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# Covalence sequences of transitive plane tessellations and transitive maps on surfaces

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**Abstract**

A covalence sequence of a vertex-transitive map is a cyclic sequence of covalences around a vertex of the map. We explain the algebraic background for a study of the relationship between covalence sequences of maps on compact orientable surfaces and covalence sequences of (infinite) plane tessellations that both exhibit the same 'level of transitivity' of automorphism groups on vertices or edges.

## 1 Introduction

By a *surface* we mean either a plane or a compact, connected, orientable 2-dimensional manifold. A *map* is any 2-cell embedded graph on some surface. We assume the reader's familiarity with basic concepts of the theory of maps on surfaces as explained e.g. in the monograph [2] and introduce just a few concepts that slightly differ from the terminology of [2]. The *covalence* of a face of a map is the length of the boundary walk of the face. If the supporting surface of the map is a plane and the map is an embedding of an infinite, 1-ended, 3-connected, planar graph with all vertex valences and all covalences finite, then the map is called a *plane tessellation.*

Choose a local orientation around a vertex $v$ of valence $d$ of a map. The cyclic sequence of length $d$ of covalences taken in the sense of the chosen local orientation

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is the *local covalence sequence* at $v$. A map is *vertex-homogeneous* [10], or simply *homogeneous*, if there is a cyclic sequence $\sigma$ of positive integers with the property that for every vertex $v$ of the map there is a choice of a local orientation at $v$ such that the corresponding local covalence sequence at $v$ coincides with $\sigma$. It follows that a homogeneous map necessarily has all vertices of the same valence, equal to the length of the covalence sequence. This is automatic if the map is vertex-transitive, that is, if the automorphism group of the map (which includes orientation reversing automorphisms) is transitive on vertices of the map. In the case when the automorphism group of a map contains a subgroup acting regularly on the vertex set we obtain the *Cayley maps* that have been studied in depth in [8] in the restricted case of no orientation reversing elements in the regular subgroup.

Covalence sequences of homogeneous tessellations have been studied in detail by M. E. Watkins and the first author [10], with emphasis on vertex-transitive and Cayley tessellations for which a complete characterization of covalence sequences was obtained. The paper also contains an indication that the part of the characterization that concerns covalence sequences of vertex-transitive tessellations extends to covalence sequences of finite, vertex-transitive maps on orientable surfaces in general.

The aim of this note is twofold. Firstly, in Sections 2 and 3 we outline rigorous foundations for proving results of the type just indicated, that is, transferring properties from tessellations to finite orientable maps (and back). This involves a discussion of universal covers of finite maps by tessellations, lifts of automorphisms, structure of automorphism groups of tessellations, and finally residual finiteness of such groups. Secondly, we apply these facts in Section 4 to characterization of covalence sequences of tessellations and finite maps of various 'levels of transitivity'.

## 2 Tessellations and universal covers

This section contains a selection of useful facts about universal covers of finite, orientable maps and their (orientation preserving) automorphism groups. The corresponding results have either become folklore or are scattered through various sources, ranging from basics in algebraic topology [7, Chapter V] through monographs on topological graph theory [2, Section 6.2] up to treatments of tessellations and their groups in [3, Chapter 7], [5, Chapter II] and [1, Chapters 5 and 8]; relevant facts are also contained in [11, Chapter 4].

For any integer $g \geq 1$ let $\mathcal{T}_g$ be the plane tessellation formed by (an infinite set of) congruent $4g$-gons, $4g$ of which meet at every vertex. The adjective 'congruent' refers here to the underlying plane $\mathcal{P}$ equipped with hyperbolic geometry if $g \geq 2$ and euclidean geometry if $g = 1$ (where one can identify $\mathcal{P}$ with the complex upper half-plane and the euclidean plane, respectively). We note that existence of such tessellations is a consequence of the celebrated Poincaré polygon theorem, cf. [5, 11]. The group of all orientation preserving automorphisms of $\mathcal{T}_g$ (which, at the same time, are isometries of $\mathcal{T}_g$) contains a subgroup $F_g$ acting regularly on faces of $\mathcal{T}_g$ and consisting just of translations (hyperbolic if $g \geq 2$). It is well known [7, 11] that $F_g$

is the Fuchsian group of signature $(g|-)$ with presentation

$$F_g = \langle a_1, b_1, \ldots, a_g, b_g |\ a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle \ . \qquad (1)$$

