

A note on the Caro-Tuza bound on the independence number of uniform hypergraphs*

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Abstract

We show some consequences of Caro and Tuza's [*J. Graph Theory* 15 (1991), 99–107] lower bound on the independence number of a K -uniform hypergraph H . This bound has the form $C_K \cdot \sum_{i=1}^n (d_i + 1)^{-1/(K-1)}$, where C_K is a constant depending only on K , and d_1, \dots, d_n are the degrees of the vertices in H . We improve on the best known bounds for C_K : in particular, we prove that $C_3 \geq \sqrt{\pi}/2$ and that $C_K \geq \exp(-\gamma/(K-1))$ for $K \geq 3$, where γ is the Euler-Mascheroni constant.

1 Introduction

A theorem due independently to Caro [1] and to Wei [6] states that for a graph $G = (V, E)$ whose vertices have respective degrees d_1, d_2, \dots, d_n , the independence

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number $\alpha(G)$ obeys

$$\alpha(G) \geq \sum_{i=1}^n \frac{1}{d_i + 1}.$$

Caro and Tuza [2] later generalized this result to K -uniform hypergraphs, and Thiele [5] generalized this result further to general hypergraphs. Caro and Tuza proved:

Theorem 1 (Caro-Tuza) *For $K \geq 2$, let $H = (V, E)$ be a K -uniform hypergraph whose vertices v_1, v_2, \dots, v_n have respective degrees d_1, d_2, \dots, d_n . Then $\alpha(H) \geq \sum_{i=1}^n p_K(d_i)$, where*

$$p_K(d) = \prod_{j=1}^d \left(1 - \frac{1}{(K-1)j + 1} \right).$$

We first consider the case $K = 3$. With the identity

$$p_3(d) = \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{2d}{2d+1} = \frac{4^d \cdot (d!)^2}{(2d+1)!}$$

and a lengthy computation involving Stirling’s approximation, it is straightforward to show $p_3(d) \sim \frac{\sqrt{\pi}}{2\sqrt{d}}$. In Section 2 we prove an even stronger statement:

Theorem 2

$$\frac{\sqrt{\pi}}{2\sqrt{d+1}} < p_3(d) < \frac{\sqrt{\pi}}{2\sqrt{d+1/2}}.$$

This point does not seem to have been made in the literature. As a consequence, we have

Theorem 3 *Let $H = (V, E)$ be a 3-uniform hypergraph whose n vertices v_1, v_2, \dots, v_n (with $n > 0$) have respective degrees d_1, d_2, \dots, d_n . Then*

$$\alpha(H) > \frac{\sqrt{\pi}}{2} \cdot \sum_{i=1}^n \frac{1}{\sqrt{d_i + 1}}.$$

$(\frac{\sqrt{\pi}}{2} = 0.886\dots)$

In Section 3 we generalize these results to K -uniform hypergraphs with $K > 3$, obtaining the following analogue to Theorem 3:

Theorem 4 *Let H be a K -uniform hypergraph for $K \geq 3$, and let d_1, d_2, \dots, d_n be the degrees of its vertices, with $n > 0$. Then*

$$\alpha(H) > e^{-\gamma/(K-1)} \cdot \sum_{i=1}^n \frac{1}{(d_i + 1)^{1/(K-1)}},$$

where $\gamma = 0.5772\dots$ is the Euler-Mascheroni constant.

The coefficient $e^{-\gamma/(K-1)}$ is an improvement on the value $1 - 1/K$ from Spencer's [4] result

$$\alpha(H) \geq \frac{n}{\delta^{1/(K-1)}} \cdot \left(1 - \frac{1}{K}\right),$$

where $\delta \geq 1$ is the average vertex degree in H .

2 The case of 3-uniform hypergraphs

In this section we will write $p(\cdot)$ for $p_3(\cdot)$. Recall Wallis' formula,

$$\frac{\pi}{2} = \prod_{i=1}^{\infty} \frac{2 \cdot \lceil i/2 \rceil}{2 \cdot \lceil i/2 \rceil + 1} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{10}{11} \cdots$$

Let W_k denote the product of the first k fractions in the formula as written above. Thus $W_1 = \frac{2}{1}$, $W_2 = \frac{2}{1} \cdot \frac{2}{3}$, etc.

