AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **52** (2012), Pages 141–162

Full friendly index sets of cylinder graphs*

Wai Chee Shiu† Fook Sun Wong

Department of Mathematics
Hong Kong Baptist University
224 Waterloo Road, Kowloon Tong
Hong Kong
China

Abstract

Let $G = (V, E)$ be a connected simple graph. A labeling $f : V \to \mathbb{Z}_2$ induces an edge labeling $f^+ : E \to \mathbb{Z}_2$ defined by $f^+(xy) = f(x) + f(y)$ for each $xy \in E$. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$ and $e_f(i) = |(f^+)^{-1}(i)|$. A labeling f is called friendly if $|v_f(1) - v_f(0)| \le 1$. For a friendly labeling f of a graph G, we define the friendly index of G under f by $i_f(G) = e_f(1) - e_f(0)$. The set $\{i_f(G) \mid f \text{ is a friendly labeling of } G\}$ is called the full friendly index set of G, denoted by $\mathrm{FFI}(G)$. In this paper, we determine the full friendly index sets of cylinder graphs $C_m \times P_n$ for even $m \ge 4$, even $n \ge 4$ and $m \le 2n$. We also list the results of other cases for $m, n \ge 4$.

1 Introduction

In this paper, all graphs are simple and connected. All undefined symbols and concepts may be looked up from [2]. Let $G = (V, E)$ be a connected simple graph. A labeling $f : V \to \mathbb{Z}_2$ induces an edge labeling $f^+ : E \to \mathbb{Z}_2$ defined by $f^+(xy) = f(x) + f(y)$ for each $xy \in E$. Throughout this paper, we will use the term 'labeling' to mean a vertex labeling whose values are taken in $\mathbb{Z}_2$. For $i \in \mathbb{Z}_2$, define $v_f(i) = |f^{-1}(i)|$ and $e_f(i) = |(f^+)^{-1}(i)|$. A labeling f is called *friendly* if $|v_f(1) - v_f(0)| \le 1$. For a friendly labeling f of a graph G, we define the *friendly index* of G under f by $i_f(G) = e_f(1) - e_f(0)$. We also refer to this value as the *friendly index* of f. Note that $i_f(G)$ can also be defined to any labeling f. Since $e_f(1) + e_f(0) = q$ the size of the graph G,

$$i_f(G) = 2e_f(1) - q = q - 2e_f(0). \tag{1}$$

* This work is partially supported by FRG, Hong Kong Baptist University.
† Corresponding author; email wcshiu@hkbu.edu.hk

The set $\{i_f(G) \mid f$ is a friendly labeling of $G\}$ is called the *full friendly index set* of G, denoted by FFI(G). It was first introduced by Shiu and Kwong [10] in 2007. Before the term FFI(G) was introduced, some of researchers studied friendly indices of graphs. Interested readers may find from [4–8]. The full friendly index sets of the graphs $P_2 \times P_n$ and $C_m \times C_n$ were found [9, 11, 12]. Recently Gao determined the full friendly index set of $P_m \times P_n$, but he used the terms 'edge difference set' instead of 'full friendly index set' and 'direct product' instead of 'Cartesian product' in [3]. In this paper, the full friendly index sets of $C_m \times P_n$ are shown for m, n even and $4 \le m \le 2n$.

The friendly index is related to the eigenvalues of a graph. Suppose A is a Laplacian matrix of a graph $G = (V, E)$ of order n. Since A is symmetric and semi-positive definite, all eigenvalues of A are real. We may list the eigenvalues of G in a descending order: $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{n-1} \ge \lambda_n = 0$. The maximum eigenvalue λ_1 is called the spectral radius of G. Let $S \subseteq V$ and $\overline{S} = V \setminus S$. Let $E(S, \overline{S})$ be the set of edges of G with one end in S and the other in $\overline{S}$. The cardinality of $E(S, \overline{S})$ is denoted by $e(S, \overline{S})$. It is clear that $e(S, \overline{S}) = e_f(1)$ for some labeling f. Then:

