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Graphs with the cycle extension property

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Abstract

A connected graph G is said to have the *cycle extension property* (or more briefly, "G is CEP") if it contains a 2-factor and for every cycle C in G with $|V(C)| \leq |V(G)| - 3$, there is a 2-factor F_C in G which includes C.

We characterize all graphs which are CEP.

1 Introduction

Considerable attention in graph theory has been paid to the theme of determining when partial structures of various types can be extended to larger structures of the same type. For example, a *well-covered* graph is one in which every independent set of vertices can be extended to an independent set of maximum size. (See [3, 5, 12, 18].) Similarly, an *equimatchable* graph is one in which every matching (i.e., set of independent edges) extends to maximum matching. (See [7, 8, 10].) The most activity in this general area has taken place in the study of matching extension. A graph G is said to be n-*extendable* if every matching of size n extends to (i.e., is a subset of) a perfect matching. (See [13, 14, 17, 21].)

In the present paper we investigate the idea of extending cycles to 2-factors in graphs. A 2-*factor* of a graph G is a collection of vertex-disjoint cycles which together span the vertices of G. (For general references to graph factors and factorizations, the reader is referred to [1, 9, 11, 15, 16].) We will say that a graph G has the *cycle* *extension property* (or more briefly, G is CEP) if G has a 2-factor and every cycle in G of length no more than $|V(G)|-3$ belongs to a 2-factor of G. We will characterize all such graphs.

As general references for graph terminology, we direct the reader to [2] and [20]. If x and y are vertices in a graph G, we will write $x \sim y$ and $x \not\sim y$ to denote their adjacency and non-adjacency respectively.

2 The main results

First, we characterize those CEP graphs which contain no cutvertex.

Theorem 1. *If G is a 2-connected CEP graph, then G is one of the following graphs:*

(1) $C_n;\ n \geq 3$

(2) $K_n;\ n \geq 3$

(3) $K_{n,n};\ n \geq 3$

(4) hamiltonian graph on at most 5 *vertices, or*

(5) one of the nine graphs shown in Figure 1.

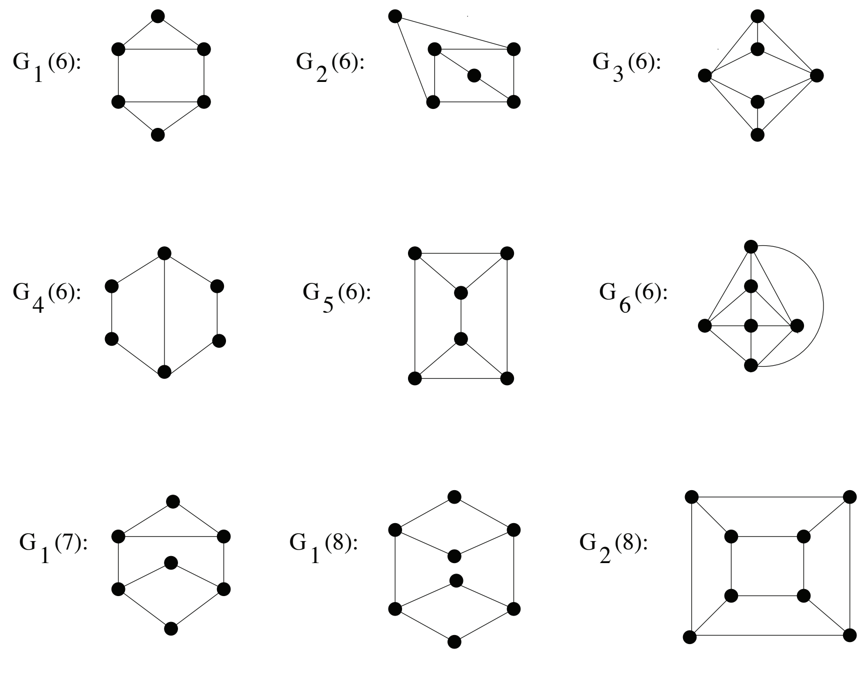

Figure 1

Proof: It is easy to show that all graphs in classes (1)–(5) are CEP graphs.

So let us assume that G is an arbitrary 2-connected CEP graph.

Suppose first that G does not contain a separating cycle. All (connected) graphs without separating cycles are characterized in Thomassen [19] and it is easily checked that the only ones which are CEP are those in classes (1)–(4) together with graphs $G_4(6)$ and $G_5(6)$ found in class (5).

So let us suppose that G is CEP and contains a separating cycle. The goal is to prove that all such graphs are also found in Figure 1. If $|V(G)| \leq 5$, then G is Hamiltonian since it must contain a 2-factor. Thus G belongs to class (4). Henceforth, therefore, we shall assume that $|V(G)| \geq 6$.

Among all such separating cycles in G, choose one, call it C_0, such that one of the components of $G - V(C_0)$ is as small as possible. Denote such a smallest component by H_1.