Remark 5 Normally Wallis' formula is given as

$$\prod_{i=1}^{\infty} \frac{(2i)^2}{(2i-1)(2i+1)},$$

giving partial products W_2, W_4, \dots , but we will find the prior version more convenient. The difference in grouping does not change its convergence properties, since

$$\lim_{k \rightarrow \infty} \frac{W_{2k+1}}{W_{2k}} = 1.$$

Proposition 6 $W_k < \frac{\pi}{2}$ for even k , and $W_k > \frac{\pi}{2}$ for odd k .

Proof: Let k be even. The next two factors in the product have the forms $\frac{i}{i-1}$ and $\frac{i}{i+1}$; their product is greater than 1. The same is true for the next two, the next two after them, etc. Letting R be the product of *all* these pairs of factors, we have $R > 1$. Since $W_k \cdot R = \frac{\pi}{2}$, it follows that $W_k < \frac{\pi}{2}$.

Similarly, when k is odd, each subsequent pair of factors has the form $\frac{i-1}{i}$ and $\frac{i+1}{i}$, whose product is less than 1. Thus the product of the remaining pairs, R , is less than 1. Since $W_k \cdot R = \frac{\pi}{2}$, we have $W_k > \frac{\pi}{2}$. □

Proof of Theorem 2: Recall

$$p(d) = \frac{2 \cdot 4 \cdot 6 \cdots (2d)}{1 \cdot 3 \cdot 5 \cdots (2d+1)}.$$

Squaring and multiplying by $2d+1$, we obtain

$$p(d)^2 \cdot (2d+1) = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots (2d) \cdot (2d)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdots (2d-1) \cdot (2d-1) \cdot (2d+1)} = W_{2d} < \frac{\pi}{2}$$

by Proposition 6. Similarly,

$$p(d)^2 \cdot (2d + 2) = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots (2d) \cdot (2d) \cdot (2d + 2)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdots (2d + 1) \cdot (2d + 1)} = W_{2d+1} > \frac{\pi}{2}.$$

Thus

$$\begin{aligned} \frac{\pi}{2(2d + 2)} &< p(d)^2 < \frac{\pi}{2(2d + 1)} \\ \frac{\sqrt{\pi}}{2\sqrt{d + 1}} &< p(d) < \frac{\sqrt{\pi}}{2\sqrt{d + 1/2}}. \end{aligned}$$

□

Theorem 2 makes it obvious that $p(d) \sim \frac{\sqrt{\pi}}{2\sqrt{d}}$.

3 The case of K -uniform hypergraphs, $K > 3$

In this section we will develop analogues of Theorems 2 and 3 that apply to K -uniform hypergraphs with $K > 3$. It is known (see [3]) that for each $K > 1$, some positive constant c_K satisfies

$$\alpha(H) \geq c_K \cdot \sum_{i=1}^n \frac{1}{(d_i + 1)^{1/(K-1)}}.$$

Recall that we defined

$$p_K(d) = \prod_{i=1}^d \frac{(K - 1)i}{(K - 1)i + 1};$$

this is the formula found by Caro and Tuza in [2]. In Section 2 we considered the case $K = 3$. For general $K \geq 2$, we have

Proposition 7

$$\lim_{d \rightarrow \infty} p_K(d) \cdot (d + 1)^{1/(K-1)} = e^{h(K-1)},$$

where

$$h(z) = \frac{1}{z} \cdot \sum_{i=2}^{\infty} \frac{(-1)^i \cdot \zeta(i)}{i} \cdot \left(\frac{1}{z^{i-1}} - 1 \right)$$

and ζ is the Riemann zeta function.

