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Unimodality of some independence polynomials via their palindromicity

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Abstract

An *independent set* in a graph G is a set of pairwise non-adjacent vertices, and the *independence number* $\alpha(G)$ is the cardinality of a maximum independent set. The *independence polynomial* of G is

$$I(G; x) = s_0 + s_1x + s_2x^2 + ... + s_\alpha x^\alpha, \alpha = \alpha(G),$$

where s_k equals the number of independent sets of size k in G (Gutman and Harary, 1983). If $s_i = s_{\alpha-i}, 0 \leq i \leq \lfloor \alpha/2 \rfloor$, then $I(G; x)$ is called *palindromic*. It is known that the graph $G \circ 2K_1$, obtained by joining each vertex of G to two new vertices, has a palindromic independence polynomial (Stevanović, 1998).

In this paper we show that for every graph G, the polynomial $I(G \circ 2K_1; x)$ is also unimodal.

1 Introduction

Throughout this paper $G = (V, E)$ is a finite, undirected, loopless and without multiple edges graph, whose vertex set is $V = V(G)$ and edge set is $E = E(G)$. $G[X]$ is the subgraph of G induced by $X \subset V$, while $G - X$ means the subgraph $G[V - X]$. The *neighborhood* of a vertex $v \in V$ is the set $N_G(v) = \{w : w \in V \ \textit{and} \ vw \in E\}$, and $N_G[v] = N_G(v) \cup \{v\}$; if there is no ambiguity on G, we use $N(v)$ and $N[v]$, respectively. A vertex v is *pendant* if its neighborhood contains only one vertex. P_n, K_n denote the cordless path on $n \geq 1$ vertices, and the complete graph on $n \geq 1$ vertices, respectively.

The *disjoint union* of the graphs G_1, G_2 is the graph $G = G_1 \cup G_2$ having as vertex set the disjoint union of $V(G_1)$, $V(G_2)$, and as edge set the disjoint union of $E(G_1)$, $E(G_2)$. In particular, nG denotes the disjoint union of $n > 1$ copies of the graph G. The *corona* of the graphs G and H is the graph $G \circ H$ obtained from G

and $|V(G)|$ copies of H, such that each vertex of G is joined to all vertices of a copy of H.

An *independent* set in G is a set of pairwise non-adjacent vertices. A largest independent set is called a *maximum independent set*, and its size is denoted by $\alpha(G)$. Let s_k equal the number of independent sets of cardinality k in G. The polynomial

$$I(G;x) = s_0 + s_1x + s_2x^2 + ... + s_\alpha x^\alpha, \alpha = \alpha(G),$$

is called the *independence polynomial* of G, (I. Gutman and F. Harary, [5]).

Theorem 1.1 *[5]* (i) $I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x)$;

(ii) $I(G;x) = I(G - v; x) + x \cdot I(G - N[v]; x)$ *holds for every* $v \in V(G)$.

A polynomial $a_0 + a_1x + a_2x^2 + ... + a_nx^n$, whose all coefficients are real, is called *(i) palindromic* (or *symmetric*) if $a_i = a_{n-i}, i = 0, 1, ..., \lfloor n/2 \rfloor$; *(ii) unimodal* if there is some $k \in \{0, 1, ..., n\}$ such that $a_0 \leq ... \leq a_{k-1} \leq a_k \geq a_{k+1} \geq ... \geq a_n$.

The unimodality of independence polynomials was studied in a number of papers, like [1, 4, 8, 12, 14, 21]. The reader is referred to [11] for a survey on this graph polynomial. Alavi, Malde, Schwenk and Erdős proved that for any permutation π of $\{1, 2, ..., \alpha\}$ there is a graph G with $\alpha(G) = \alpha$ such that $s_{\pi(1)} < s_{\pi(2)} < ... < s_{\pi(\alpha)}$, [1]. Nevertheless, for trees, they stated the following conjecture, which is still open.

