

The modular product and existential closure

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Abstract

In this article we study the modular product graph operation, denoted by \diamond , with particular emphasis on when the operation preserves the property of a graph being 3-existentially closed. We characterise when $G\diamond H$ is 3-e.c. provided that H is 3-e.c., and we establish two new infinite classes of 3-e.c. graphs of the form $G\diamond H$, for which G need not be 3-e.c. itself.

1 Introduction

In 1963, Erdős and Rényi [9] introduced the concept of existential closure, whereby a graph G with vertex set V is said to be n -existentially closed, or n -e.c., if for each proper subset S of V with cardinality $|S| = n$ and each subset T of S , there exists some vertex x not in S that is adjacent to each vertex of T but to none of the vertices of $S \setminus T$.

Since this property was first introduced, only a handful of classes of graphs have been shown to be n -e.c. for arbitrary (but fixed) values of n . Among these classes are sufficiently large Paley graphs [4], a family of strongly regular graphs described by Cameron and Stark [7], as well as graphs arising from affine planes and resolvable designs as described by Baker et al. [2, 3]. Recently, the block intersection graphs of combinatorial designs were considered [10, 13]. Aside from finite designs, in [12, 14] block intersection graphs of infinite designs were also considered. In [14] it was shown that any infinite t - (v, k, λ) design with finite block size behaves similarly to finite designs in the sense that if the block intersection graph is n -e.c., then n is bounded, namely $n \leq t + 1$. However, in [12] an infinite design having infinite block size and whose block intersection graph is isomorphic to the Rado graph R (which was shown by Erdős and Rényi to be asymptotically almost surely n -e.c. for all $n \geq 1$ [9]) was constructed.

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The scarcity of other readily recognised families of n -e.c. graphs for arbitrary n has motivated research into classes of graphs that are n -e.c. for small values of n ; however, it is not easy to find explicit examples of such graphs even for $n = 3$. It has been shown that every 3-e.c. graph has at least 24 vertices and examples of 3-e.c. graphs of order 28 have been found [5, 11]. In 2001, Hadamard matrices of order $4m$ with odd $m > 1$ were used to obtain 3-e.c. graphs of order $16m^2$ [6]. Also in 2001, Baker et al. presented new 3-e.c. graphs arising from collinearity graphs of partial planes resulting from affine planes [1]. Recently, another construction of 3-e.c. graphs of order at least p^d for prime $p \geq 7$ and $d \geq 5$ was presented using quadrances [16]. Also it was confirmed that there are only two STS(19) with 3-e.c. block intersection graphs [8, 10].

As part of an effort to find explicit examples of n -e.c. graphs, Bonato and Cameron examined several common binary graph operations to see which operations preserve the n -e.c. property for $n \geq 1$ [5]. They showed that the symmetric difference of two 3-e.c. graphs is a 3-e.c. graph. Baker et al. subsequently introduced another graph construction which is 3-e.c. preserving [1].

In this article, we take a different approach to the construction in [1] that enables us to relax the requirement that the two graphs considered be both 3-e.c. We formulate the construction as a binary non-commutative graph operation denoted by the symbol \diamond and we determine necessary and sufficient conditions for the graph $G \diamond H$ to be 3-e.c., given that H itself is a 3-e.c. graph. We then use this operation to construct new classes of 3-e.c. graphs of the form $G \diamond H$ when G is not necessarily a 3-e.c. graph. In particular, the classes that we consider are those for which G is either a complete multipartite graph or a strongly regular graph. The graph G for which we show that $G \diamond H$ is 3-e.c. can have as few as four vertices, which represents an improvement in comparison to when G is required to be 3-e.c.

2 The Modular Product and a Characterisation Theorem

If G and H are two graphs, then we let $G \diamond H$ represent the graph with vertex set $V(G) \times V(H)$ in which two vertices (x, u) and (y, v) are adjacent if

- (a) $xy \in E(G)$ and $uv \in E(H)$, or
- (b) $xy \notin E(G)$ and $uv \notin E(H)$.

