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Peg solitaire on the windmill and the double star graphs

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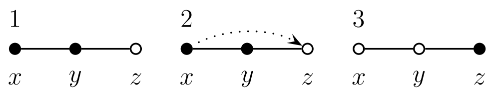
Abstract

In a recent work by Beeler and Hoilman, the game of peg solitaire is generalized to arbitrary boards. These boards are treated as graphs in the combinatorial sense. In this paper, we extend this study by considering the windmill and the double star. Simple necessary and sufficient conditions are given for the solvability of each graph. We also discuss an open problem concerning the range of values for which a graph has a terminal state with k pegs.

1 Introduction

Peg solitaire is a table game which traditionally begins with "pegs" in every space except for one which is left empty (i.e., a "hole"). If in some row or column two adjacent pegs are next to a hole (as in Figure 1), then the peg in x can jump over the peg in y into the hole in z. The peg in y is then removed. The goal is to remove every peg but one. If this is achieved, then the board is considered solved [1, 6]. For more information on traditional peg solitaire, refer to [1, 5, 6, 7].

Figure 1: A Typical Jump in Peg Solitaire



In [3], peg solitaire is generalized to graphs. A graph, $G = (V, E)$, is a set of vertices, V, and a set of edges, E. Because of the restrictions of peg solitaire, we will assume that all graphs are finite undirected graphs with no loops or multiple edges. In particular, we will *always* assume that graphs are connected. For all undefined graph theory terminology, refer to West [8]. In particular, P_n and K_n will denote the path and the complete graph on n vertices, respectively. The complete bipartite graph with $V = X \cup Y$, $|X| = n$, and $|Y| = m$ is denoted $K_{n,m}$. A vertex of degree one is a *pendant*. A vertex that is adjacent to every other vertex is called *universal* [8].

If there are pegs in vertices x and y and a hole in z, then we allow x to jump over y into z provided that $xy, yz \in E$. The peg in y is then removed. A graph G is *solvable* if there exists some vertex s so that, starting with a hole in s, there exists an associated terminal state consisting of a single peg. A graph G is *freely solvable* if for all vertices s so that, starting with a hole in s, there exists an associated terminal state consisting of a single peg. It is not always possible to solve a graph. A graph G is *k-solvable* if there exists some vertex s so that, starting with a hole in s, there exists an associated terminal state consisting of k nonadjacent pegs. In particular, a graph is *distance 2-solvable* if there exists some vertex s so that, starting with a hole in s, there exists an associated terminal state consisting of two pegs that are distance 2 apart. Graphs such as the path, cycle, the complete graph, the complete bipartite graph, and the hypercube were determined to be solvable in [3]. Those results relevant to the constructions in this paper are given in the following proposition.

Proposition 1.1 *[3]*

(i) The star $K_{1,n}$ is $(n-1)$-solvable.

(ii) The path P_n is freely solvable if and only if $n = 2$.

(iii) P_n is solvable if and only if n is even or $n = 3$.

(iv) P_n is distance 2-solvable if n is odd and $n > 3$.

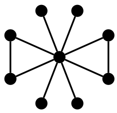
(v) The complete graph K_n is freely solvable for $n \geq 2$.

The goal of this paper is to expand on the results presented in [3]. In particular, this paper represents the first serious attempt to characterize solvable graphs and freely solvable trees. Because the more general problem of characterizing all solvable graphs may be untractable, we instead consider only the specific cases of the windmill and the double star graphs in this paper. Simple necessary and sufficient conditions are given for the solvability of these graphs.

2 Windmill Graphs

In this section, we consider the *windmill graph*. The windmill graph has a universal vertex u which is adjacent to B blades consisting of two vertices each. The blade vertices $b_1,...,b_{2B}$ satisfy $b_{2i-1}b_{2i} \in E$ for $i = 1, ..., B$. The windmill with B blades is denoted $W(B)$. We will also consider a natural variation of the windmill. In this variant, the universal vertex u is adjacent to P pendant vertices, $p_1,...,p_P$ in addition to the B blades. The windmill variant with parameters P and B is denoted $W(P, B)$ (see Figure 2).