Since G is 2-connected, there are at least two vertices of attachment for H_1 on cycle C_0. Choose two such vertices of attachment u and v such that the path from u to v on C_0 (taken clockwise, say) is as short as possible. Now choose u' and v' in H_1 such that $u' \sim u$ and $v' \sim v$. (Note that it may be that $u' = v'$.)

Let Q be a shortest $u'-v'$ path in H_1. Then $C_0' = C_0 - C_0(u,v) + uu' + vv' + Q$ is a cycle and if $V(H_1) - V(Q) \neq \emptyset$, C_0' contradicts the choice of C_0. Hence $V(H_1) = V(Q)$. But since Q was chosen as a *shortest* $u' - v'$ path in H_1, H_1 contains no cycles and hence since G is CEP, $|V(H_1)| \leq 2$. But since C_0 is a separating cycle, there is a second component of $G - V(C_0)$ different from H_1, call it H_2, and again since G is CEP, it follows that $|V(H_1)| = |V(H_2)| = 1$ and $|V(C_0)| = |V(G)| - 2$.

Let us denote the two singleton components of $G - V(C_0)$ by u and v. Each of u and v must be joined to two different vertices on C_0 since G is 2-connected.

Let us denote cycle C_0 by $x_0, x_1, \ldots, x_{n-3}$ viewed clockwise around the cycle. Let the two singleton components separated by C_0 be u and v. Henceforth, if G is CEP and C_i in G is a cycle of length at most $|V(G)| - 3$, F_i will denote a 2-factor containing C_i.

Claim 1: Cycle C_0 has no chords.

Suppose, to the contrary, that x_0x_k is a chord of C_0 (in which case we call x_0x_k a *k-chord*) and, moreover, assume that this chord is such that the cycle $C_1 = x_0x_1 \cdots x_kx_0$ is a shortest possible cycle formed by a chord, together with a subpath of C_0.

We make the following observations (in which the roles of u and v may be interchanged):

(i) Since $C_2 = x_0x_kx_{k+1} \cdots x_{n-3}x_0$ extends to a 2-factor F_2 of G, each of u and v has at least two distinct neighbors in $A = \{x_1, x_2, \ldots, x_{k-1}\}$, and thus $k \geq 3$;

(ii) Again, since cycle C_2 must extend to a 2-factor, and since C_1 was chosen as a shortest cycle formed by a chord and a subpath of C_0, $\{x_1, x_{k-1}\} \subseteq N_G(u) \cup N_G(v)$;

(iii) Since $C_3 = x_0 \cdots x_a u x_b \cdots x_k x_{k+1} \cdots x_{n-3} x_0$ must extend to a 2-factor, it follows that whenever $\{x_a, x_b\} \subseteq N_G(u) \cap A$ and $b \geq a+3$, $\{x_{a+1}, x_{b-1}\} \subseteq N_G(v)$;

(iv) Again, since cycle C_2 must extend to a 2-factor, without loss of generality we have either:

(1) $\{x_1, x_{k-1}\} \subseteq N_G(u)$ and $\{x_2, x_{k-2}\} \subseteq N_G(v)$;

(2) $\{x_1, x_j\} \subseteq N_G(u)$ and $\{x_{j-1}, x_{k-1}\} \subseteq N_G(v)$ for some $x_j \in A - \{x_1\}$; or

(3) $\{x_1, x_j\} \subseteq N_G(u)$ and $\{x_{j+1}, x_{k-1}\} \subseteq N_G(v)$ for some $x_j \in A - \{x_1\}$.

Moreover, in light of Observation (iii) above, we see that option (iv)(3) is only required if $N_G(u) \cap A \subseteq \{x_1, x_2, x_3\}$.

Case 1.1 ($k = 3$): Suppose x_0x_3 is a 3-chord. Then $|V(G)| \geq 8$. Since cycle $C_2 = x_0x_3x_4 \cdots x_{n-3}x_0$ must extend, edges ux_1, ux_2, vx_1 and vx_2 must be present.

1.1.1: Suppose $n = 8$. By symmetry, we may suppose that both u and v are adjacent to x_4 and x_5 as well. Then cycle $C_4 = x_0x_1x_2vx_5x_0$ implies that $u \sim x_3$ and then by symmetry, $u \sim x_0, v \sim x_3$ and $v \sim x_0$ as well. However, then cycle $C_5 = ux_1vx_5x_0u$ does not extend by the minimality of k.

1.1.2: So suppose $n \geq 9$. If $v \sim x_6$, cycle $C_6 = ux_2vx_6 \cdots x_{n-3}x_0x_1u$ does not extend, so $v \not\sim x_6$. Also note that by the minimality of k, path $x_4x_5x_6$ does not contain a chord.