Thus for fixed K ,

$$p_K(d) \sim \frac{e^{h(K-1)}}{d^{1/(K-1)}}.$$

Proof: We can rewrite $p_K(d) \cdot (d + 1)^{1/(K-1)}$ as $\prod_{i=1}^d a_i$, where

$$\begin{aligned} a_i &= \frac{(K-1)i}{(K-1)i+1} \cdot \left(\frac{i+1}{i}\right)^{1/(K-1)} \\ &= \left(1 + \frac{1}{(K-1)i}\right)^{-1} \cdot \left(1 + \frac{1}{i}\right)^{1/(K-1)}. \end{aligned}$$

Then

$$\begin{aligned} \log a_i &= \frac{1}{K-1} \cdot \log\left(1 + \frac{1}{i}\right) - \log\left(1 + \frac{1}{(K-1)i}\right) \\ &= \frac{1}{K-1} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j \cdot i^j} - \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j \cdot ((K-1)i)^j} \\ &= \frac{1}{K-1} \cdot \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j \cdot i^j} - \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j \cdot ((K-1)i)^j}, \end{aligned}$$

since the terms for $j = 1$ cancel out. These were the troublesome terms, since they belong to the harmonic series, which does not converge.

The next step is to calculate $\sum_{i=1}^{\infty} \log a_i$, but we must be careful. The series for $\log a_i$ converges absolutely when $i > 1$ but *not* when $i = 1$. We must therefore pay closer attention to that series.

$$\begin{aligned} \sum_{i=1}^{\infty} \log a_i &= \sum_{i=1}^{\infty} \left(\frac{1}{K-1} \cdot \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j \cdot i^j} - \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j \cdot ((K-1)i)^j} \right) \\ &= \sum_{i=1}^{\infty} \left(\sum_{j=2}^{\infty} \left(\frac{(-1)^{j+1}}{(K-1) \cdot j \cdot i^j} - \frac{(-1)^{j+1}}{j \cdot ((K-1)i)^j} \right) \right) \\ &= \sum_{j=2}^{\infty} \left(\frac{(-1)^{j+1}}{(K-1) \cdot j} - \frac{(-1)^{j+1}}{j \cdot (K-1)^j} \right) \\ &\quad + \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} \left(\frac{(-1)^{j+1}}{(K-1) \cdot j \cdot i^j} - \frac{(-1)^{j+1}}{j \cdot ((K-1)i)^j} \right) \\ &= \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j} \left(\frac{1}{K-1} - \frac{1}{(K-1)^j} \right) \\ &\quad + \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j \cdot i^j} \left(\frac{1}{K-1} - \frac{1}{(K-1)^j} \right) \tag{1} \\ &= \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j} \left(\frac{1}{K-1} - \frac{1}{(K-1)^j} \right) \\ &\quad + \sum_{j=2}^{\infty} \sum_{i=2}^{\infty} \frac{(-1)^{j+1}}{j \cdot i^j} \left(\frac{1}{K-1} - \frac{1}{(K-1)^j} \right) \tag{2} \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=2}^{\infty} \left(\sum_{i=1}^{\infty} \frac{(-1)^{j+1}}{j \cdot i^j} \left(\frac{1}{K-1} - \frac{1}{(K-1)^j} \right) \right) \\
&= \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j} \left(\frac{1}{K-1} \cdot \sum_{i=1}^{\infty} \frac{1}{i^j} - \frac{1}{(K-1)^j} \cdot \sum_{i=1}^{\infty} \frac{1}{i^j} \right) \\
&= \frac{1}{K-1} \cdot \sum_{j=2}^{\infty} \frac{(-1)^{j+1} \cdot \zeta(j)}{j} \left(1 - \frac{1}{(K-1)^{j-1}} \right) \\
&= h(K-1).
\end{aligned}$$

We have repeatedly used the identity $\sum_{k=1}^{\infty} (x_k + y_k) = (\sum_{k=1}^{\infty} x_k) + (\sum_{k=1}^{\infty} y_k)$, which is valid whenever the sums $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ are finite. To get from (1) to (2), we used the fact that the double sum is absolutely convergent (in fact, $\sum_{i=2}^{\infty} \sum_{j=2}^{\infty} 1/(j \cdot i^j) = 1 - \gamma$).