Theorem 1.1 ([1, Lemma 4.1]) *Let $G = (V, E)$ be a graph of order n, and let $S \subseteq V$ with $|S| = s$. Then*

$$\frac{s(n-s)\lambda_{n-1}}{n} \le e(S, \overline{S}) \le \frac{s(n-s)\lambda_1}{n}.$$

Thus we have

Corollary 1.1 *Let $G = (V, E)$ be a graph of order n and size q, and let f be any friendly labeling of G. Then*

$$\lambda_{n-1} \le \frac{(i_f(G) + q)n}{2\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil} \le \lambda_1.$$

Proof: Let $S = f^{-1}(1)$. Then $\overline{S} = f^{-1}(0)$. Since f is friendly, $s(n-s) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, where $s = |S|$. By Theorem 1.1 and (1) we have the corollary. $\square$

The *bipartition width* bw(G) is defined as

$$\text{bw}(G) = \min\{e(S, \overline{S}) \mid S \subseteq V, |S| = \lfloor \tfrac{n}{2} \rfloor\}.$$

It is clear that bw(G) is the minimum value of $e_f(1)$, where f is a friendly labeling. That is, bw$(G) = \frac{1}{2}(q + \min\{i_f(G) \mid f$ is friendly$\})$, where q is the size of G. It is also related to the Laplacian matrix of G (see [1]). The maximum value of $i_f(G)$ is related with max-cut problem and isoperimetric problem (see [1]).

For $m \ge 3$ and $n \ge 2$, given a cycle C_m and a path P_n with vertex sets $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$, respectively, the Cartesian product $C_m \times P_n$ is a graph with vertex set consisting of mn vertices labeled (i, j), where $1 \le i \le m$

and $1 \leq j \leq n$. Two vertices (i, j) and (h, k) are adjacent in $C_m \times P_n$ if either $i = h$ and v_j is adjacent to v_k in P_n, or $j = k$ and u_i is adjacent to u_h in C_m. Note that $C_m \times P_n$ is a graph of order mn and size $2mn - m$. It is also called a *cylinder graph* (or for short a *cylinder*). In this paper, the vertices (i, j) are often denoted as $u_{i,j}$ (or u_{ij}).

Shiu and Wong [11] determined the maximum and minimum values of the friendly index of cylinders in 2009.

Theorem 1.2 ([11]) *The maximum value of the friendly index of $C_m \times P_n$ is*

$$\begin{cases} 2mn - m & \text{if } m \text{ is even,} \\ 2mn - m - 2n & \text{if } m \text{ is odd.} \end{cases}$$

The next theorem is a combined version of the results in [11].

Theorem 1.3 *The minimum value of the friendly index of $C_m \times P_n$ is:*

1. *For $m \geq 2n$,*
$$\begin{cases} 4n + m - 2mn & \text{if } m \text{ is even,} \\ 4n + m - 2mn + 2 & \text{if } m \text{ is odd.} \end{cases}$$

2. *For $m \leq 2n - 1$,*
$$\begin{cases} 3m - 2mn & \text{if } n \text{ is even,} \\ 3m - 2mn + 4 & \text{if } n \text{ is odd.} \end{cases}$$

2 Nonexistent friendly indices of cylinders

In this section we will prove that some values, which lie between the extreme friendly indices of cylinders, do not appear in the friendly index set of cylinders in various conditions. Before discussing this matter, we have to introduce some concepts and some useful results.

For a fixed labeling f, a vertex v is called a *k-vertex* if $f(v) = k$ and an edge e is called a *k-edge* if $f^+(e) = k$. A path P (a cycle C, respectively) is called *mixed* (under f), if there are two vertices $u, v \in V(P)$ ($V(C)$, respectively) such that $f(u) = 1$ and $f(v) = 0$. Clearly, a mixed cycle or a mixed path contains at least one 1-edge. A cycle C is called a 1-*pure cycle* (under f), if $f(u) = 1$ for any vertex $u \in V(C)$. A cycle C is called a 0-*pure cycle* (under f), if $f(u) = 0$ for any vertex $u \in V(C)$. The definitions of 0-*pure path* and 1-*pure path* are similar.