Suppose, for some $i, 7 \leq i \leq n-3$, (a) $v \not\sim x_6, \ldots, x_{i-1}$ and (b) path $x_4x_5x_6 \cdots x_{i-1}$ does not have a chord.

Suppose $v \sim x_i$. Then $C_7 = ux_2x_3x_0x_{n-3} \cdots x_ivx_1u$ cannot be extended to a 2-factor covering vertex x_{i-1}. So $v \not\sim x_i$.

Assume next that $x_4x_5x_6 \cdots x_i$ has a chord e. Then e must join x_i to some x_j, where, by the minimality of k, $4 \leq j \leq i-3$. But then $C_8 = ux_2x_3x_4 \cdots x_jx_i \cdots x_{n-3}x_0x_1u$ cannot be extended to cover vertex v. So $x_4x_5x_6 \cdots x_i$ has no chord.

So by induction, v is not adjacent to any of $x_6, x_7, \ldots, x_{n-3}$ and path $x_4x_5 \cdots x_{n-3}$ has no chord. Then cycle $C_9 = ux_2x_3x_0x_1u$ extends and so v is adjacent to both x_4 and x_5 in the 2-factor F_9. But then F_9 cannot cover x_{n-3}, a contradiction. So C_0 contains no 3-chord.

Case 1.2 ($k = 4$): Suppose C_0 contains a 4-chord x_0x_4. So $|V(G)| \geq 10$. Cycle $C_{10} = x_0x_4x_5 \cdots x_{n-3}x_0$ extends and so u, v, x_1, x_2 and x_3 must lie on a common 5-cycle Z. Moreover, by the minimality of k, $x_1 \not\sim x_3$. Without loss of generality, we may assume $Z = ux_1x_2vx_3u$.

If $v \sim x_7$, then by the minimality of k, cycle $C_{11} = ux_3x_2vx_7x_8 \cdots x_{n-3}x_0x_1u$ cannot extend. So $v \not\sim x_7$. Moreover, also by the minimality of k, $x_5x_6x_7$ has no chord.

Now assume for some $i, 8 \leq i \leq n-3$, (a) $v \not\sim x_7, \ldots, x_{i-1}$ and (b) $x_5x_6x_7 \cdots x_{i-1}$ has no chord.

Suppose $v \sim x_i$. Then cycle $C_{12} = ux_3x_2vx_ix_{i+1} \cdots x_{n-3}x_0x_1u$ implies there must be a chord joining x_{i-1} to x_4. In the 2-factor extension F_{13} of $C_{13} = ux_3x_4x_{i-1}x_i \cdots x_{n-3}$ x_0x_1u, since by the minimality of k, $x_5 \not\sim x_2$, F_{13} must use edge x_5v. So consider cycle $C_{14} = ux_3x_2vx_5x_4x_{i-1}x_i \cdots x_{n-3}x_0x_1u$. If $i \geq 10$, then F_{14} cannot cover vertex x_6 by (b) and we have a contradiction.

Suppose next that $i = 9$. Then cycle $C_{15} = x_0x_1x_2x_3x_4x_5vx_9 \cdots x_{n-3}x_0$ must use edge ux_8 in order to extend. But then $C_{16} = ux_1x_2x_3x_4x_0x_{n-3} \cdots x_8u$ cannot extend, since we cannot cover vertex x_7 by the minimality of k.

So suppose $i = 8$. But in this case, chord x_4x_7 contradicts the minimality of k. So suppose $v \not\sim x_i$.

Assume now that $x_5x_6x_7 \cdots x_i$ has a chord e. Then by (b), e must join x_i to some vertex $x_j, 5 \leq j \leq i-4$. But then $C_{17} = x_0x_1x_2x_3x_4x_5 \cdots x_jx_ix_{i+1} \cdots x_{n-3}x_0$ does not extend, since F_{17} cannot cover vertex v. So $x_5x_6 \cdots x_i$ has no chord.

So by induction, $v \not\sim x_7, x_8, \ldots, x_{n-3}$ and path $x_5x_6x_7 \cdots x_{n-3}$ has no chord. Then cycle $C_{18} = x_0x_1x_2x_3x_4x_0$ implies that 2-factor F_{18} must cover vertex v using the two edges x_5v and x_6v. But then, if $n \geq 11$, cycle $C_{19} = ux_3x_2vx_5x_4x_0x_1u$ does not extend, a contradiction.

So suppose $n = 10$. Then cycle $C_{20} = x_0x_1x_2vx_5x_6x_7x_0$ requires that edge ux_4 be used in its extension. But then $C_{21} = ux_4x_0x_1x_2vx_3u$ cannot extend by the minimality of k.

Case 1.3 ($k \geq 5$): Suppose C_0 has a k-chord x_0x_k. Then $|V(G)| \geq 2k+2$.

We shall consider the three cases indicated in Observation (iv) separately.