Exponentiating, we conclude $\prod_{i=1}^{\infty} a_i = \exp(h(K-1))$. \square

As was the case for $K = 3$, our asymptotic bound also yields a lower bound:

Proposition 8 For $K \geq 3$,

$$p_K(d) \cdot (d+1)^{1/(K-1)} > e^{h(K-1)}.$$

Thus

$$p_K(d) > \frac{e^{h(K-1)}}{(d+1)^{1/(K-1)}}.$$

Proof: We will show that each factor a_i (from the proof of Proposition 7) is less than 1; as a result, the partial products will decrease toward the limit given in Proposition 7. The statement $a_i < 1$ is equivalent to

$$\frac{(K-1)i}{(K-1)i+1} \cdot \left(\frac{i+1}{i} \right)^{1/(K-1)} < 1,$$

which in turn is equivalent to

$$\left(\frac{(K-1)i}{(K-1)i+1} \right)^{K-1} \cdot \frac{i+1}{i} < 1$$

and

$$((K-1)i)^{K-1} \cdot (i+1) < ((K-1)i+1)^{K-1} \cdot i.$$

This latter statement is easily shown to be true. The left-hand side equals $(K-1)^{K-1} \cdot i^K + (K-1)^{K-1} \cdot i^{K-1}$. The expansion of the right-hand side will contain (among others) the terms $(K-1)^{K-1} \cdot i^K$ and $\binom{K-1}{1} \cdot (K-1)^{K-2} \cdot i^{K-2} \cdot i$. Thus the right-hand side exceeds the left-hand side. \square

Since for a K -uniform hypergraph H we have $\alpha(H) \geq \sum_{i=1}^n p_K(d_i)$, Theorem 1 and Proposition 8 demonstrate

Theorem 9 For $K \geq 3$, let $H = (V, E)$ be a K -uniform hypergraph whose n vertices v_1, v_2, \dots, v_n (with $n > 0$) have respective degrees d_1, d_2, \dots, d_n . Then

$$\alpha(H) > e^{h(K-1)} \cdot \sum_{i=1}^n \frac{1}{(d_i + 1)^{1/(K-1)}}.$$

Proof of Theorem 4: The identity $\sum_{i=2}^\infty (-1)^i \cdot \zeta(i)/i = \gamma$ lets us rewrite $h(z)$ as

$$\frac{1}{z} \cdot \left[\left(\sum_{i=2}^\infty \frac{(-1)^i \cdot \zeta(i)}{i \cdot z^{i-1}} \right) - \gamma \right].$$

The parenthesized summation is an alternating series whose terms are descending in magnitude; thus the sign of the limit is the same as the sign of its first term, which is positive. We thereby have

$$h(K - 1) > -\frac{\gamma}{K - 1},$$

which together with Theorem 9 proves Theorem 4. □

4 Conclusion

We have shown that for any non-empty K -uniform hypergraph H with vertex degrees d_1, \dots, d_n , the statement

$$\alpha(H) \geq c_K \cdot \sum_{i=1}^n \frac{1}{(d_i + 1)^{1/(K-1)}}$$

holds true when we set $c_3 = \sqrt{\pi}/2$ and $c_K = e^{-\gamma/(K-1)}$ for $K > 3$. The value $e^{-\gamma/(K-1)}$ is a close approximation to the true value given by Propositions 7 and 8; it would be nice to know whether the exact value is expressible in a closed form.

It is also of interest to determine how close to optimal these constants are; since the case of a hypergraph with no edges (every degree is zero) forces $c_K \leq 1$, we can alternatively consider hypergraphs where every degree is at least some constant D and try to bound the corresponding constant $c_{K,D}$. In this case, we can show fairly easily that the optimal value of $c_{K,D}$ is at most $e + o(1)$, where the $o(1)$ term depends only on K . Consider a complete K -uniform hypergraph H on n vertices. Clearly $\alpha(H) = K - 1$. On the other hand, each vertex has degree $\binom{n-1}{K-1}$, so that it must hold true that

$$c_{K,D} \cdot \frac{n}{\left(\binom{n-1}{K-1} + 1\right)^{1/(K-1)}} \leq K - 1.$$

Hold K constant and let n go to ∞ . Then $\binom{n-1}{K-1}$ behaves as $n^{K-1}/(K-1)!$, so that the left-hand side is asymptotic to $c_{K,D} \cdot (K-1)!^{1/(K-1)}$. From Stirling's formula it is easily seen that

$$K!^{1/K} \rightarrow \frac{K}{e},$$

and $c_{K,D} \leq e + o(1)$.

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