Lemma 2.1 ([10, Corollary 5]) *Let f be a labeling of a cycle C. If C contains a 1-edge under f, then the number of 1-edges in C is a positive even number.*

Corollary 2.1 *Let f be a labeling of an even cycle C. If C contains a* 0*-edge under f, then the number of* 0*-edges in C is a positive even number.*

Consider the cylinder $C_m \times P_n$. For $1 \le i \le m$, the path $u_{i,1}u_{i,2}\cdots u_{i,n}$ is called a *vertical path* and for $1 \le i \le n$, the cycle $u_{1,j}u_{2,j}\cdots u_{m,j}u_{1,j}$ is called a *horizontal cycle*. The cycle $u_{1,1}u_{2,1}\cdots u_{m,1}u_{1,1}$ is called the *inner cycle*, and the cycle $u_{1,n}u_{2,n}\cdots u_{m,n}u_{1,n}$ is called the *outer cycle*. Each of them is called a *boundary cycle* of $C_m \times P_n$.

From (1) we have

$$i_f(C_m \times P_n) = 2e_f(1) - 2mn + m = 2mn - m - 2e_f(0) \tag{2}$$

for any labeling f.

Theorem 2.1 *For even m with $m \ge 4$ and $n \ge 2$, there is no friendly labeling f such that $e_f(1) = 2mn - m - p$, where $p = 1, 2, 3$.*

Proof: This is equivalent to showing that $e_f(0) \ne 1, 2, 3$ for any friendly labeling f. Note that, since m is even, $C_m \times P_n$ is bipartite without a cut edge.

Suppose $e_f(0) = 1$. Let C be a cycle, which must be even, in $C_m \times P_n$ containing the 0-edge. By Corollary 2.1, this is impossible.

Suppose $e_f(0) = 2$. If there is a 0-edge e which is not on a boundary cycle, then there exist two distinct 4-cycles (cycles of length 4) A and B containing e. For such cycles, by Corollary 2.1, there exist other 0-edges $e_1 \in E(A)$ and $e_2 \in E(B)$. Since $e_1 \ne e_2$, there are at least three 0-edges. This contradicts $e_f(0) = 2$. If two 0-edges are on a boundary cycle, then each of the 4-cycles (not the boundary cycles) containing such 0-edges contains only one 0-edge. It is impossible. If two 0-edges lie on different boundary cycles, then each boundary cycle must contain another 0-edge by Corollary 2.1. This contradicts $e_f(0) = 2$.

Suppose $e_f(0) = 3$. By Corollary 2.1, the total number of 0-edges lying on two boundary cycles is even. So there is a 0-edge, say e, which does not lie on a boundary cycle. Then there exist two distinct 4-cycles A and B containing e. For such cycles, by Corollary 2.1, there exist other 0-edges $e_1 \in E(A)$ and $e_2 \in E(B)$. Suppose the subgraph T induced by these three 0-edges is not isomorphic to $K_{1,3}$. Since $m \ge 4$, we can find an even cycle containing these 0-edges. It contradicts Corollary 2.1. If e_1 and e_2 do not lie on a boundary cycle, then we can find a 4-cycle containing only one 0-edge. It also contradicts Corollary 2.1. So T is isomorphic to $K_{1,3}$ and e_1 and e_2 lie on a boundary cycle. Without loss of generality, we may assume that the vertices of T are 1-vertices and two of these vertices lie on the inner cycle. Since there are no other 0-edges, the total numbers of 0-vertices and 1-vertices in each horizontal cycle are the same except in the inner cycle. Consider the total number of 0-vertices and 1-vertices in the inner cycle. There are two more 1-vertices than 0-vertices. So f is no longer friendly. Hence the proof is completed. □

Lemma 2.2 *For even m, if $C_m \times P_n$ contains a vertical mixed path under a friendly labeling f, then the number of vertical mixed paths is at least two.*