Case 1.3.1: Suppose first that (iv)(1) applies, i.e. $\{x_1, x_{k-1}\} \subseteq N_G(u)$ and $\{x_2, x_{k-2}\}$ $\subseteq N_G(v)$.

If $v \sim x_{k+3}$, then cycle $C_{22} = vx_{k+3}x_{k+4} \cdots x_{n-3}x_0x_1ux_{k-1}x_{k-2} \cdots x_2v$ does not extend by the minimality of k, so $v \not\sim x_{k+3}$. Moreover path $x_{k+1}x_{k+2}x_{k+3}$ has no chord by the minimality of k.

Assume now that for some $i, k+4 \leq i \leq n-3$, (a) $v \not\sim x_{k+3}, \ldots, x_{i-1}$ and (b) path $x_{k+1}x_{k+2} \cdots x_{i-1}$ has no chord.

If $v \sim x_i$, then $C_{23} = vx_ix_{i+1} \cdots x_{n-3}x_0x_kx_{k-1}ux_1x_2 \cdots x_{k-2}v$ does not extend, since $i \geq k+4$ and by hypothesis (b), so $v \not\sim x_i$. Suppose that path $x_{k+1} \cdots x_i$ has a chord e. Then e must join x_i to some $x_{i'}, k+1 \leq i' \leq i-4$. Then $C_{24} =$ $x_0x_1x_2x_3x_4x_5 \cdots x_i'x_i \cdots x_{n-3}x_0$ cannot extend to a 2-factor which covers vertex v (since $|N_G(v) \cap (V(G) - V(C_4))| \leq 1$).

So by induction, we have proved that (a) $v \not\sim x_{k+3}, \ldots, x_{n-3}$ and (b) $x_{k+1}x_{k+2} \cdots$ x_{n-3} has no chord. But then if $C_{25} = x_0x_1 \cdots x_kx_0$, then 2-factor F_{25} must use edges vx_{k+1} and vx_{k+2}. However, then since $|V(G)| \geq 2k+2 \geq 12$, cycle $C_{26} =$ $ux_{k-1}x_{k-2} \cdots x_2vx_{k+1}x_kx_0x_1u$ does not extend by (b).

Case 1.3.2: Suppose next that (iv)(2) applies, i.e. $\{x_1, x_j\} \subseteq N_G(u)$ and $\{x_{j-1}, x_{k-1}\}$ $\subseteq N_G(v)$ for some $x_j \in A - \{x_1\}$.

By Observation (iii), and since $k \geq 5$, we must have $x_{k-2} \in N_G(u)$ and $x_2 \in N_G(v)$. Moreover, $x_3 \in N_G(u)$, since if $k = 5$, $k-2 = 3$ and if $k \geq 6$, since $\{x_2, x_{k-1}\} \subseteq N_G(v)$ we may apply Observation (iii) again.

If $v \sim x_{k+3}$, then cycle $C_{27} = vx_{k+3}x_{k+4} \cdots x_{n-3}x_0x_1ux_{k-2}x_{k-3} \cdots x_2v$ does not extend, since path $x_{k-1}x_kx_{k+1}x_{k+2}x_{k+3}$ has no chord by the minimality of k. So $v \not\sim x_{k+3}$.

Assume now that for some $i, k+4 \leq i \leq n-3$, (a) $v \not\sim x_{k+3}, \ldots, x_{i-1}$ and (b) path $x_{k+1}x_{k+2} \cdots x_{i-1}$ has no chord.

Suppose $v \sim x_i$. Then by (b), cycle $C_{28} = vx_ix_{i+1} \cdots x_{n-3}x_0x_kx_{k-1}x_{k-2} \cdots x_3ux_1x_2v$ does not extend by hypothesis (b). So $v \not\sim x_i$. Suppose $x_{k+1}x_{k+2} \cdots x_i$ has a chord e. Then by (b), and the minimality of k, e must join x_i to some $x_{i'}, k+1 \leq i' \leq i-4$. But then cycle $C_{29} = ux_{k-2}x_{k-3} \cdots x_2vx_{k-1}x_k \cdots x_{i'}x_i \cdots x_{n-3}x_0x_1u$, cannot extend to cover vertex x_{i-1}.

So by induction, (a) $v \not\sim x_{k+3}, \ldots, x_{n-3}$ and $x_{k+1}x_{k+2} \cdots x_{n-3}$ does not have a chord. But then cycle $C_{30} = ux_{k-2}x_{k-3} \cdots x_2vx_{k-1}x_kx_0x_1u$ does not extend.

Case 1.3.3: Finally, we assume that Observation (iv)(3) applies i.e. $\{x_1, x_j\} \subseteq N_G(u)$ and $\{x_{j+1}, x_{k-1}\} \subseteq N_G(v)$ for some $x_j \in A - \{x_1\}$. Moreover, we have $N_G(u) \cap A \subseteq \{x_1, x_2, x_3\}$ and so $x_j = x_2$ or $x_j = x_3$.