Conjecture 1.2 *[1] The independence polynomial of every tree is unimodal.*

It is easy to see that if $\alpha(G) \leq 3$ and $I(G;x)$ is palindromic, then $I(G;x)$ is unimodal as well. However, there exist graphs, whose independence polynomials are palindromic and non-unimodal; e.g., $I(K_{52}+3K_4+4K_1;x) = 1+68x+\mathbf{54}x^2+68x^3+x^4$. Recall that $G_1 + G_2$ is the graph (called the *Zykov sum*) with $V(G_1) \cup V(G_2)$ as a vertex set and $E(G_1) \cup E(G_2) \cup \{v_1v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}$ as an edge set.

Theorem 1.3 *[2] If P and Q are both unimodal and palindromic, then $P \cdot Q$ is unimodal and palindromic.*

The palindromicity of matching polynomial of a graph was investigated in [10], while for independence polynomial we cite [7, 13, 15, 16, 20]. There are several ways to build graphs with palindromic independence polynomials; e.g., see [3, 6, 19].

Theorem 1.4 *[19] The polynomial $I(G \circ 2K_1; x)$ is palindromic for every graph G.*

It is known that if all the roots of $I(G;x)$ are real, then: *(i)* $I(G;x)$ is unimodal (by a theorem of Newton, [9]); *(ii)* $I(G \circ 2K_1; x)$ has only real roots, [17]. Therefore, $I(G \circ 2K_1; x)$ is unimodal and palindromic for each claw-free graph G, since, in this case, $I(G;x)$ has only real roots, [4]. In particular, the same is true for $I(P_n \circ 2K_1; x)$, [21].

In this note we prove that $I(G \circ 2K_1; x)$ is unimodal for every graph G.

2 Results

It is clear that the sum of two unimodal (palindromic) polynomials of the same degree is unimodal (palindromic, respectively). The following lemma, which will play a key role in what follows, is less evident.

Lemma 2.1 *[3] Let $p(x)$ and $q(x)$ be polynomials of degree r and $r-1$ respectively, for some $r \geq 2$, and let $p(0) \neq 0$ and $q(0) = 0$. If $p(x)$ and $q(x)$ are palindromic and unimodal, then so is $p(x) + q(x)$.*

Now we can prove our assertion. It is worth mentioning that, in a similar manner, Schwenk showed that the matching polynomial of a graph is unimodal, [18].

Theorem 2.2 *The polynomial $I(G \circ 2K_1; x)$ is unimodal and palindromic, for every graph G.*

Proof. The palindromicity of $I(G \circ 2K_1; x)$ follows from Theorem 1.4.

We show that $I(G \circ 2K_1; x)$ is unimodal by induction on the order n of G.

If $n = 1$, then $G = K_1$ and $I(G \circ 2K_1; x) = I(P_3; x) = 1 + 3x + x^2$, which is clearly unimodal.

If $n = 2$, then either $G = K_2$ and

$$I(G \circ 2K_1; x) = I(K_2 \circ 2K_1; x) = 1 + 6x + 10x^2 + 6x^3 + x^4,$$

or $G = 2K_1$ and

$$I(G \circ 2K_1; x) = I(2K_1 \circ 2K_1; x) = \left(1 + 3x + x^2\right)^2.$$

In both cases, $I(G \circ 2K_1; x)$ is clearly unimodal.

Let G be a graph of order $n \geq 3$.

If $E(G) = \emptyset$, then Theorem 1.1*(i)* implies that $I(G \circ 2K_1; x) = (1 + 3x + x^2)^n$, which is unimodal, according to Theorem 1.3.

Assume that $E(G) \neq \emptyset$, and let $v \in V(G)$ be with $N_G(v) = \{u_i : 1 \leq i \leq k\}$.

