AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **52** (2012), Pages 209–214

The tight bound on the number of $\overrightarrow{C_3}$-free vertices on regular 3-partite tournaments*

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Abstract

Let T be a 3-partite tournament. We say that a vertex v is $\overrightarrow{C_3}$-free if v does not lie on any directed triangle of T. Let $F_3(T)$ be the set of the $\overrightarrow{C_3}$-free vertices in a 3-partite tournament and $f_3(T)$ its cardinality. In a recent paper, it was proved that if T is a regular 3-partite tournament, then $f_3(T) < \frac{n}{9}$, where n is the order of T. In this paper, we prove that $f_3(T) \leq \lfloor \frac{n}{12} \rfloor$. We also prove that this bound is tight by giving an infinite family of regular 3-partite tournaments having exactly $\lfloor \frac{n}{12} \rfloor$ $\overrightarrow{C_3}$-free vertices.

1 Introduction

Let c be a non-negative integer. A *c-partite* or *multipartite tournament* is a digraph obtained from a complete c-partite graph by substituting each edge with exactly one arc. It is easy to see that a *tournament* is a c-partite tournament with exactly c vertices. Let D be an oriented graph. The vertex set and the arc set of an oriented graph D are denoted by $V(D)$ and $A(D)$, respectively. Let $S \subset V(D)$. We denote by $D[S]$ the subdigraph induced by S in D. The *out-neighborhood* (*in-neighborhood*, respectively) $N^+(x)$ ($N^-(x)$, respectively) of a vertex x is the set $\{y \in V(D) \mid xy \in A(D)\}$ ($\{y \in V(D) \mid yx \in A(D)\}$, respectively). The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are the *out-degree* and the *in-degree* of x, respectively. An oriented graph D is *r-regular* if $d^-(v) = d^+(v) = r$ for every

* Research supported by PROMEP UAM-PTC-187 and CONACyT-México under projects 83917 and 83856.
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$v \in V(D)$. Let $X, Y \subseteq V(D)$. Then X *dominates* Y, denoted by $X \Longrightarrow Y$, if $Y \subseteq N^+(x)$ for every $x \in X$. An *m-cycle* of an oriented graph is a directed cycle of length m. Let T be a c-partite tournament. We say that a vertex v is $\overrightarrow{C_3}$*-free* if v does not lie on any directed triangle of T. Let $F_3(T)$ be the set of the $\overrightarrow{C_3}$-free vertices in a c-partite tournament and $f_3(T)$ its cardinality.

The structure of cycles in multipartite tournaments has been extensively studied. A very recent survey on this topic [3] appeared with several interesting open problems. For instance, the study of cycles whose length does not exceed the number of partite sets leads to various extensions and generalizations of classic results on tournaments. Zhou et al. [4] proved that every vertex of a regular c-partite tournament with at least four partite sets ($c \geq 4$) is contained in a cycle of length m for each $m \in \{3, 4, \ldots, c\}$. Volkmann [2] provided the following infinite family of $4p$-regular 3-partite tournaments which shows that the previous theorem is not valid for regular 3-partite tournaments.

Example 1 (Volkmann [2])**.** *Let T_p be the $4p$-regular 3-partite tournament of order $12p$ with the partite sets $U = U_1 \cup U_2$, $V = V_1 \cup V_2$ and $W = W_1 \cup W_2$ such that:*

1. $|W_1| = p$, $|W_2| = 3p$ *and* $|U_1| = |U_2| = |V_1| = |V_2| = 2p$.
2. *The sets $U_1 \cup V_1$ and $U_2 \cup V_2$ generate p-regular bipartite tournaments T_1 and T_2, respectively.*
3. $V_1 \Longrightarrow U_2$ *and* $U_1 \Longrightarrow V_2$.
4. $V(T_1) \Longrightarrow W_1 \Longrightarrow V(T_2)$ *and* $V(T_2) \Longrightarrow W_2 \Longrightarrow V(T_1)$.

In [1], Figueroa et al. posed the following question: *What is the maximum number of $\overrightarrow{C_3}$-free vertices on regular 3-partite tournaments?*. They proved the following.

Theorem 1. *[1] Let T be a regular 3-partite tournament of order n. Then $f_3(T) < n/9$.*

In this paper, we prove that if T is a regular 3-partite tournament, then $f_3(T) \leq \lfloor \frac{|V(T)|}{12} \rfloor$. In order to prove that this upper bound is tight, we show that, for every n, there is an infinite family of regular 3-partite tournaments of order n having exactly $\lfloor \frac{|V(T)|}{12} \rfloor$ $\overrightarrow{C_3}$-free vertices. The family of Volkmann (Example 1) is a particular case of our construction, whenever $n \equiv 0 \pmod 4$.

2 Tripartite regular tournaments

Let T be a 3-partite regular tournament and P be a partite set of T, with $u, v \in V(T)$ and $S \subseteq V(T)$. Throughout this note we use the following notation:

$$P^+ := (\bigcap_{s\in F_3(T)} N^+(s)) \cap P,$$

$$P^- := (\bigcap_{s\in F_3(T)} N^-(s)) \cap P \quad \text{and}$$

$$P^* := P \backslash (P^+ \cup P^-).$$

As a consequense of the proof of Theorem 1 (for details see [1]) we have the following proposition.