It so happens that $G \diamond H$ is the complement of a construction that was introduced by Vizing in 1974 [17]. In keeping with [15, 18], we shall refer to $G \diamond H$ as the modular product of G and H .

Unless stated otherwise, we shall generally assume that the graph G has a loop at each vertex and also that H is 3-e.c. When describing the graph $G \diamond H$, for each vertex $x \in V(G)$ let H_x be the subgraph of $G \diamond H$ that is isomorphic to H and consists of all vertices of the form (x, u) where $u \in V(H)$. Since the vertices of H_x can be considered to be indexed by $V(G)$, we will often use the notation u_x to denote the

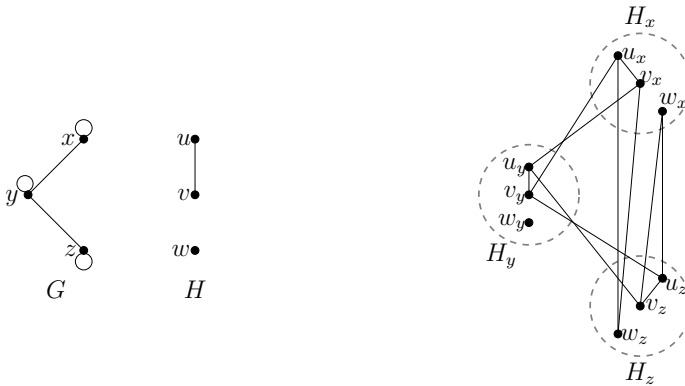


Figure 1: $G \diamond H$.

vertex (x, u) . Two vertices $u_x = (x, u) \in H_x$ and $v_y = (y, v) \in H_y$ will be said to be congruent if $u = v$; otherwise they are incongruent. An example of $G \diamond H$ is illustrated in Figure 1, for $G = K_{1,2}$ and $H = \overline{K_{1,2}}$.

It can be easily deduced that $\overline{G \diamond H} = G \diamond \overline{H}$ where \overline{G} is the simple complement of G (the complement of a loop is a non-loop and the complement of a non-loop remains a non-loop). Also, note that when G has a loop at every vertex, $G \diamond H$ is isomorphic to the graph $G(H)$ as described in [1] in which the following theorem was proved:

Theorem 2.1 [1] *If the graphs G and H are both 3-e.c., then the graph $G \diamond H$ is also 3-e.c.*

We devote the remainder of this section to the development and proof of a characterisation of 3-e.c. graphs of the form $G \diamond H$ where H is 3-e.c. but G is not necessarily so. This characterisation will help us to find smaller 3-e.c. graphs by simplifying the process of checking when $G \diamond H$ is 3-e.c.

For a graph G , given a set $S \subset V(G)$ and a subset T of S , we say a vertex $x \in V(G) \setminus S$ is a T -solution with respect to S if x is adjacent to every vertex in T and to none in $S \setminus T$. A solution for S is said to exist if there is a T -solution for every $T \in P(S)$, where $P(S)$ denotes the power set of S . Observe that if a solution exists for every n -subset of V , then G is n -e.c.

We say a graph G is weakly n -existentially closed, or n -w.e.c., if for any set S with $|S| = n$ and any $T \subseteq S$, there exists a vertex in $V(G)$ that is adjacent to each vertex in T and to no vertex in $S \setminus T$, or there exists a vertex that is adjacent to each vertex in $S \setminus T$ and to no vertex in T . Such a vertex is called a weak T -solution with respect to S . Note that $K_1 \diamond H = H$ and it can easily be confirmed that if $|V(G)| \in \{2, 3\}$, then G cannot be 3-w.e.c. So we henceforth assume that $|V(G)| \geq 4$.

For a graph G and a vertex $x \in V(G)$ we define $N[x] = \{y \in V(G) \mid xy \in E(G)\}$, and for a set A of vertices we let $N[A] = \bigcup_{x \in A} N[x]$ and $N'[A] = \bigcap_{x \in A} N[x]$. Also, for a set of vertices $A \subseteq V(G \diamond H)$ and for each $a \in V(G)$, we let $A_a = \{u_a \in V(H_a) \mid \text{there is some } x \in V(H) \text{ such that } u_x \in A\}$.