Figure 2: The Windmill Variant — $W(4, 2)$



Theorem 2.1 *The windmill $W(B)$ is solvable for all B. The windmill $W(B)$ is freely solvable if and only if $B \neq 2$.*

Proof. We first show that $W(2)$ is not freely solvable. Suppose that the initial hole is in u. Without loss of generality, the first move is to jump from b_4 over b_3 into u. Similarly, the second is to jump from b_2 over u into b_3. Since the final two pegs are in nonadjacent vertices, $W(2)$ is not solvable when the initial hole is in u.

We now show that $W(B)$ is solvable for all B when the initial hole is in b_{2B}. If $B = 1$, then $W(1) \cong K_3$ is freely solvable. Assume that for some $B \geq 1$, $W(B)$ is solvable when the initial hole is in b_{2B}.

Consider $W(B+1)$. Place the initial hole in b_{2B+2}. Jump from b_{2B} over u into b_{2B+2}. Next, jump from b_{2B+2} over b_{2B+1} into u. Ignoring the holes in b_{2B+1} and b_{2B+2}, the remaining graph is $W(B)$ with a hole in b_{2B}. The claim then follows by induction.

To show $W(B)$ is freely solvable for $B \geq 3$, it is sufficient to show that it is solvable with the initial hole in u. Consider $W(3)$ with the initial hole in u. Jump from b_6 over b_5 into u. Next jump from b_4 over u into b_5 and from b_1 over b_2 into u. Now, jump from b_5 over u into b_4. Finally, jump from b_4 over b_3 into u. Similarly, for $W(4)$, we jump from b_8 over b_7 into u. Next, jump from b_6 over u into b_7 and from b_4 over b_3 into u. Now jump from b_2 over u into b_6. Jump from b_6 over b_5 into u and from b_7 over u into b_2. Finally, jump from b_2 over b_1 into u.

Suppose that for some $B \geq 3$, $W(B)$ is solvable with the initial hole in u and consider $W(B+2)$. First jump from b_{2B} over b_{2B-1} into u. Next jump from b_{2B+4} over u into b_{2B} and from b_{2B+1} over b_{2B+2} into u. Finally, jump from b_{2B+3} over u into b_{2B-1}. Ignoring the holes in the blades, the remaining graph is $W(B)$ with a hole in u. The claim then follows by induction. ■

Theorem 2.2 *(i) The windmill variant $W(P, B)$ is solvable if and only if $P \leq 2B$.*

(ii) *$W(P,B)$ is freely solvable if and only if $P \leq 2B-1$ and $(P,B) \neq (0,2)$.*

(iii) *$W(P,B)$ is distance 2-solvable if and only if $P = 2B+1$.*

(iv) *$W(P,B)$ is $(P-2B+1)$-solvable if $P > 2B+1$.*

Proof. Note that if there is at most one peg in each blade and the center is empty, then no moves are possible. If the center is empty, then the only possible move is to jump from b_{2i-1} over b_{2i} into u.

Suppose b_{2i-1} and b_{2i} are both empty. Jumping from p_j over u into b_{2i-1} and jumping from p_ℓ over u into b_{2i} will "refill" the blade. Thus, each blade vertex will remove at most one peg from a pendant vertex. Hence $P \leq 2B$ is necessary for $W(P,B)$ to be solvable. Moreover, an algorithm that removes two pendants for each blade will result in the minimum number of pegs.

To solve $W(P,B)$, begin with the hole in p_{P-1}. Jump from p_P over u into p_{P-1}. For the remaining moves, do the following:

(i) If u is empty, then jump from b_{2i-1} over b_{2i} into u for the largest possible i.

(ii) If u is not empty, jump from p_j over u into either b_{2B-1} or b_{2B} for the largest possible j. If however this move results in the final four pegs in b_1, b_2, b_{2B-1}, and b_{2B}, instead do the following: First, jump from b_2 over u into b_{2B} (or b_{2B-1}). Jump from b_{2B-1} over b_{2B} into u and from p_1 over u into b_2. Finally, jump from b_2 over b_1 into u.

(iii) If all pendants are empty, then the result follows from Theorem 2.1.

This algorithm will remove two pendants for each blade. Thus $W(P,B)$ is solvable when $P \leq 2B$. If $P > 2B$, then the above algorithm will leave $P-2B+1$ pegs in $p_1, \ldots, p_{P-2B}$, and b_{2B-1}. In particular, if $P = 2B+1$, then these two pegs will be distance 2 apart.