Proposition 1. *Let T be a regular 3-partite tournament with $F_3(T) \neq \emptyset$. Then the following hold.*

a) There exists a partite set P_0 such that $F_3(T) \subset P_0$.

b) There exists a partite set $P_1 \neq P_0$ such that $P_1 = P_1^+ \cup P_1^-$.

c) If P_2 is the remaining partite set of T, then $P_1^- \Longrightarrow P_2^+ \cup P_2^*$ and $P_2^- \cup P_2^* \Longrightarrow P_1^+$.

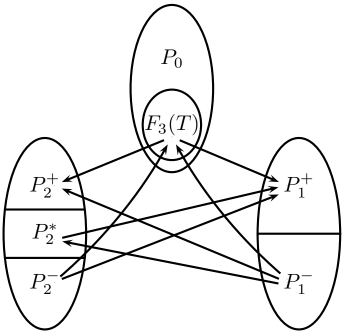

Figure 1: The structure of a 3-partite tournament with $\overrightarrow{C_3}$-free vertices.

Theorem 2. *Let T be a r-regular 3-partite tournament. Then $f_3(T) \leq \lfloor \frac{r}{4} \rfloor$.*

Proof. Let P_0, P_1 and P_2 be the partite sets of T and $|P_0| = |P_1| = |P_2| = r$, with

$$r = 4s + k \text{ and } 0 \leq k \leq 3. \tag{1}$$

Throughout this proof we will write F_3 (respectively, f_3) instead of $F_3(T)$ (respectively, $f_3(T)$). By Proposition 1, we may assume that $F_3 \subseteq P_0$. Without lost of generality, we assume that $|P_1^+| \geq |P_1^-|$. Let $|P_1^+| = p_1$ and $|P_2^+| = p_2$.

For a contradiction, suppose that $f_3 \geq \lfloor \frac{r}{4} \rfloor + 1 = s + 1$.

Let $v^+ \in P_1^+$ and $w^+ \in P_2^+$. Then $F_3 \cup P_2^* \cup P_2^- \subseteq N^-(v^+)$ and $F_3 \cup P_1^- \subseteq N^-(w^+)$. Moreover, since T is r-regular, $|P_i| = r$, $i = 0, 1, 2$ and

$$|N^-(v^+)| \geq |F_3 \cup P_2^* \cup P_2^-| \geq s + 1 + (r - p_2) = r - (p_2 - s - 1). \tag{2}$$

$$|N^-(w^+)| \geq |F_3 \cup P_1^-| \geq s + 1 + (r - p_1) = r - (p_1 - s - 1). \tag{3}$$

By (2), every $v^+ \in P_1^+$ has at least $r - (p_2 - s - 1)$ in-neighbors in $F_3 \cup P_2^* \cup P_2^-$. Since $d^-(v^+) = r$, $N^+(v^+) \cup N^-(v^+) \subset P_0 \cup P_2$ and $P_0 \cup P_2 \setminus (F_3 \cup P_2^* \cup P_2^-) = P_2^+ \cup P_0 \setminus F_3$ we have that (see Figure 1),

$$\text{each vertex } v^+ \in P_1^+ \text{ has at most } p_2 - s - 1 \text{ in-neighbors in } P_2^+ \cup P_0 \setminus F_3. \tag{4}$$

analogously ,

$$\text{each vertex } w^+ \in P_2^+ \text{ has at most } p_1 - s - 1 \text{ in-neighbors in } P_1^+ \cup P_0 \setminus F_3. \tag{5}$$

Notice that the bipartite tournament $T_2 = T[P_1^+ \cup P_2^+]$ has p_1p_2 arcs. By (4), the set of arcs from P_2^+ to P_1^+ has cardinality at most $p_1(p_2 - s - 1)$, and by (5) the set of arcs from P_1^+ to P_2^+ has cardinality at most $p_2(p_1 - s - 1)$. Then,

$$p_1(p_2 - s - 1) + p_2(p_1 - s - 1) \geq p_1p_2,$$

that is,

$$p_1p_2 - (p_1 + p_2)(s + 1) \geq 0. \tag{6}$$

Let $m = p_1 + p_2$. Since $p_2 = m - p_1$, by inequality (6), we obtain the quadratic inequality

$$(p_1)^2 - m(p_1) + m(s + 1) \leq 0. \tag{7}$$

Since the discriminant of inequality (7) must be non-negative, $(m(m-4(s+1)) \geq 0)$ and $m > 0$ and thus

$$m \geq 4(s + 1) > 4s + k = r.$$

Note that $m \leq d^+(x) = r$ for $x \in F_3$; then we have a contradiction to the fact that $m > r$. So $f_3(T) \leq \lfloor \frac{r}{4} \rfloor$. □

The following example shows that the bound of Theorem 2 is tight for $r = 4p + 3$ (see Figure 2).

