $\deg c = 4$, the equation (1) becomes $a = \frac{(2p+1)(p+1)+2i}{p+1}$, $1 \le i \le p$. Since p is even, it follows that a is not an integer, which is a contradiction. □

Example 3.2. *A* $(28,1)$*-SEAMT labeling of* $\mathcal{F}_{2,13}$ *and a* $(32,1)$*-SEAMT labeling of* $\mathcal{F}_{2,15}$ *are given in Figures 1 and 2.*

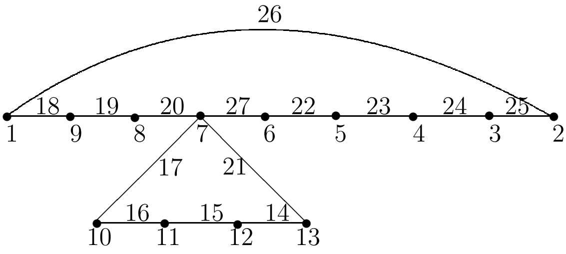

Figure 1: $\mathcal{F}_{2,13}$ has a $(28,1)$-SEAMT labeling.

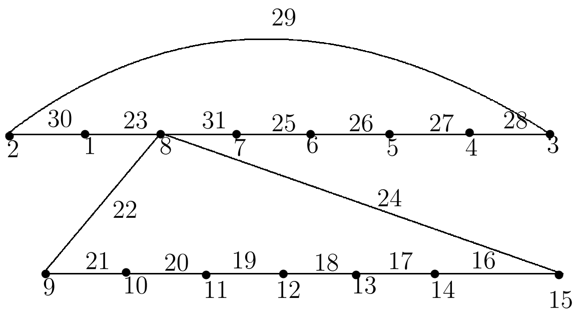

Figure 2: $\mathcal{F}_{2,15}$ has a $(32,1)$-SEAMT labeling.

4 Conclusion and Scope

We have proved that the friendship graph F_n does not admit an $(a,2)$-SEAMT labeling when n is even and $n \not\equiv 4 \pmod{12}$. The problem still remains open for other values of n. We have also proved that the generalized friendship graph $\mathcal{F}_{2,p}$, where $p \geq 5$, admits a $(2p+2,1)$-SEAMT labeling if and only if p is odd. The problem remains open for graphs $\mathcal{F}_{m,p}$ when $m \geq 3$.

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