With these notations, note that a graph G is 3-w.e.c. if and only if for every 3-subset $A \subset V(G)$, the following two items hold

- (1) $N'[A] \neq \emptyset$ or $V(G) \setminus N[A] \neq \emptyset$, and
- (2) for every vertex $t \in A$, $N[t] \setminus N[A \setminus \{t\}] \neq \emptyset$ or $N'[A \setminus \{t\}] \setminus N[t] \neq \emptyset$.

We are now ready to state and prove a characterisation theorem.

Theorem 2.2 *Let G be a graph with $|V(G)| \geq 4$ and with loops at every vertex of $V(G)$ and let H be a 3-e.c. graph. The graph G is 3-w.e.c. if and only if $G \diamond H$ is 3-e.c.*

Proof Suppose that H is 3-e.c. and G is 3-w.e.c. In order to show that $G \diamond H$ is 3-e.c., for an arbitrary set of three vertices $S = \{u_x, v_y, w_z\} \subset V(G \diamond H)$ we show that there exists a T -solution for each $T \in P(S)$. Note that since $\overline{G \diamond H} = G \diamond \overline{H}$, then if there is a T -solution in $G \diamond H$ for $|T| = 0$ (1, resp.), then there is a T -solution for $|T| = 3$ (2, resp.). To see this, suppose that there is an \emptyset -solution in $G \diamond H$, and since \overline{H} is 3-e.c., there is an \emptyset -solution in $G \diamond \overline{H}$ and hence in $\overline{G \diamond H}$, too. This implies that there is an S -solution in $G \diamond H$. A similar argument holds for the case $|T| = 1$.

Let $A = \{x, y, z\}$ and $B = \{u, v, w\}$. So $1 \leq |A|, |B| \leq 3$. If $|A| = 1$, then since H is 3-e.c., there exists an S -solution. Now we consider the remaining possibilities for B and A .

Case 1. Suppose that $|A| = 3$. First consider the case $T = \emptyset$. Let a be weak \emptyset -solution with respect to A . If $a \in V(G) \setminus N[A]$, then if t is an S_a -solution with respect to S_a , t_a is an \emptyset -solution with respect to S . If $a \in N'[A]$, then if t is an \emptyset -solution with respect to S_a , t_a is an \emptyset -solution with respect to S .

If $T = \{u_x\}$, then let a be a weak $\{x\}$ -solution with respect to A . If $a \in N[x] \setminus N[A \setminus \{x\}]$, then if t is an S_a -solution with respect to S_a , t_a is a $\{u_x\}$ -solution with respect to S . If $a \in N'[A \setminus \{x\}] \setminus N[x]$, then if t is an \emptyset -solution with respect to S_a , t_a is a $\{u_x\}$ -solution with respect to S . Similar arguments hold for $T \in \{\{v_y\}, \{w_z\}\}$.

Case 2. Next suppose that $|A| = 2$. We argue this case in two subcases depending on whether the vertices of S are congruent or incongruent.

Case 2.a. First suppose that the vertices of S are incongruent; $S = \{u_x, v_y, w_y\}$.

We first consider the case $T = \emptyset$. Let a be weak \emptyset -solution with respect to A . If $a \in V(G) \setminus N[A]$, then if t is an S_a -solution with respect to S_a , t_a is an \emptyset -solution with respect to S . If $a \in N'[A]$, then if t is an \emptyset -solution with respect to S_a , t_a is an \emptyset -solution with respect to S .

If $T = \{u_x\}$, then let a be a weak $\{x\}$ -solution with respect to A . If $a \in N[x] \setminus N[y]$, then if t is an S_a -solution with respect to S_a , t_a is a $\{u_x\}$ -solution with

respect to S . If $a \in N[y] \setminus N[x]$, then if t is an \emptyset -solution with respect to S_a , t_a is a $\{u_x\}$ -solution with respect to S .