Proof: Assume there is only one vertical mixed path. If there are x vertical 1-pure paths and b 1-vertices in the vertical mixed path, then $v_f(1) = xn + b$. Note that $1 \le b < n$. Since f is friendly, we have $v_f(1) = v_f(0) = \frac{mn}{2}$, and then $2(xn+b) = mn$. We get $x = \frac{mn-2b}{2n} = \frac{m}{2} - \frac{b}{n} \notin \mathbb{Z}$. It is impossible. □

Lemma 2.3 *For even n, if $C_m \times P_n$ contains a horizontal mixed cycle under a friendly labeling f, then the number of horizontal mixed cycles is at least two.*

Proof: The proof is similar to Lemma 2.2. □

Lemma 2.4 *For even n with $m \le 2n$ and $m \ge 3$, $n \ge 4$, if $C_m \times P_n$ contains a horizontal pure cycle and a horizontal mixed cycle under a friendly labeling f, then $e_f(1) \ge m+4$ if m is odd; $e_f(1) \ge m+3$ if m is even and $m = 2n$; and $e_f(1) \ge m+4$ if m is even and $m \le 2n-2$.*

Proof: Let r and s be the numbers of horizontal 1-pure cycles and horizontal 0-pure cycles, respectively. By definition, we have $0 \le r, s \le \frac{n}{2}$.

When either $r = 0$ or $s = 0$ but not both. Then, without loss of generality, we may assume $r \ne 0$ and $s = 0$; otherwise, consider the labeling $1 - f \pmod 2$ instead of f. In this case, the number of horizontal mixed cycles is $n - r$, and hence there exist at least $\lceil \frac{mn}{2(n-r)} \rceil$ vertical mixed paths since $\frac{mn}{2}$ 0-vertices lies in $n-r$ horizontal mixed cycles. Therefore, by Lemma 2.1 we have $e_f(1) \ge 2(n-r) + \lceil \frac{mn}{2(n-r)} \rceil$.

Assume that $1 \le r \le \frac{n-2}{2}$. Otherwise f is not friendly. Then

$$e_f(1) \ge 2(n-r) + \lceil \tfrac{mn}{2(n-r)} \rceil \ge 2(n - \tfrac{n-2}{2}) + \lceil \tfrac{m(n-r+r)}{2(n-r)} \rceil = n + 2 + \lceil \tfrac{m}{2} + \tfrac{mr}{2(n-r)} \rceil. \quad (3)$$

For even m, (3) becomes

$$e_f(1) \ge n + 2 + \tfrac{m}{2} + \lceil \tfrac{mr}{2(n-r)} \rceil \ge n + 2 + \tfrac{m}{2} + 1 = n + 3 + \tfrac{m}{2}.$$

Thus, if $m = 2n$, then $n+3+\frac{m}{2} = m+3$; and if $m \le 2n-2$, then $n+3+\frac{m}{2} \ge m+4$.

For odd m, the condition $m \le 2n$ becomes $m \le 2n-1$ and Eq. (3) becomes

$$e_f(1) \ge n + 2 + \lceil \tfrac{m}{2} + \tfrac{mr}{2(n-r)} \rceil \ge n + 2 + \lceil \tfrac{m}{2} \rceil = n + 2 + \tfrac{m+1}{2}.$$

If $m \le 2n-3$, then from the above inequality we have $e_f(1) \ge n+2+\frac{m+1}{2} \ge m+4$.

If $m = 2n-1$, then from (3) we have

$$\begin{aligned} e_f(1) &\ge 2(n-r) + \lceil \tfrac{mn}{2(n-r)} \rceil \ge n + 2 + \lceil \tfrac{m(m+1)}{2m+2-4} \rceil = n + 2 + \lceil (\tfrac{m+1}{2})(\tfrac{m}{m-1}) \rceil \\ &\ge \tfrac{m+1}{2} + 2 + \tfrac{m+1}{2} + 1 = m + 4. \end{aligned}$$

When $r \neq 0$ and $s \neq 0$. There exist m vertical mixed paths. By the hypothesis and Lemma 2.3, the number of horizontal mixed cycles is at least 2. By Corollary 2.1, we have $e_f(1) \geq m + 2(2) = m + 4$.