We now determine when $W(P,B)$ is freely solvable. If the initial hole is in b_{2B}, then the initial jump is from p_P over u into b_{2B}. This reduces to the case above. If the initial hole is in u, our first jump is from b_{2B} over b_{2B-1} into u. We have two possible options for our next move:

(i) Jump from b_{2B-2} over u into b_{2B} and from b_{2B-4} over b_{2B-5} into u. Ignoring the holes in b_{2B-4} and b_{2B-5}, this reduces the case of $W(P,B-2)$.

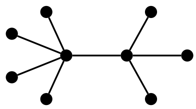
(ii) Jump from p_P over u into b_{2B-1}. We are then forced to jump from b_{2B-2} over b_{2B-3} into u. Ignoring the holes in b_{2B-2} and b_{2B-3}, this reduces to the case of $W(P-1,B-1)$.

Thus $P \leq 2B-1$ is necessary for $W(P,B)$ to be freely solvable. The other necessary condition follows from Theorem 2.1. ∎

3 Double Star Graphs

One of the problems of [3] is to characterize solvable and freely solvable trees. An important first step in this characterization is the special case of the double star. The *double star* consists of two adjacent vertices u_ℓ and u_r. The vertex u_ℓ is adjacent to L pendant vertices denoted ℓ_1,...,ℓ_L. Similarly, u_r is adjacent to R pendant vertices denoted r_1,...,r_R. Without loss of generality, assume that $L \geq R$. The double star with parameters L and R is denoted $DS(L, R)$ (see Figure 3).

Figure 3: The Double Star — $DS(4, 3)$



The double star is important in two respects. As shown in [3], to determine if a graph is (freely) solvable, it suffices to show that it has a (freely) solvable spanning subgraph. Fortunately, it is relatively easy to determine whether a graph has a double star as a spanning subgraph. Second, the results concerning the double star are particularly useful for the constructions in [2].

Theorem 3.1 *(i) The double star $DS(L, R)$ is freely solvable if and only if $L = R$ and $R \neq 1$.*

(ii) $D(L, R)$ is solvable if and only if $L \leq R + 1$.

(iii) $DS(L, R)$ is distance 2-solvable if and only if $L = R + 2$.

(iv) $DS(L, R)$ is $(L - R)$-solvable if $L \geq R + 3$.

Proof. Note that if both u_ℓ and u_r are empty, then no moves are possible. Further, if we jump from r_i over u_r into r_j (or from ℓ_i over u_ℓ into ℓ_j), then this ends the game unless the initial hole is in r_j. However, if we jump from r_i over u_r into u_ℓ, this allows a subsequent jump from ℓ_i over u_ℓ into u_r. Hence a pendant on one side of the double star can eliminate at most one from the other side. The peg in u_ℓ can eliminate one additional peg in $\{\ell_1, ..., \ell_L\}$. Hence $L \leq R + 1$ is necessary. Further, any algorithm in which each pendant in $\{r_1, ..., r_R\}$ eliminates one in $\{\ell_1, ..., \ell_L\}$ will be optimal.

Suppose that $L > R$ and that the initial hole is in u_ℓ. The first jump is from r_R over u_r into u_ℓ. Ignoring the hole in r_R, this graph is $DS(L, R - 1)$ with a hole in u_r. For this to be solvable, we must have $L \leq (R - 1) + 1$ by above. Since $L \geq R$ by hypothesis, $L = R$ is necessary for $DS(L, R)$ to be freely solvable. The other necessary condition follows from [3].

We now show that $DS(L,R)$ is solvable if $L = R$ or $L = R+1$. If $R = 0$, then $DS(0,0)$ and $DS(1,0)$ are isomorphic to P_2 and P_3, respectively. These graphs are solvable with the initial hole in u_r [3]. Assume that for some $R = k$, $DS(k,k)$ and $DS(k+1,k)$ are solvable with the initial hole in u_r. Consider $DS(k+1,k+1)$ and $DS(k+2,k+1)$.