If $v \sim x_{k+3}$, then consider cycle $C_{31} = vx_{k+3}x_{k+4} \cdots x_{n-3}x_0x_1ux_{j'}x_{j'+1} \cdots x_{k-1}v$, where $j' = 2$ if $ux_2 \in E(G)$, and $j' = 3$ otherwise. We claim that C_{31} does not extend. To see this, note that path $x_kx_{k+1}x_{k+2}$ has no chord and x_2 is adjacent to neither x_k nor x_{k+1}, since x_0x_k is a shortest chord of C_0. So if $j' = 2$, F_{31} cannot cover x_k, x_{k+1} and x_{k+2}, whereas, if $j' = 3$, F_{31} cannot cover x_2. So C_{31} does not extend, a contradiction, and hence $v \not\sim x_{k+3}$.

Assume now that for some $i, k+4 \leq i \leq n-3$, (a) $v \not\sim x_{k+3}, \ldots, x_{i-1}$ and (b) path $x_{k+1}x_{k+2} \cdots x_{i-1}$ has no chord.

If $v \sim x_i$, then $C_{32} = vx_ix_{i+1} \cdots x_{n-3}x_0x_1ux_{j'}x_{j'+1} \cdots x_{k-1}v$ does not extend, since $i \geq k+4$ and by hypothesis (b) together with the assumption that x_2 cannot be adjacent to x_k since x_0x_k is a shortest chord of C_0. (Here again $j' = 2$ if $ux_2 \in E(G)$ and $j' = 3$ otherwise.) So $v \not\sim x_i$.

Suppose that path $x_{k+1} \cdots x_i$ has a chord e. Thus e must join x_i to some $x_{i'}, k+1 \leq i' \leq i-4$. Then $C_{33} = x_0x_1x_2x_3x_4x_5 \cdots x_i'x_i \cdots x_{n-3}x_0$ cannot extend to a 2-factor which covers vertex v.

So by induction, we have proved that (a) $v \not\sim x_{k+3}, \ldots, x_{n-3}$ and (b) $x_{k+1}x_{k+3} \cdots x_{n-3}$ has no chord. But then cycle $C_{34} = x_0x_1 \cdots x_kx_0$, requires 2-factor F_{34} to use edges vx_{k+1} and vx_{k+2}. However, then since $|V(G)| \geq 2k+2 \geq 12$, cycle

$$C_{35} = vx_{k+2}x_{k+3} \cdots x_{n-3}x_0x_1ux_{j'}x_{j'+1}v$$

cannot extend to a 2-factor, again since x_0x_k is a shortest chord.

This completes the proof of Claim 1.

We now proceed to complete the proof of Theorem 2.1. Suppose G contains a separating cycle. Then we have shown it contains a chordless separating cycle C_0 with $|V(C_0)| = |V(G)| - 2$. Let $V(G) - V(C_0) = \{u, v\}$. Since G is 2-connected, each of u and v must be adjacent to at least two vertices of cycle C_0. Without loss of generality, let us then choose vertices $w \in \{u, v\}$ and $y, z \in V(C_0)$ such that $w \sim y$ and $w \sim z$ and then among all such triples, choose one in which the distance between y and z on C_0 is minimum. Then again without loss of generality, we may label the vertices of C_0 clockwise, say, as $y = x_0, x_1, \ldots, x_k = z, x_{k+1}, \ldots, x_{n-3}, x_0$, where $n = |V(G)|$. Moreover, we may assume, without loss of generality, that $u \sim x_0$ and $u \sim x_k$.

If $k \geq 3$, and $\Sigma_1 = x_0 u x_k x_{k+1} \cdots x_{n-3} x_0$, then, since cycle Σ_1 extends to a 2-factor, $v \sim x_1$ and $v \sim x_{k-1}$. But then the triple $\{v, x_1, x_{k-1}\}$ contradicts the choice of triple $\{u, x_0, x_k\}$. So $1 \leq k \leq 2$.

Case 1: Suppose $k = 2$.

Case 1.1: Suppose $|V(G)| = 6$. Then $u \not\sim x_1, x_3$ by the choice of triple $\{u, x_0, x_k\}$.

Case 1.1.1: Suppose $v \sim x_0$. Then $v \not\sim x_1, x_3$, again by the choice of triple $\{u, x_0, x_k\}$. So then $v \sim x_2$ and $G \cong K_{2,4}$ which is not CEP as it has no 2-factor.

Case 1.1.2: So suppose $v \not\sim x_0$. Then by symmetry, $v \not\sim x_2$ either. So $v \sim x_1$ and $v \sim x_3$ and $G \cong G_2(6)$.