Summarizing all cases, we have $e_f(1) \geq m + 4$ if m is odd; $e_f(1) \geq m + 3$ if m is even and $m = 2n$; and $e_f(1) \geq m + 4$ if m is even and $m \leq 2n - 2$. □

Theorem 2.2 *For even m, n with $m \leq 2n$, $m \geq 4$ and $n \geq 4$, for any friendly labeling f of $C_m \times P_n$, then*

(1) $e_f(1) = m$ or $e_f(1) \geq m + 2$ when $m = 2n$;

(2) $e_f(1) = m$ or $e_f(1) = m + 2$ or $e_f(1) \geq m + 4$ when $m = 2n - 2$;

(3) $e_f(1) = m$ or $e_f(1) \geq m + 4$ when $m \leq 2n - 4$.

Proof: Let a and b be the numbers of horizontal mixed cycles and vertical mixed paths, respectively. Since both m and n are even, by Lemmas 2.2 and 2.3, $a \neq 1$ and $b \neq 1$.

If $a = 0$, then there are m identical vertical mixed paths. Thus $e_f(1)$ is a multiple of m, i.e. $e_f(1) = mk$ for some integer $k \geq 1$. Since $m \geq 4$, either $e_f(1) = m$ or $e_f(1) \geq 2m \geq m + 4$.

If $2 \leq a < n$, then $C_m \times P_n$ contains a horizontal pure cycle and a horizontal mixed cycle. By Lemma 2.4, we have $e_f(1) \geq m + 3$ if $m = 2n$ and $e_f(1) \geq m + 4$ if $m \leq 2n - 2$.

Suppose $a = n$. Then

$$e_f(1) \geq 2n + b. \tag{4}$$

For $2n \geq m + 4$, $e_f(1) \geq m + 4$.

For $2n = m+2$, from (4), $e_f(1) \geq m+2+b$. If $b \geq 2$, then $e_f(1) \geq m+4$. If $b = 0$, then $e_f(1) = 2nk = (m + 2)k$ for some integer $k \geq 1$, which implies $e_f(1) \neq m + 3$ and $m + 1$.

For $2n = m$, from (4), we have $e_f(1) \geq m + b \geq m + 2$ if $b \geq 2$. If $b = 0$, then $e_f(1) = mk$ which implies that $e_f(1) \neq m + 1$. □

Lemma 2.5 *For even m, n with $m \geq 2n \geq 8$. If $C_m \times P_n$ contains a vertical pure path and a vertical mixed path under a friendly labeling f, then $e_f(1) \geq 2n + 2$.*

Proof: Let r and s be the numbers of vertical 1-pure paths and vertical 0-pure paths, respectively. By definition, we have $0 \leq r, s \leq \frac{m}{2}$. When either $r = 0$ or $s = 0$ but not both, then, without loss of generality, we may assume $r \neq 0$ and $s = 0$, otherwise, consider $1 - f \pmod 2$ instead of f. In this case, the number of vertical mixed paths is $m - r$, and hence there exist at least $\lceil \frac{mn}{2(m-r)} \rceil$ horizontal mixed cycles since $\frac{mn}{2}$ 0-vertices lie in $m-r$ vertical paths. Therefore, $e_f(1) \geq 2\lceil \frac{mn}{2(m-r)} \rceil + m - r$. Note that

$$2\lceil \tfrac{mn}{2(m-r)} \rceil + m - r \geq 2(\tfrac{n}{2}) + 2\lceil \tfrac{nr}{2(m-r)} \rceil + m - \tfrac{m}{2} \geq 2n + 2\lceil \tfrac{nr}{2(m-r)} \rceil \geq 2n + 2.$$

When $r \neq 0$ and $s \neq 0$. There exist n horizontal mixed cycles. By the hypothesis and Lemma 2.2, the number of vertical mixed paths is at least 2, then $e_f(1) \geq 2n+2$. Combining all cases, we have $e_f(1) \geq 2n + 2$. □

Theorem 2.3 *For even m, n with $m \geq 2n \geq 8$, for any friendly labeling f of $C_m \times P_n$, either $e_f(1) = 2n$ or $e_f(1) \geq 2n + 2$.*

Proof: Let a and b be the numbers of horizontal mixed cycles and vertical mixed paths, respectively. By Lemma 2.2 and 2.3, we know that $a \neq 1$ and $b \neq 1$. If $b = 0$, then there are $a = n$ identical horizontal mixed cycles. Hence, by Lemma 2.1, $e_f(1)$ is a multiple of $2n$. So either $e_f(1) = 2n$ or $e_f(1) \geq 4n \geq 2n + 4$.