Any compact, orientable surface $\mathcal{S}_g$ of genus $g \geq 1$ can be identified with the quotient space $\mathcal{P}/F_g$. The corresponding natural projection $p: \ \mathcal{P} \to \mathcal{S}_g$ sending any point in $\mathcal{P}$ onto its orbit under the action of $F_g$ is a smooth (that is, branch-point free) regular covering and the group $F_g$ is isomorphic to the fundamental group of $\mathcal{S}_g$. The most important property of the covering $p$ we will use here is the following [11]:

**Fact 1** *If $C$ is any closed, contractible curve on $\mathcal{S}_g$ with $g \geq 1$, then $p$ restricted to any connected component of $p^{-1}(C)$ is a bijection. On the other hand, if $C$ is any closed, non-contractible curve on $\mathcal{S}_g$, then every connected component of $p^{-1}(C)$ is a two-way infinite (topological) path on $\mathcal{P}$.*

Taking inverse images, the projection $p$ enables one to lift maps. Indeed, if $M$ is a map with underlying graph $G$ on $\mathcal{S}_g$ where $g \geq 1$, then $\tilde{M} = p^{-1}(M)$ is a tessellation of the plane $\mathcal{P}$ with underlying graph $p^{-1}(G)$. We will call $\tilde{M}$ the *universal cover* of $M$. To have a different view on the way the universal cover arises, let us first realize that the image of the 1-skeleton of the tessellation $\mathcal{T}_g$ on $\mathcal{P}$ under the projection $p$ is a bouquet of $2g$ loops embedded on $\mathcal{S}_g$ with a single face and determines a standard representation of $\mathcal{S}_g$ in the form of an $4g$-sided identification polygon. The universal cover $\tilde{M} = p^{-1}(M)$ can now be visualized in $\mathcal{T}_g$ as 'multiple copies' of the 'drawing' of $M$ within such polygons representing $\mathcal{S}_g$ (with appropriate side identifications), cf. [11].

Along with lifting finite maps we will be interested in lifting their automorphisms. Let $M$ be a map on $\mathcal{S}_g$ and let $p: \ \tilde{M} \to M$ be the natural projection as above. An automorphism $A$ of $M$ is said to *lift* to an automorphism $\tilde{A}$ of $\tilde{M}$ if $p\tilde{A} = Ap$, where the composition is read from right to left. Necessary and sufficient conditions for an automorphism to lift are known in several forms and here we use the one of [7] which translates to our specific case as follows: The automorphism $A$ lifts if and only if, for each closed walk $W$ in the underlying graph of $M$, all components of $p^{-1}(W)$ are closed walks of the same length as $W$ if and only if all components of $p^{-1}(A(W))$ are closed walks of the same length as the walk $A(W)$. But from Fact 1 one sees that this condition is satisfied since the closed walks $W$ and $A(W)$ are either both contractible, in which case all components of their pre-images have the same length as $W$ (and $A(W)$), or both non-contractible, and then every component of their pre-images are two-way infinite paths. Therefore, every automorphism of $M$ lifts onto an automorphism of its universal cover. Moreover, by the general theory [7], for every automorphism $A$ of $M$, any two vertices $u, v$ of $M$, and any choice of vertices $\tilde{u} \in p^{-1}(u)$ and $\tilde{v} \in p^{-1}(v)$ there is a lift $\tilde{A}$ of $A$ such that $\tilde{A}(\tilde{u}) = \tilde{v}$. In particular, lifts of the identity automorphism of $M$ are in a one-to-one correspondence with the pre-images $p^{-1}(v)$ for any vertex $v$ of $M$. The collection of lifts of the identity form the *covering transformation group* which, in our case, is isomorphic to $F_g$. The set of all lifts of all automorphisms of $M$ constitute a group of automorphisms of the universal cover $\tilde{M}$ which can be regarded as a lift of the automorphism group $Aut(M)$ of $M$ and, again, by [7], the covering transformation group is a normal subgroup of this lift. Summarizing, we have:

**Fact 2** *If $M$ is a map on an orientable surface of positive genus, then every automorphism of $M$ lifts to an automorphism of its universal cover $\tilde{M}$. This way, the automorphism group $Aut(M)$ of $M$ lifts onto a group $K$ of automorphisms of the universal cover and can be written as the product $K = F_g \cdot Aut(M)$ with $F_g \lhd K$. In particular, if $M$ is an arc-transitive, an edge-transitive, a vertex-transitive, or a Cayley map, then so is $\tilde{M}$.*

## 3 Quotient maps and residual finiteness

So far we have looked at building tessellations from finite orientable maps of positive genus. The process can be reversed. Let $\mathcal{T}$ be a tessellation of $\mathcal{P}$ and let $H$ be a subgroup of the group $Aut^+(\mathcal{T})$ consisting just of orientation preserving automorphisms of $\mathcal{T}$. Assume further that $H$ acts freely on vertices, edges, and faces of $\mathcal{T}$ and has a finite number of orbits on the vertex set of $\mathcal{T}$. (Such assumptions are sufficient for our purposes since they greatly simplify the matter but are not necessary for development of a general theory outlined in [4].) The set of vertices, edges, and faces of the *quotient map* $\mathcal{T}/H$ is the set of orbits of $H$ on vertices, edges, and faces of $\mathcal{T}$, and incidence in the quotient map is given by non-empty intersection of orbits. The associated projection $p:\ \mathcal{T} \to \mathcal{T}/H$ that sends any map element $x$ (which may be a vertex, and edges, or a face) onto the orbit $H(x)$ of $x$ under the action of $H$ is a smooth covering. It follows from our assumptions that the supporting surface for the map $\mathcal{T}/H$ is compact, orientable, and has positive genus. Hence, by [2, 4, 11], the quotient map $\mathcal{T}/H$ can be identified with a map $M$ on some surface $\mathcal{S}_g$ with $g \geq 1$, the projection $p$ is exactly the one encountered in the previous section, the tessellation $\mathcal{T}$ can be identified with the universal cover $\tilde{M}$ of $M$, and the group $H$ must be isomorphic to $F_g$.

This viewpoint enables one to 'pull down' automorphisms of $\mathcal{T}$ onto suitable finite orientable maps. Let $G$ be a subgroup of $Aut(\mathcal{T})$ and let $H$ be a *normal* subgroup of $G \cap Aut^+(\mathcal{T})$. Then, either $H$ is normal in $G$, or $H$ has two conjugacy classes in $G$. In the first case we let $H_o = H$ and in the second case $H_o$ will be the intersection of the two conjugacy classes; either way $H_o$ is the core of $H$ in $G$. Now, any automorphism $B^* \in G$ acts on the (orientable) quotient map $M = \mathcal{T}/H_o$ by sending any element $H_o(x)$ of the quotient onto $B^*(H_o(x)) = H_o(B^*(x))$. Denoting this automorphism of $M$ by $B$, one can check that $B^*$ can be identified with the lift $\tilde{B}$ of $B$ as introduced earlier. Of course, $B$ preserves orientation if and only if $B^*$ does. Further, the supporting surface of $M$ is compact if and only if $H$ has a finite number of orbits on the vertex set of $\mathcal{T}$. Loosely speaking, *any group $G$ of automorphisms of $\mathcal{T}$ with a finite number of vertex orbits projects onto a finite, orientable, normal quotient of $\mathcal{T}$ by the core of a suitable subgroup.*

Groups of automorphisms of tessellations have a remarkable property that arises by bringing algebra and geometry together. Namely, by the available resources (e.g. [4, 5, 11]), a tessellation $\mathcal{T}$ with a subgroup $G$ of $Aut(\mathcal{T})$ having a finite number of orbits on vertices can be viewed as a rigid geometric object, with $G$ acting as a group of hyperbolic or euclidean isometries on $\mathcal{T}$. It follows that $G$ can be represented as a matrix group over a field which is a finite extension of the rationals. One can thus apply the well known theorem on representation of general groups by matrix groups [6] and conclude that $G$ (and, in fact, any of its subgroups) is a *residually finite group*, which means that for any finite subset $S$ of elements of $G$ not containing the identity there is a subgroup $G_S$ of finite index in $G$ and disjoint from $S$. Since the core of a subgroup of finite index in $G$ has also a finite index, $G_S$ may be assumed to be normal. An equivalent way of stating that $G$ is residually finite is to say that for any finite subset $S$ of $G$ not containing the identity there is an epimorphism $f$ from $G$ onto some finite group such that the restriction of $f$ onto $S$ is an injective function. Summing up, we obtain:

**Fact 3** *Any subgroup of the full automorphism group of a tessellation $\mathcal{T}$ is residually finite. If $H$ is a subgroup of $Aut^+(\mathcal{T})$ acting freely on vertices, edges, and faces of $\mathcal{T}$ with a finite number of orbits on the vertex set, then the quotient $\mathcal{T}/H$ is a map on some compact surface of positive genus, smoothly covered by $\mathcal{T}$. Moreover, if such a group $H$ is normal in $G \cap Aut^+(\mathcal{T})$ for some group $G$ of automorphisms of $\mathcal{T}$ and $H_o$ is the core of $H$ in $G$, then every automorphism in $G$ projects onto an automorphism of $M = \mathcal{T}/H_o$. In particular, if $\mathcal{T}$ is an arc-transitive, and edge-transitive, a vertex-transitive or a Cayley tessellation, then so is $M$.*

## 4 Application to covalence sequences of maps

On the basis of the theory outlined in the previous two sections we now show that, essentially, the study of covalence sequences of homogeneous tessellations with various 'levels of symmetry' is equivalent of the study of such sequences for finite, orientable maps. The central result is:

**Theorem 1** *Let $\sigma$ be a finite, cyclic sequence of positive integers. Then, $\sigma$ is a covalence sequence of a finite, homogeneous map on a compact surface of positive genus if and only if $\sigma$ is a covalence sequence of a homogeneous tessellation admitting an action of a group of automorphisms with a finite number of orbits on vertices.*

**Proof.** Let $M$ be a homogeneous map on an orientable surface of positive genus such that the covalence sequence of $M$ is $\sigma$. By Fact 2, its universal cover $\tilde{M}$ is homogeneous and has the same covalence sequence $\sigma$. Moreover, the associated group of covering transformations $F_g$ acts on $\tilde{M}$ with a finite number of orbits (corresponding to the finite set of vertices of $M$).

Conversely, let $\mathcal{T}$ be a homogeneous tessellation with covalence sequence $\sigma$, admitting a subgroup $G$ of $Aut(\mathcal{T})$ acting on the vertex set of $\mathcal{T}$ with a finite number of orbits, say, $O_1, O_2, \ldots, O_t$. For each $i$, $1 \leq i \leq t$, take a vertex $u_i \in O_i$ and denote by $V_i$ the set of all vertices $v$ of $\mathcal{T}$ such that $v$ is on the boundary of some face incident with $u_i$. Further, let $S$ be the subset of $G$ consisting of all non-identity automorphisms $f$ such that $f(v) = w$ for some $v, w \in V_i$ where $1 \leq i \leq t$. Clearly, the set $S$ must be finite. By Fact 3 (residual finiteness) there exists a normal subgroup $H$ of $G \cap Aut^+(\mathcal{T})$ of finite index such that $H$ avoids $S$. But then the quotient $M = \mathcal{T}/H$ is a map on some orientable surface of positive genus and has the same covalence sequence $\sigma$, since by $H$ avoiding $S$, the cyclic sequence of face lengths around each vertex (in some local orientation) in the quotient remains preserved. □

The same approach immediately yields results for various levels of transitivity of maps.