Case 1.2: Suppose now that $|V(G)| \geq 7$. Hence $n-3 \geq 4$ and 4-cycle $\Sigma_2 = u x_0 x_1 x_2 u$ implies that $v \sim x_3$ and $v \sim x_{n-3}$. Then by the choice of triple $\{u, x_0, x_k\}$, $v \not\sim x_0$, $v \not\sim x_2$, $u \not\sim x_3$ and $u \not\sim x_{n-3}$.

If $n - 3 \geq 6$, then cycle $\Sigma_3 = u x_2 x_3 v x_{n-3} x_0 u$ does not extend to a 2-factor. Hence $n - 3 \leq 5$.

Case 1.2.1: Suppose $n - 3 = 5$; that is, $|V(G)| = 8$. If $u \sim x_3$, then cycle $\Sigma_4 = u x_3 x_2 u$ does not extend. So we may suppose that $u \not\sim x_3$ and by symmetry, $u \not\sim x_5$.

Suppose $u \sim x_4$. Suppose further that $v \sim x_0$. Then cycle $\Sigma_5 = u x_2 x_3 v x_0 u$ does not extend. Hence $v \not\sim x_0$ and by symmetry, $v \not\sim x_2$. Suppose $v \sim x_4$. Then cycle $\Sigma_6 = u x_4 v x_3 x_2 u$ does not extend. So $v \not\sim x_4$. So, finally, suppose $v \sim x_1$. Then $G \cong G_2(8)$.

So suppose $u \not\sim x_4$. Thus $\deg_G(u) = 2$. If $v \sim x_1$, then $\Sigma_7 = v x_1 x_2 x_3 v$ does not extend, so $v \not\sim x_1$. If $v \sim x_4$, then $\Sigma_8 = v x_3 x_4 v$ does not extend, so $v \not\sim x_4$. So $\deg_G(v) = 2$ also. But then $G \cong G_1(8)$.

Case 1.2.2: Suppose $n - 3 = 4$ and hence $|V(G)| = 7$.

Then $\Sigma_2 = u x_2 x_1 x_0 u$ implies $v \sim x_3$ and x_4. But then triple $\{v, x_3, x_4\}$ contradicts the choice of triple $\{u, x_1, x_2\}$.

Case 2: Suppose $k = 1$. Since $|V(G)| \geq 6$, cycle $\Sigma_9 = u x_1 x_0 u$ implies that $v \sim x_2$ and x_{n-3}. If $n - 3 \geq 6$, then cycle $\Sigma_{10} = u x_1 x_2 v x_{n-3} x_0 u$ does not extend, so we may suppose $n - 3 \leq 5$.

Case 2.1: Suppose $n-3=5$; i.e., $|V(G)|=8$. Suppose first that $u \sim x_2$. Then $\Sigma_{11} = ux_2vx_5x_0u$ does not extend. So $u \not\sim x_2$ and by symmetry, $u \not\sim x_5$.

Now suppose $u \sim x_3$. Then $\Sigma_{12} = ux_3x_2x_1u$ implies that $v \sim x_0$ and x_4.

If $u \sim x_4$, then $\Sigma_{13} = ux_0vx_4u$ does not extend. So $u \not\sim x_4$. But then $\Sigma_{14} = x_0x_1x_2vx_5x_0$ does not extend.

So $u \not\sim x_3$ and by symmetry, $u \not\sim x_4$. So $\deg_G(u) = 2$. But then $\Sigma_{14} = x_0x_1x_2vx_5x_0$ does not extend.

Case 2.2: Suppose $n-3=4$; i.e., $|V(G)|=7$. Suppose first that $v \sim x_0$. Then $\Sigma_{15} = x_0vx_4x_0$ implies that $u \sim x_3$. However, then $\Sigma_{16} = ux_3x_4x_0u$ implies that $v \sim x_1$. But then $\Sigma_{17} = ux_1vx_0u$ does not extend.

So we may assume that $v \not\sim x_0$ and by symmetry that $v \not\sim x_1$ as well. Suppose $v \sim x_3$. Then $\Sigma_8 = vx_3x_4v$ implies that $u \sim x_2$ and by symmetry $u \sim x_4$. But then $\Sigma_{18} = ux_2vx_4u$ does not extend.

So we may assume that $v \not\sim x_3$. Suppose $u \sim x_2$. Then $\Sigma_{19} = ux_2x_1u$ does not extend. So $u \not\sim x_2$ and by symmetry, $u \not\sim x_4$. If $u \sim x_3$, then $\Sigma_{13} = ux_3x_2x_1u$ does not extend, so we may suppose $u \not\sim x_3$. But then $G \cong G_1(7)$.

Case 2.3: So suppose $n-3=3$, i.e., $|V(G)|=6$. Suppose first that $u \sim x_2$. Then $\Sigma_{19} = ux_2x_1u$ implies that $v \sim x_0$. Suppose $u \sim x_3$. Then $\Sigma_4 = ux_2x_3u$ implies that $v \sim x_1$ and $G \cong G_6(6)$.