If $2 \leq b < m$, then we have $m - b$ vertical pure paths and b vertical mixed paths. By Lemma 2.5, we have $e_f(1) \geq 2n + 2$.

If $b = m$ and $a = 0$, then the number of 1-edge is a multiple of m as a result of all horizontal cycles are pure cycles. So $e_f(1) = mk \neq 2n + 1$ for even m.

If $b = m$ and $a \geq 2$, then the number of $e_f(1) \geq m + 2a \geq 2n + 4$. Hence, combining all cases, we have either $e_f(1) = 2n$ or $e_f(1) \geq 2n + 2$. □

Lemma 2.6 *For odd n with $m \leq 2n - 1$ and $m \geq 4$, $n \geq 5$, if $C_m \times P_n$ contains a horizontal pure cycle and a horizontal mixed cycle under a friendly labeling f, then $e_f(1) = m + 2$ or $e_f(1) \geq m + 4$.*

Proof: Let r and s be the numbers of horizontal 1-pure cycles and horizontal 0-pure cycles, respectively. By definition, we have $0 \leq r, s \leq \frac{n-1}{2}$.

When either $r = 0$ or $s = 0$ but not both. Similar to the proof of Lemma 2.4, we may assume $r \neq 0$ and $s = 0$ and get $e_f(1) \geq 2(n - r) + \lceil \frac{mn}{2(n-r)} \rceil$. Note that if $1 \leq r \leq \frac{n-3}{2}$, then

$$\begin{aligned} e_f(1) &\geq 2(n-r) + \lceil \tfrac{mn}{2(n-r)} \rceil \geq 2(n - \tfrac{n-3}{2}) + \lceil \tfrac{m}{2} + \tfrac{mr}{2(n-r)} \rceil \\ &> n + 3 + \tfrac{m}{2} \geq \tfrac{m+1}{2} + 3 + \tfrac{m}{2} = m + \tfrac{7}{2}. \end{aligned}$$

If $r = \frac{n-1}{2}$, then

$$\begin{aligned} e_f(1) &\geq 2(n-r) + \lceil \tfrac{mn}{2(n-r)} \rceil = n + 1 + \lceil \tfrac{mn}{n+1} \rceil \geq n + 1 + \lceil \tfrac{5m}{6} \rceil \\ &\geq \tfrac{m+1}{2} + 1 + \tfrac{5m}{6} = m + \tfrac{2m+9}{6} \\ &\geq m + \tfrac{19}{6} \quad \text{if } m \geq 5. \end{aligned}$$

If $m = 4$ ($r = \frac{n-1}{2}$), then

$$e_f(1) \geq (n+1) + \lceil \tfrac{4n}{n+1} \rceil = (n+1) + 3 + \lceil \tfrac{n-3}{n+1} \rceil \geq 10 > m + 4.$$

So both cases guarantee that $e_f(1) \geq m + 4$.