Example 2. *Let p be a non negative integer. We denote by T_{4p+3} the $4p+3$-regular 3-partite tournament with partite sets $U = U_1 \cup U_2 \cup \{u_1, u_2, u_3\}$, $V = V_1 \cup V_2 \cup \{v_1, v_2, v_3\}$ and $W = W_1 \cup W_2 \cup W_3 \cup \{w, w_1, w_2, w_3\}$ such that:*

1. *$|U_1| = |U_2| = |V_1| = |V_2| = |W_3| = 2p$ and $|W_1| = p - 1$, $|W_2| = p$.*
2. *The sets $U_1 \cup V_1$ and $U_2 \cup V_2$ generate p-regular bipartite tournaments T_1 and T_2, respectively.*
3. *$V_1 \Longrightarrow U_2 \cup \{u_1, u_2, u_3\}$ and $U_1 \Longrightarrow V_2 \cup \{v_1, v_2, v_3\}$.*
4. *$V(T_1) \Longrightarrow W_1 \cup \{w\}$ and $V(T_2) \Longrightarrow W_2 \cup W_3 \cup \{w_1, w_2, w_3\}$.*
5. *$N^+(u_1) = V_2 \cup W_3 \cup \{w_1, w_2, v_3\}$.*

6. $N^+(u_2) = V_2 \cup W_1 \cup W_2 \cup \{w_1, w_2, w_3, v_3\}$.

7. $N^+(u_3) = V_2 \cup W_3 \cup \{w, w_3, v_3\}$.

8. $N^+(v_1) = N^+(v_2) = U_2 \cup W_1 \cup W_2 \cup \{w, u_1, u_2, u_3\}$.

9. $N^+(v_3) = U_2 \cup W_3 \cup \{w_1, w_2, w_3\}$.

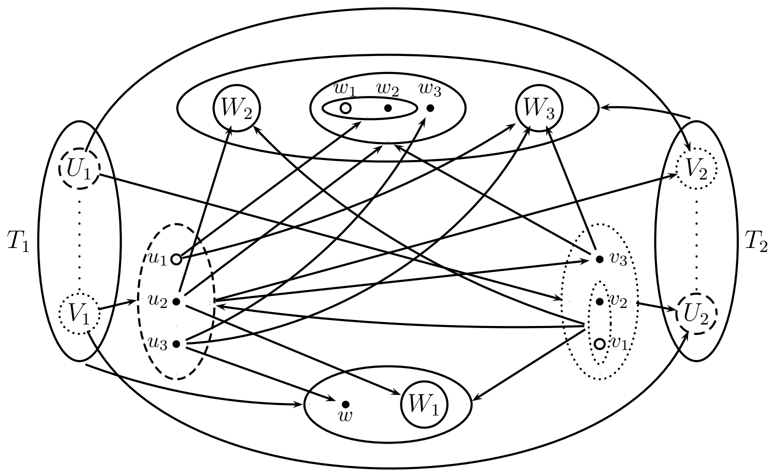

Figure 2: Family of r-regular 3-partite tournaments with $f_3 = \lfloor \frac{r}{4} \rfloor$.

In Figure 2, we exhibit the family of 3-partite tournaments of Example 2. In order to simplify Figure 2, the only arcs shown are the arcs from the dotted vertex set and from the dashed vertex set. The dotted lines represent a p-regular bipartite tournament.

Theorem 3. *For every non-negative integer r, there exists a r-regular 3-partite tournament T_r such that $f_3(T_r) = \lfloor \frac{r}{4} \rfloor$.*

Proof. It is not difficult to verify that T_{4p+3} of Example 2 is an r-regular 3-partite tournament for $r = 4p + 3$ such that $f_3(T) = |W_1 \cup \{w\}| = p = \lfloor \frac{r}{4} \rfloor$. For $r = 4p + 2$, $T_r = T_{4p+3} \setminus \{u_1, v_1, w_1\}$ is a r-regular 3-partite tournament with $f_3(T) = |W_1 \cup \{w\}| = p = \lfloor \frac{r}{4} \rfloor$. For $r = 4p + 1$, $T_r = T_{4p+3} \setminus \{u_2, u_3, v_2, v_3, w_2, w_3\}$ is a r-regular 3-partite tournament with $f_3(T) = |W_1 \cup \{w\}| = p = \lfloor \frac{r}{4} \rfloor$. Finally, for $r = 4p$, $T_r = T_{4p+3} \setminus \{u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, w_3\}$ and T_r is isomorphic to the 3-partite tournament T_p of Example 1. □

References

[1] A. P. Figueroa, B. Llano and R. Zuazua, The number of $\overrightarrow{C_3}$-free vertices on 3-partite tournaments, *Discrete Math.* 310 no. 19 (2010), 2482–2488.

[2] L. Volkmann, Cycles in multipartite tournaments: Results and problems, *Discrete Math.* 245 (1–3) (2002), 19–53.

[3] L. Volkmann, Multipartite tournaments: a survey, *Discrete Math.* 307 (24) (2007), 3097–3129.

[4] G. Zhou, T. Yao and K. M. Zhang, A note on cycles in regular multipartite tournaments, *J. Nanjing Univ. Math. Biq* 15 (1998), 73–75.

(Received 17 June 2011; revised 5 July 2011)