If $T = \{v_y\}$, then let a be a weak $\{x\}$ -solution with respect to A . If $a \in N[y] \setminus N[x]$, then if t is a $\{u_a, v_a\}$ -solution with respect to S_a , t_a is a $\{v_y\}$ -solution with respect to S . If $a \in N[x] \setminus N[y]$, then if t is a $\{w_a\}$ -solution with respect to S_a , t_a is a $\{v_y\}$ -solution with respect to S . A similar argument holds for $T = \{w_y\}$.

Case 2.b. Now suppose that S contains congruent vertices; $S = \{u_x, u_y, w_y\}$. In this case, the only difference with Case 2.a. is in finding a $\{w_y\}$ -solution. Let a be weak \emptyset -solution with respect to A . If $a \in V(G) \setminus N[A]$, then if t is a $\{u_a\}$ -solution with respect to S_a , t_a is a $\{w_y\}$ -solution with respect to S . If $a \in N'[A]$, then if t is a $\{w_a\}$ -solution with respect to S_a , t_a is a $\{w_y\}$ -solution with respect to S .

Note that for any 3-subset $S \subset V(G \diamond H)$, and any $a \in V(G)$, since H_a is isomorphic to H and hence is 3-e.c., then for each $T' \subseteq S_a$ there exists a T' -solution with respect to S_a . Observe for each case considered in this argument, the solutions found are in $V(G \diamond H) \setminus S$. As there is a solution for an arbitrary set of three vertices of $G \diamond H$, we conclude that $G \diamond H$ is 3-e.c.

To prove the converse implication, suppose that $G \diamond H$ is 3-e.c. but G is not 3-w.e.c. Assume that $A = \{x, y, z\} \subset G$ for which there is no weak T -solution for some $T \subseteq A$. Let $S = \{u_x, u_y, u_z\}$.

As an initial case, suppose that there is no weak \emptyset -solution. If every vertex of G is in the neighbourhood of at least one and at most two of the vertices in A , then every vertex of $G \diamond H$ is adjacent to at least one and at most two of the vertices in S , and so there is no vertex of $G \diamond H$ that is an S -solution with respect to S .

Now suppose that there is no weak T -solution for some $T \subseteq A$ with $|T| = 1$. Without loss of generality suppose that $N[x] \setminus N[\{y, z\}] = \emptyset$ and $N[\{y, z\}] \setminus N[x] = \emptyset$. So, any vertex in $N[x]$ is also in $N[\{y, z\}]$ and any vertex in $N[\{y, z\}]$ is also in $N[x]$. These imply that any vertex in $N[x]$ is in $N[y]$ or $N[z]$ and any vertex in $V(G) \setminus N[x]$ is in at most one of $N[y]$ and $N[z]$. Thus any vertex of $G \diamond H$ that is adjacent to u_x is also adjacent to u_y or to u_z and so there is no $\{u_x\}$ -solution with respect to S .

In each case we establish the contradiction that the graph $G \diamond H$ is not 3-e.c., and the argument is complete. □

Theorem 2.1 now becomes a corollary of Theorem 2.2.

Proof of Theorem 2.1 Since G is 3-e.c., it is also 3-w.e.c. □

In general, given graphs G and H such that H is 3-e.c., in order to determine whether or not $G \diamond H$ is 3-e.c., we need to examine the existence of $8 \binom{|V(G)| + |V(H)|}{3}$ T -solutions. However, by applying Theorem 2.2, we only need to examine if G is 3-w.e.c., and hence at most $8 \binom{|V(G)|}{3}$ sets would need to be compared with the empty set.

Having shown that the modular product can produce a 3-e.c. graph given a 3-w.e.c. graph and a 3-e.c. graph, we now find graphs G that are 3-w.e.c. We focus our

attention on cases in which G is either a complete multipartite graph or a strongly regular graph.

3 Weakly 3-e.c. Complete Multipartite Graphs

In this section we show that most of the complete multipartite graphs are 3-w.e.c.