When $r \neq 0$ and $s \neq 0$. There exist m vertical mixed paths. Since n is odd and f is friendly, there is at least one mixed cycle. If there exist two mixed cycles, then $e_f(1) \geq m + 2(2) = m + 4$. If there is exactly one mixed cycle, then as f is friendly, the number of 0-pure cycles and 1-pure cycles are the same. And also there are $\lceil \frac{m}{2} \rceil$ 1-vertices and $\lfloor \frac{m}{2} \rfloor$ 0-vertices contained in the mixed cycle as we may assume the number of 1-vertices is more than 0-vertices. So there are two kinds of identical mixed paths. The first kind contains $\lceil \frac{m}{2} \rceil$ identical mixed paths and the second kind contains $\lfloor \frac{m}{2} \rfloor$ identical mixed paths. Suppose the i-th kind mixed path contains x_i 1-edges, where $x_i \geq 1$. Then $e_f(1) = \lceil \frac{m}{2} \rceil x_1 + \lfloor \frac{m}{2} \rfloor x_2 + 2z$, where $2z$ is the number of 1-edges contained in the mixed cycle. If $x_1 + x_2 = 1$, then $e_f(1) = m + 2z \neq m + 3$. If $x_1 + x_2 \geq 3$, then $e_f(1) \geq m + \lfloor \frac{m}{2} \rfloor + 2z \geq m + 4$ as $z \geq 1$. So the proof is completed. □

Remark 2.1 *For odd n with $n \geq 3$, if $C_3 \times P_n$ contains a horizontal pure cycle and a horizontal mixed cycle under a friendly labeling f, then $e_f(1)$ can equal to $m + 3 = 6$.*

Theorem 2.4 *For even m, odd n with $m \leq 2n - 2$, $m \geq 4$ and $n \geq 5$, let f be a friendly labeling of $C_m \times P_n$. Then $e_f(1) = m + 2$ or $e_f(1) \geq m + 4$.*

Proof: Let the numbers of horizontal mixed cycles and vertical mixed paths of graph be a and b, respectively. By Lemma 2.2, we know that $b \neq 1$. Also since f is friendly and n is odd, $a \neq 0$.

If $1 \leq a < n$, then by Lemma 2.6, we have $e_f(1) = m + 2$ or $e_f(1) \geq m + 4$.

For $a = n$, all horizontal cycles are mixed. If $b = 0$, then $e_f(1) = 2nk \neq m + 3$ as m is even. If $b \geq 2$, then $e_f(1) = 2nk + b \geq 2n + 2 = m + 4$.

The proof is completed. □

Lemma 2.7 *For even m, odd n with $m \geq 2n \geq 10$. If $C_m \times P_n$ contains a horizontal pure cycle and a horizontal mixed cycle under a friendly labeling f, then $e_f(1) \geq 2n + 2$.*

Proof: Let r and s be the numbers of horizontal 1-pure cycles and horizontal 0-pure cycles, respectively. By definition, we have $0 \leq r, s \leq \frac{n}{2}$. When either $r = 0$ or $s = 0$ but not both. Similar to the proof of Lemma 2.4, we have $e_f(1) \geq 2(n-r) + \lceil \frac{mn}{2(n-r)} \rceil$. For $1 \leq r \leq \frac{n-3}{2}$, we have

$$\begin{aligned} e_f(1) &\geq 2(n-r) + \lceil \tfrac{mn}{2(n-r)} \rceil \geq 2(n - \tfrac{n-3}{2}) + \lceil \tfrac{n^2}{n-1} \rceil \\ &\geq n + 3 + n + 1 = 2n + 4. \end{aligned}$$

For $r = \frac{n-1}{2}$, we have

$$e_f(1) \geq n + 1 + \lceil \tfrac{mn}{n+1} \rceil \geq n + 1 + \lceil 2n - 2 + \tfrac{2}{n+1} \rceil = 3n \geq 2n + 5.$$

When $r \neq 0$ and $s \neq 0$. By a similar argument as in the proof of Lemma 2.4 we have $e_f(1) \geq m + 2 \geq 2n + 2$. Thus $e_f(1) \geq 2n + 2$. □

Theorem 2.5 *For even m, odd n with $m \geq 2n \geq 10$, for any friendly labeling f of $C_m \times P_n$, either $e_f(1) = 2n$ or $e_f(1) \geq 2n + 2$.*