Notice that $G \circ 2K_1 - v$ is the disjoint union of $2K_1$ and $(G - v) \circ 2K_1$. The graph $G \circ 2K_1 - N_{G \circ 2K_1}[v]$ consists of the disjoint union of $k(2K_1)$ (i.e., the "new" neighbors of $N_G(v)$ in $G \circ 2K_1 - V(G)$) and $(G - N_G[v]) \circ 2K_1$.

Applying Theorem 1.1, we obtain

$$I(G \circ 2K_1; x) = I(G \circ 2K_1 - v; x) + x \bullet I(G \circ 2K_1 - N_{G \circ 2K_1}[v]; x) = p(x) + q(x),$$

where

$$p(x) = (1 + x)^2 \bullet I((G - v) \circ 2K_1; x) \text{ and}$$
$$q(x) = x \bullet (1 + x)^{2k} \bullet I((G - N_G[v]) \circ 2K_1; x).$$

Firstly, $\alpha\left(G \circ 2K_1\right) = \alpha\left(G \circ 2K_1 - v\right) = 2n$, because the graph $G \circ 2K_1$ has a unique maximum independent set, say S, consisting of only pendant vertices, and $S \cap V\left(G\right) = \emptyset$, while $v \in V\left(G\right)$.

Secondly, it is easy to see that

$$\alpha\left(G \circ 2K_1 - N_{G \circ 2K_1}\left[v\right]\right) = \left|S - \{a_1, a_2\}\right| = \alpha\left(G \circ 2K_1\right) - 2,$$

where $\{a_1, a_2\} = S \cap N_{G \circ 2K_1}\left(v\right)$.

Consequently, we infer that $\deg p = \deg q + 1 \geq 2$.

According to Theorem 1.4, the polynomial $I\left(\left(G - v\right) \circ 2K_1; x\right)$ is palindromic, and by induction hypothesis, $I\left(\left(G - v\right) \circ 2K_1; x\right)$ is also unimodal. Theorem 1.3 assures that $p\left(x\right)$ is palindromic and unimodal as well.

Similarly, one can deduce that $\left(1 + x\right)^{2k} \bullet I\left(\left(G - N_G\left[v\right]\right) \circ 2K_1; x\right)$ is palindromic and unimodal.

Since $p\left(0\right) = 1$, while $q\left(0\right) = 0$, we finally obtain that $I(G \circ 2K_1; x) = p\left(x\right) + q\left(x\right)$ is unimodal and palindromic, according to Lemma 2.1. ■

It is easy to see that $G \circ 2K_1$ is a tree if and only if G is a tree, and consequently, Theorem 2.2 gives some support to Conjecture 1.2.

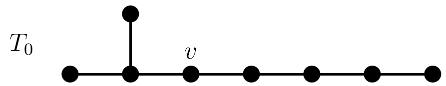

Figure 1: A tree with a palindromic independence polynomial.

Notice that the independence polynomial of the tree T_0 from Figure 1 is palindromic and unimodal, as

$$I(T_0; x) = I(T_0 - v; x) + x \bullet I(T_0 - N[v]; x) = 1 + 8x + 21x^2 + 21x^3 + 8x^4 + x^5.$$

Taking into account the structure of T_0 and that $\alpha\left(T_0\right) = 5$, we propose the following.

Problem 2.3 *Show that if a tree H has $\alpha\left(H\right) \geq 6$ and $I\left(H; x\right)$ is palindromic, then $H = T \circ 2K_1$, for some tree T.*

By inspection, one can deduce that there is no tree T with $\alpha(T) = 3$, such that $I(T; x)$ is palindromic. The trees K_1, $K_1 \circ 2K_1$ and $K_2 \circ 2K_1$ have $\alpha\left(K_1\right) = 1$, $\alpha\left(K_1 \circ 2K_1\right) = 2$, $\alpha\left(K_2 \circ 2K_1\right) = 4$, and their independence polynomials are palindromic and unimodal. Therefore, if Problem 2.3 has an affirmative answer, then one can assert that whenever a tree T has a palindromic independence polynomial, then its $I(T; x)$ is unimodal as well.

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