Theorem 3.1 *The complete i -partite graph $K_{\ell_1, \ell_2, \dots, \ell_i}$ with $\ell_j \geq 2$ for all $j \in \{1, 2, \dots, i\}$ is 3-w.e.c.*

Proof Let X and Y be two distinct parts in the obvious partition of $K_{\ell_1, \ell_2, \dots, \ell_i}$. Consider a set of vertices $A = \{x, y, z\}$ of $K_{\ell_1, \ell_2, \dots, \ell_i}$. If all three vertices of A are in the same part, say $A \subseteq X$, any vertex in Y is a weak \emptyset -solution with respect to A . Also $x \in N[x] \setminus N[\{y, z\}]$, and similarly $y \in N[y] \setminus N[\{x, z\}]$ and $z \in N[z] \setminus N[\{x, y\}]$, and so there exists a weak T -solution for any $T \subset A$ with $|T| = 1$.

If x is a vertex in a part, say $x \in X$, and y and z are in another part, say $\{y, z\} \subseteq Y$, then $x \in N'[A]$ and so there is a weak \emptyset -solution with respect to A . Also note that since $\ell_j \geq 2$, then there exists a vertex $r \in X \setminus \{x\}$, and hence $r \in N'[\{y, z\}] \setminus N[x]$. Also, $z \in N'[\{x, z\}] \setminus N[y]$, and $y \in N'[\{x, y\}] \setminus N[z]$.

It now remains to consider the case when each vertex in A is in a distinct part. Suppose that x', y' and z' are vertices of $V(K_{\ell_1, \ell_2, \dots, \ell_i}) \setminus A$ and in the same parts as x, y and z respectively. We have $x \in N'[A]$ and so there exists a weak \emptyset -solution with respect to A . Also $x' \in N'[\{y, z\}] \setminus N[x]$, $y' \in N'[\{x, z\}] \setminus N[y]$, and $z' \in N'[\{x, y\}] \setminus N[z]$ and so is a weak T -solution for any $T \subset A$ with $|T| = 1$. So, A has a weak solution and the graph $K_{\ell_1, \ell_2, \dots, \ell_i}$ is 3-w.e.c. \square

It follows from Theorem 3.1 that every bipartite graph $K_{\ell, m}$ with $\ell, m \geq 2$ is 3-w.e.c. The only remaining bipartite graphs to consider are of the form $K_{1, m}$ with $m \geq 3$. Let $A = \{x, y, z\} \subset V(K_{1, m})$. We will show that there is a weak solution for A . If all the vertices of A are in the same part, then the argument is similar to the corresponding case in the proof of Theorem 3.1. Now without loss of generality suppose that x is the singleton part, and y, z and r are in the part with m vertices. So, $x \in N'[A]$, $y \in N'[\{x, y\}] \setminus N[z]$, $z \in N'[\{x, z\}] \setminus N[y]$ and $r \in N[x] \setminus N[\{y, z\}]$ and so $K_{1, m}$ is 3-w.e.c.

We have shown that $K_{2,2} \diamond H$ and $K_{1,3} \diamond H$ are 3-e.c. if H is 3-e.c., thereby producing two non-isomorphic 3-e.c. graphs of order $4|V(H)|$. Since the smallest 3-e.c. graph that is known to date has order 28 [11], this order of $4|V(H)|$ is much smaller than $28|V(H)|$ if both graphs were required to be 3-existentially closed (as was required in [1]).

4 Weakly 3-e.c. Strongly Regular Graphs

A k -regular graph G in which each pair of adjacent vertices has exactly λ common neighbours, and each pair of non-adjacent vertices has exactly μ common neighbours

is called a strongly regular graph; we say that G is a $SRG(v, k, \lambda, \mu)$ with $v = |V(G)|$. In this section we recognise a few classes of strongly regular graphs that possess the 3-w.e.c. adjacency property.

Theorem 4.1 *The empty graph G with $|V(G)| \geq 4$ is 3-w.e.c.*

Proof Let $A = \{x, y, z\} \subset V(G)$ and $t \in V(G) \setminus A$. Obviously, $t \in V(G) \setminus N[A]$ which establishes the existence of a weak \emptyset -solution. Also $x \in N[x] \setminus N[\{y, z\}]$, $y \in N[y] \setminus N[\{x, z\}]$, and $z \in N[z] \setminus N[\{x, y\}]$ which establish the existence of a weak T -solution with respect to A for any set $T \subset A$ with $|T|=1$. □

By Theorem 4.1, in addition to the two 3-e.c. graphs $K_{2,2} \diamond H$ and $K_{1,3} \diamond H$, we obtain $\overline{K}_4 \diamond H$ as another 3-e.c. graph on $4|V(H)|$ vertices. We now characterise another family of 3-w.e.c. strongly regular graphs.