In each case, start with the initial hole in u_r. In the first case, jump from ℓ_{k+1} over u_ℓ into u_r and from r_{k+1} over u_r into u_ℓ. Ignoring the holes in ℓ_{k+1} and r_{k+1}, the remaining graph is $DS(k,k)$. In the second case, jump from ℓ_{k+2} over u_ℓ into u_r and from r_{k+1} over u_r into u_ℓ. Ignoring the holes in ℓ_{k+2} and r_{k+1}, the remaining graph is $DS(k+1,k)$ with a hole in u_r. In either case, the claim follows by induction.

To show that $DS(k,k)$ is freely solvable when $k \neq 1$, we need only check the case where the initial hole is in a pendant, say r_{k-1}. Begin by jumping from r_k over u_r into r_{k-1}. Ignoring the hole in r_k, the remaining graph is $DS(k,k-1)$ with a hole in u_r. This is solvable by above.

If $L > R+1$, note that r_1,...,r_R will eliminate ℓ_1,...,ℓ_R by above. The subgraph induced by the remaining pegs and u_r is isomorphic to $K_{1,L-R+1}$, which is $(L-R)$-solvable [3]. In particular, if $L = R+2$, then the final two pegs are distance 2 apart. ■

The double star suggests an interesting open problem. Namely, what role, if any, does the maximum degree of a tree (or in general a graph) have in its solvability?

4 Solvability Range

An interesting open problem involves the *solvability range* for a given graph. A *terminal state* $T \subset V$ is a set of nonadjacent vertices that have pegs at the end of the game. The *solvability range* of a graph G is the set of all k such that $k = |T|$ and T is a terminal state of G. We denote the solvability range of a graph G by $SR(G)$.

In analyzing the terminal states of a graph, the following theorem is useful.

Theorem 4.1 *[3] Suppose that S is a starting state of G with associated terminal state T. Define the sets S' and T' by reversing the roles of "pegs" and "holes" in S and T, respectively. It follows that T' is a starting state of G with associated terminal state S'.*

It seems to be the case that the solvability range for most graphs (including the windmill) is an interval. In other words, $SR(G)$ contains every integer between its minimum value and its maximum value (see [4] for more information on the maximum number of pegs that can be in a terminal state of a graph). A natural conjecture is that the solvability range for *all* graphs is an interval. However, this is not the case, as we will show in the following theorem.

Theorem 4.2 *The solvability range for the double star $DS(L,R)$ is:*

(i) $SR(DS(L,R)) = \{1, ..., L+R-2, L+R\}$ *if* $L = R$*;*

(ii) $SR(DS(L,R)) = \{L-R, ..., L+R-2, L+R\}$ *if* $L > R$*.*

Proof. Let m be the minimum number of pegs in any terminal state of the double star, as given in Theorem 3.1. For elements $k \in \{m, ..., L+R-2\}$, simply use the algorithm presented in Theorem 3.1 until there are $k+1$ pegs remaining on the graph. Instead of jumping from r_i over u_r into u_ℓ (or from ℓ_i over u_ℓ into u_r), we jump from r_i over u_r into r_j for any empty pendant r_j. This will result in a terminal state with k pegs. For a terminal state with $L+R$ pegs, we begin with our initial hole in r_R. Our only jump is from u_ℓ over u_r into r_R. As $L+R$ is the independence number of $DS(L,R)$, this is the maximum number of pegs in any terminal state of $DS(L,R)$.

We claim that there is no terminal state with $L+R-1$ pegs. If there were, then we would be able to solve the double star from a starting state with three pegs by Theorem 4.1. If all three of these pegs were in pendant vertices, then no moves are possible. If $S = \{u_\ell, u_r, r_R\}$, then any jump would result in two non-adjacent pegs. A similar argument holds if $S = \{u_\ell, u_r, \ell_L\}$ or if there are pegs in exactly two pendants. ■

Thus, there is a "gap" or *lacunae* in the solvability range of the double star between $L+R-2$ and $L+R$. This gap naturally suggests several open problems regarding the solvability range:

(i) Can a lacunae in $SR(G)$ consist of more than one integer?

(ii) Can $SR(G)$ contain multiple disjoint lacunae?

(iii) Given any set of non-negative integers A, is there a connected graph G such that $SR(G) = A$? What if we allow disconnected graphs?

(iv) Alternately, given any set of non-negative integers A, is there a connected graph G such that A and $SR(G)$ are disjoint? What if we allow disconnected graphs?

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