So suppose $u \not\sim x_3$. If $v \sim x_1$, then $\Sigma_{20} = x_1x_2vx_1$ does not extend, so $v \not\sim x_1$. But then $G \cong G_3(6)$.

So finally suppose $u \not\sim x_2$. By symmetry, then, we may also suppose $u \not\sim x_3, v \not\sim x_0$ and $v \not\sim x_1$. But then $G \cong G_1(6)$.

This completes the proof of the theorem. □

In summary, then, we have shown that those 2-connected CEP graphs which have no separating cycle are those making up classes (1)–(3) together with all but three of the graphs in Class (4) and $G_4(6)$ and $G_5(6)$ from Class (5), while those which do have a separating cycle (necessarily of size $|V(G)|-2$) are $G_1(6), G_2(6), G_3(6), G_6(6), G_1(7)$, $G_1(8)$ and $G_2(8)$ and the remaining 3 graphs in Class (4).

Now we characterize those CEP graphs which contain a cutvertex. Let $\mathcal{G}$ denote the family of connected graphs G such that G consists of a union of vertex-disjoint chordless cycles $\mathcal{C}$ with $|\mathcal{C}| \geq 3$, together with a set of edges $\mathcal{E}$ joining pairs of cycles from $\mathcal{C}$ in a "tree-like" fashion such that

(i) each edge in $\mathcal{E}$ joins two different cycles in $\mathcal{C}$,

(ii) no two distinct edges of $\mathcal{E}$ join the same pair of cycles in $\mathcal{C}$, and

(iii) there exist no other cycles in G (or equivalently, the blocks of G are precisely $\mathcal{C} \cup \mathcal{E}$).

(See e.g. Figure 2.)

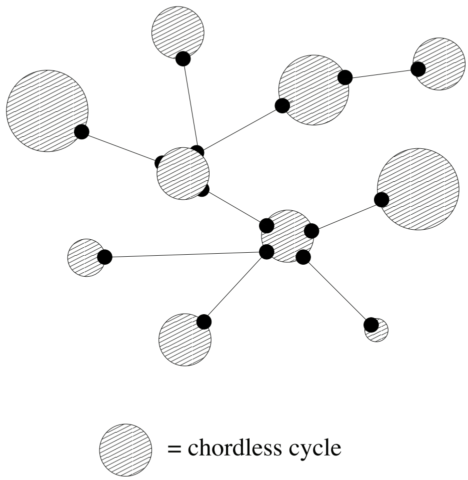

Figure 2

We note first that clearly each member of $\mathcal{G}$ is CEP.

Now let $\mathcal{G}'$ denote the class of all CEP graphs which contain a cutvertex.

Theorem 2.2: $\mathcal{G} = \mathcal{G}'$.

Proof: Clearly $\mathcal{G} \subseteq \mathcal{G}'$. It therefore remains to show that $\mathcal{G}' \subseteq \mathcal{G}$. We propose to prove this by induction on $|V(G)|$ for $G \in \mathcal{G}'$.

Let G belong to $\mathcal{G}'$. By definition of CEP and Theorem 2.1, we may assume that $|V(G)| \geq 6$. Moreover, if $|V(G)| = 6$, we claim that $G \in \mathcal{G}$. To see this, note that by definition, G contains a cutvertex v. Since G is CEP, it has a 2-factor F which, in turn, must contain a cycle C_v containing v. But then since C_v extends to 2-factor F, it must be that $|V(C_v)| = 3$. Moreover, since C_v extends to a 2-factor, there must be at least one cycle C' in $G - V(C_v)$ and since $|V(G)| = 6$, C' must be a triangle also and hence $V(G) = V(C_v) \cup V(C')$. Since v is a cutvertex of G, v is adjacent to one, two or three of the vertices of C'. If v is adjacent to two or more vertices of C', we must have a triangle C'' in G which contains v and two additional vertices of C'. But then C'' does not extend to a 2-factor, a contradiction.

So vertex v is adjacent to exactly one vertex of C' and G consists of two disjoint triangles joined by a single edge and hence belongs to the class $\mathcal{G}$.

We proceed by induction on $|V(G)|$. So suppose $G \in \mathcal{G}'$ and $|V(G)| \geq 7$. Consider the block-cutvertex tree $BC(G)$ of G. Since G contains a cutvertex, $BC(G)$ is a tree with at least two endvertices. Let us call the blocks of G which contain a cycle, *cyclic* and the rest K_2-*blocks*. Note that since G contains a 2-factor, all blocks of G are either cyclic or K_2-blocks and, moreover, the endvertices of $BC(G)$ must all correspond to cyclic blocks of G.

Let B_e be the (cyclic) endblock of G corresponding to an endvertex of $BC(G)$. Then B_e contains exactly one cutvertex of G; call it v.