Theorem 4.2 *If G is a $SRG(v, k, \lambda, \mu)$ such that*

- (i) $v \geq \max\{3k - \lambda - 2\mu + 2, 3k - 3\mu + 4\}$ and
- (ii) $k \geq \max\{2\lambda + 3, \lambda + \mu + 2, 2\mu + 1\}$,

then G is 3-w.e.c.

Proof Let $A = \{x, y, z\} \subset V(G)$. We first show that there is a weak \emptyset -solution with respect to A .

If at least two pairs of the vertices of A are adjacent, then $N'[A] \neq \emptyset$ and so there is a weak \emptyset -solution with respect to A .

If only one pair of the vertices in A is adjacent, say x and y , then x and y have λ common neighbours, whereas x and z (resp. y and z) have μ common neighbours. Considering that the degree of each vertex is k , then by the principle of inclusion and exclusion we have $|N[A]| = 3k - \lambda - 2\mu + 1 + |N'[A]|$. Now if $|N'[A]| \neq \emptyset$, then clearly there is a weak \emptyset -solution with respect to A . Otherwise $|N'[A]| = \emptyset$ and $|N[A]| = 3k - \lambda - 2\mu + 1$, and since $v > 3k - \lambda - 2\mu + 1$ by (i), then there is a vertex in $V(G) \setminus A$ that is not a neighbour of x, y , or z ; hence $V(G) \setminus N[A] \neq \emptyset$ and there is a weak \emptyset -solution with respect to A .

Finally, if no pair of the vertices in A is adjacent, then since G is k -regular and since every pair of the vertices of A have μ common neighbours, then by the principle of inclusion and exclusion $|N[A]| = 3k - 3\mu + 3 + |N'[A]|$. Again, if $|N'[A]| \neq \emptyset$, then there is a weak \emptyset -solution with respect to A . Otherwise $|N'[A]| = \emptyset$ and $|N[A]| = 3k - 3\mu + 3$, and since $v > 3k - 3\mu + 3$ by (i), then there exists a vertex in $V(G) \setminus A$ that is non-adjacent to every vertex in A ; hence $V(G) \setminus N[A] \neq \emptyset$ which establishes the existence of a weak \emptyset -solution with respect to A .

Now it only remains to show that there is a weak T -solution with respect to A for any $T \subset A$ with $|T| = 1$. Without loss of generality let $t = x$. If x is adjacent

to both y and z , then x and y (resp. x and z) have λ common neighbours. Since $\deg(x) = k$ and $k > 2\lambda + 2$ by (ii), then there exists a vertex different from y and z which is adjacent to x and non-adjacent to both y and z . This implies that $N[x] \setminus N[\{y, z\}] \neq \emptyset$.

If x is adjacent to one of y or z , say y , then x and y have λ common neighbours, and x and z have μ common neighbours. Again, since $\deg(x) = k$ and $k > \lambda + \mu + 1$ by (ii), then there exists a vertex different from y which is adjacent to x and non-adjacent to both y and z , and hence $N[x] \setminus N[\{y, z\}] \neq \emptyset$. The case that x is non-adjacent to both y and z can be argued similarly.

So, there is a weak solution for A and G is 3-w.e.c. \square

Note that the graphs that satisfy the conditions of Theorem 4.2 tend to be sparse. Examples of such graphs are the Petersen graph (a SRG(10, 3, 0, 1)), the Clebsch graph (a SRG(16, 5, 0, 2)), the Hoffman-Singleton graph (a SRG(50, 7, 0, 1)), the Gewirtz graph (a SRG(56, 10, 0, 2)), the M22 graph (a SRG(77, 16, 0, 4)), the Brouwer-Haemers graph (a SRG(81, 20, 1, 6)), the Higman-Sims graph (a SRG(100, 22, 0, 6)), the Local McLaughlin graph (a SRG(162, 56, 10, 24)), and the $n \times n$ square rook's graph (a SRG($n^2, 2n - 2, n - 2, 2$)) for large enough n . Next we present a family of 3-w.e.c. that are dense.