**Theorem 2** *Let $\sigma$ be a finite, cyclic sequence of positive integers. Then, $\sigma$ is a covalence sequence of a finite, orientable, arc-transitive (edge-transitive, vertex-transitive, and Cayley) map on a compact surface of positive genus if and only if $\sigma$ is a covalence sequence of an arc-transitive (edge-transitive, vertex-transitive, and Cayley, respectively) tessellation.*

**Proof.** As we have mentioned in the Introduction, a sketchy proof of the 'vertex-transitive' part of this statement can be found in [10, Theorem 5.1]. The two earlier sections now provide details for other levels of transitivity. In fact, it is sufficient to recall the proof of Theorem 1 together with the notation introduced there. The first part of this proof carries over almost word-by-word and the corresponding level of transitivity of the universal cover follows from Fact 2. In particular, in the Cayley case, if a subgroup $J$ of automorphisms acts regularly on vertices of a map $M$ on an orientable surface of genus $g \geq 1$, then the lifted group $J \cdot F_g$ acts regularly on the vertex set of the universal cover (see the discussion preceding Fact 2).

The second part of the proof of Theorem 1 applies with a minor modification, since the group $G$ of automorphisms of $\mathcal{T}$ may now contain orientation reversing elements. Therefore, instead of dividing out by the subgroup $H$ obtained from residual finiteness we use the core $H_o$ of $H$ in $G$, also of finite index in $G$, to form the quotient. The required level of transitivity of the quotient obtained by dividing out $\mathcal{T}$ by the subgroup $H_o$ arises as a consequence of Fact 3; for example, if $G$ is regular on the vertex set of the tessellation, then $G/H_o$ acts regularly on the vertex set of the quotient. □

The two theorems say that the decision problems for the existence of a homogeneous *tessellation* and a homogeneous *finite, orientable map* with a prescribed covalence sequence are equivalent. The point, however, is that it is much easier to work with tessellations than with finite maps, both on an intuitive as well as on a mathematically precise level. We illustrate this by two examples.

Suppose one wants to check if there is a finite, orientable, arc-transitive map with covalence sequence $\sigma = (\ell_1, \ell_2, \ldots, \ell_1, \ell_2)$ of even length $k \geq 4$ for distinct $\ell_1, \ell_2 \geq 3$. Even on an intuitive level it is not hard to visualise a tessellation of a hyperbolic plane consisting of $\ell_1$-gons and $\ell_2$-gons alternating at every vertex (of valence $k$), from which one can conclude that there are appropriate reflections in axes passing through vertices and midpoints of edges which make this tessellation arc-transitive. By Theorem 2, there exists a finite, orientable, arc-transitive map with covalence sequence $\sigma$. In fact, since there are an infinite number of choices for the subgroup $H$ in the proof of Theorem 2, there exist an infinite number of finite, orientable, arc-transitive maps with covalence sequence $\sigma$. But if one would want to build an *example* of such a finite map from scratch, it would be far from obvious where to start. The reader is invited to test this by, say, trying to construct a finite, orientable, arc-transitive map with covalence sequence $(19, 6, 19, 6)$ to realise the difficulties of doing this without resorting to tessellations.

If one allows the two entries $\ell_1$ and $\ell_2$ to be equal and drops the assumption of even length of the covalence sequence, one arrives at sequences of the form $\omega = (\ell, \ell, \ldots, \ell)$ of length $k$; to avoid maps on a sphere we will assume that $1/k+1/\ell \leq 1/2$. Existence of finite, orientable, arc-transitive maps with covalence sequence $\omega$ for arbitrary $k, \ell$ as above can again be obtained by Theorem 2 using a tessellation $T(k, \ell)$ of a plane (Euclidean or hyperbolic, depending on whether $1/k + 1/\ell$ is equal to or less than $1/2$) by $\ell$-gons, $k$ of which meet at every vertex. As an aside, arc-transitivity of $T(k, \ell)$ is implied by the presence of a $k$-fold rotational symmetry of $T(k, \ell)$ around any vertex and a 2-fold rotational symmetry of $T(k, \ell)$ about the midpoint of every edge. It follows that the corresponding finite, orientable, arc-transitive maps will be *orientably regular*, that is, for any two incident vertex-edge pairs in the map there will be exactly one orientation preserving automorphism of the map taking the first pair onto the second. Thus, by Theorem 2, for every pair $(k, \ell)$ such that $1/k + 1/\ell \leq 1/2$ there exist (infinitely many) finite, orientably regular maps of valence $k$ and covalence $\ell$. This result, well known in the theory of regular maps, was re-discovered many times; see [9] for details and for a proof that is a special version of the proof of Theorem 2.

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