Claim 1: B_e is a chordless cycle.

Since $B_e \neq K_2$, there exists a cycle C_v in B_e containing vertex v. (See Figure 3.)

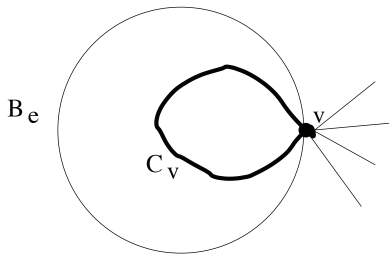

Figure 3

Suppose there is such a C_v with $|V(C_v)| < |V(B_e)|$. Among all such C_v's, choose one with a component H of $B_e - V(C_v)$ to be as small as possible. Since B_e is 2-connected, there are at least two vertices of attachment for H on cycle C_v. Choose two such distinct vertices u and w such that the path from u to w on C_v (taken clockwise, say) is as short as possible. (Note that it may be that $v \in \{u, w\}$.) Choose vertices u' and w' in $V(H)$ such that u' is adjacent to u and w' is adjacent to w. (Note that u' and w' may be the same vertex.)

Let Q be a shortest $u'w'$ path in H. (See Figure 4.)

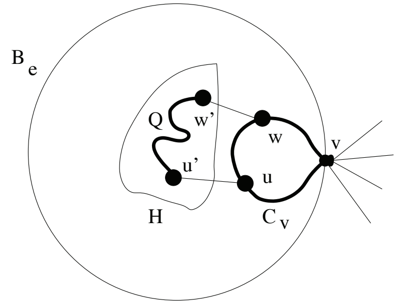

Figure 4

Then $C'_v = C_v - C_v(u, w) + uu' + ww' + Q$ is also a cycle in B_e through vertex v. So $V(H) = V(Q)$ by the choice of C_v and H. But path Q is chordless, so $C_v = V(B_e)$. Moreover, cycle C_v has no chords. That is, B_e is a chordless cycle as claimed.

Claim 2: Any block of G different from B_e which contains vertex v must be a K_2-block.

Suppose, to the contrary, that B' is a cyclic block of G which contains v and $B' \neq B_e$. Then B' contains a cycle C^* which in turn contains v. But then C^* does not extend to a 2-factor of G since $B_e - v$ cannot be covered.

Hence, since G is CEP, $|V(C^*)| \geq |V(G)| - 2$ and hence $|V(C^*)| = |V(G)| - 2$ and B_e must be a triangle. So G consists of a triangle B_e and a cyclic block B' also containing v such that B' contains $|V(G)| - 2$ vertices and $V(B_e) \cap V(B') = \{v\}$. Moreover, the cyclic block B' is Hamiltonian and hence 2-connected. In addition, since $|V(G)| \geq 7$, $|V(B')| \geq 5$. Now if C^* contains a chord e, we can find a cycle C_e in B' which contains v and e such that $|V(C_e)| < |V(C^*)| = |V(B')|$. But then $|V(C_e)| \leq |V(G)| - 3$, but does not extend to a 2-factor, a contradiction. So C^* does not contain a chord and hence B' is also a chordless cycle. But then the triangle B_e does not extend to a 2-factor, a contradiction. This finishes the proof of Claim 2.

So B_e is a chordless cycle containing a cutvertex v of G and v is incident only with K_2-blocks (other than B_e). Moreover, since B_e is an endvertex of $BC(G)$, v must belong to exactly one K_2-block which we shall call vw. Now let H be obtained from G by deleting $V(B_e)$ and edge vw. (See Figure 5.)

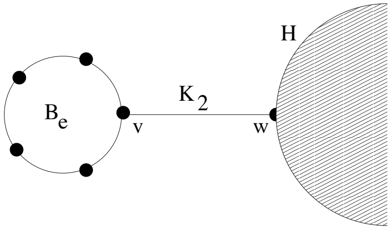

Figure 5

Claim 3: H is CEP.

This is clear.

Suppose H contains a cutvertex v_H of H. Then by the induction hypothesis, $H \in \mathcal{G}$, since $|V(H)| < |V(G)|$. But then G also belongs to $\mathcal{G}$ also and we are done.

So suppose H contains no cutvertex of itself; i.e., it is 2-connected. Hence H belongs to one of the five families listed in the statement of Theorem 2.1. If H is a cycle, then $G \in \mathcal{G}$ as desired. On the other hand, if H is any other graph from the remaining four families, it is easy to check that no matter how B_e is attached to H by a single edge, the resulting graph G is not CEP, a contradiction. This completes the proof of Theorem 2.2. □

Remark: The CEP graphs with cutvertices are thus members of a class of graphs called *Husimi trees* as defined by Harary and Uhlenbeck [4]. The reader is cautioned, however, that there exist different and, in fact, inequivalent definitions of Husimi trees in the literature. (Cf. [6].)

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