Theorem 4.3 *If G is a SRG($v, v - 2, v - 4, v - 2$) with $v \geq 4$, then G is 3-w.e.c.*

Proof Note that since G is $(v - 2)$ -regular, for each set of three vertices of G at least two pairs of the vertices are adjacent. Let $A = \{x, y, z\} \subset V(G)$ be a set of three vertices, and without loss of generality suppose that x is adjacent to both y and z . Since $\deg(x) = v - 2$, there exists a vertex $r \in V(G) \setminus A$ such that $rx \notin E(G)$ and $\{ry, rz\} \subseteq E(G)$. Note that $x \in N[A]$ and so there is a weak \emptyset -solution with respect to A . It only remains to show that there is a weak T -solution with respect to A for any $T \subset A$ with $|T| = 1$. We will consider two cases depending on whether or not $yz \in E(G)$.

As a first case, suppose that $yz \notin E(G)$. So $y \in N'[\{x, y\}] \setminus N[z]$, $z \in N'[\{x, z\}] \setminus N[y]$ by symmetry, and $r \in N'[\{y, z\}] \setminus N[x]$. Second, suppose that $yz \in E(G)$ and without loss of generality let $t = x$. We have $r \in N'[\{y, z\}] \setminus N[x]$, and by symmetry similar arguments establish the cases $t = y$ and $t = z$. So there is a weak T -solution with respect to A for any $T \subset A$ with $|T| = 1$.

So, there is a weak solution for A and G is 3-w.e.c. \square

Note that for each even $v \geq 4$, SRG($v, v - 2, v - 4, v - 2$) is the complement of a perfect matching on v vertices.

5 Discussion

Now that we are able to recognise some classes of graphs G that are 3-w.e.c. and hence enabling us to construct new 3-e.c. graphs $G \diamond H$ given that H is 3-e.c., in this

section we discuss some graphs G for which G is not 3-w.e.c. In Theorem 3.1 we showed that $K_{2,2}$ is 3-w.e.c. By observing that $K_{2,2}$ is isomorphic to C_4 , it is natural to ask which values of m result in 3-w.e.c. C_m . As it happens $m = 4$ is unique in this regard.

Proposition 5.1 *The cycle C_m of order m is 3-w.e.c. if and only if $m = 4$.*

Proof Suppose that we have labelled the vertices of C_m in the clockwise order by $1, 2, \dots, m$. The graph C_4 is isomorphic to $K_{2,2}$ for which we have shown $K_{2,2}$ is 3-w.e.c. If $m \neq 4$, then for $A = \{1, 2, 3\}$ there is no weak $\{2\}$ -solution with respect to A because $N'[\{1, 3\}] \setminus N[2] = \emptyset$ and $N[2] \setminus N[\{1, 3\}] = \emptyset$. \square

It is also natural to ask whether it might be possible to use the modular product to obtain graphs that are 4-e.c.

Proposition 5.2 *If G and H are two graphs such that $|V(G)| \geq 2$ and H is 4-e.c., then $G \diamond H$ cannot be 4-e.c.*

Proof Let H be a 4-e.c. graph, and let G be any graph. Consider $S = \{u_x, u_y, v_x, v_y\}$ a set of four vertices of $G \diamond H$ such that $x \neq y$, u_x and u_y are congruent, and v_x and v_y are also congruent. For $T = \{u_x, u_y, v_x\}$ there is no T -solution. \square

We conclude this section with two open problems that warrant further investigation.

Problem 5.1 *Other than graph complementation, no graph operation has yet been found that preserves the n -e.c. property for $n \geq 4$. Find an n -e.c. preserving (binary) graph operation for $n = 4$ and then for higher values of n .*

Problem 5.2 *Produce n -e.c. graphs using some graph operations such that none of the graph(s) in the operation needs to be n -e.c.*

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