Proof: Let a and b be the numbers of horizontal mixed cycles and vertical mixed paths, respectively. Note that since n is odd, $a \neq 0$. By Lemma 2.2, $b \neq 1$. If $1 \leq a < n$, then by Lemma 2.7, we have $e_f(1) \geq 2n + 2$. For $a = n$, all cycles are mixed. If $b = 0$, then $e_f(1) = k(2n) \neq 2n + 1$. If $b \geq 2$, then $e_f(1) \geq 2n + 2$. All cases suggest that $e_f(1) = 2n$ or $e_f(1) \geq 2n + 2$. □

Theorem 2.6 *For odd m, even n with $m \leq 2n - 1$, $m \geq 5$ and $n \geq 4$, let f be a friendly labeling of $C_m \times P_n$. Then $e_f(1) = m$ or $e_f(1) \geq m + 4$ when $m \leq 2n - 3$; and $e_f(1) = m$ or $e_f(1) \geq m + 2$ when $m = 2n - 1$.*

Proof: Let a and b be the numbers of horizontal mixed cycles and vertical mixed paths, respectively.

If $a = 0$, then there are m identical vertical mixed paths. Thus $e_f(1)$ is a multiple of m, i.e. $e_f(1) = mk$ for some integer $k \geq 1$. Since $m \geq 4$, $e_f(1) \neq m + p$ for $p = 1, 2, 3$.

If $1 \leq a < n$, then $C_m \times P_n$ contains a pure cycle and a mixed cycle. By Lemma 2.4, we have $e_f(1) \geq m + 4$.

Suppose $a = n$. Since m is odd, there is at least one mixed path. So $b \geq 1$, and

$$e_f(1) \geq 2n + b. \tag{5}$$

For $2n = m + 1$, $e_f(1) \geq 2n + 1 = m + 2$. For $2n \geq m + 3$, from (5), $e_f(1) \geq m + 3 + b \geq m + 4$. The proof is completed. □

Theorem 2.7 *For odd m, n with $m \leq 2n - 1$, $m \geq 5$ and $n \geq 5$, let f be a friendly labeling of $C_m \times P_n$. Then $e_f(1) \geq m + 2$ when $m = 2n - 1$ and $e_f(1) = m + 2$ or $e_f(1) \geq m + 4$ when $m \leq 2n - 3$.*

Proof: Let a and b be the numbers of horizontal mixed cycles and vertical mixed paths, respectively. Since both m and n are odd, $a \neq 0$ and $b \neq 0$. If $1 \leq a < n$, then by Lemma 2.6, we have $e_f(1) = m + 2$ or $e_f(1) \geq m + 4$.

For $a = n$, all cycles are mixed. Thus

$$e_f(1) \geq 2n + b \geq 2n + 1 \geq m + 2.$$

Note that if $m \leq 2n-3$, then the above inequality becomes $e_f(1) \geq 2n+b \geq 2n+1 \geq m + 4$. The proof is completed. □

Remark 2.2 For odd m with $m \geq 2n+1$, we cannot exclude other friendly indices.

3 Some Tools

We shall show some tools for finding friendly indices of the cylinders $C_m \times P_n$ for $m \geq 4$ and $n \geq 4$. We will not discuss small values of m and n.

Lemma 3.1 *For any m and n, let $G = C_m \times P_n$. Suppose $u, v \in V(G)$ are non-adjacent. Let f be a labeling with $f(u) = 0$ and $f(v) = 1$. Suppose that u is incident with x 1-edges and v is incident with y 1-edges. After swapping the labels of u and v, we let the new labeling be g. Then $e_f(1) - e_g(1) = 2x + 2y - \deg(u) - \deg(v)$.*

Proof: By assumption, u is incident with $\deg(u) - x$ 0-edges. Similarly, v is incident with $\deg(v) - y$ 0-edges. There are $x + y$ 1-edges incident with either u or v. After swapping the labels of u and v, there are $\deg(u) - x$ 1-edges incident with u and $\deg(v) - y$ 1-edge incident with v. Then there are $\deg(u) + \deg(v) - x - y$ 1-edges incident with u or v. Thus $e_f(1) - e_g(1) = (x + y) - (\deg(u) + \deg(v) - x - y) = 2x + 2y - \deg(u) - \deg(v)$. □

Lemma 3.2 *For any m and n, let $G = C_m \times P_n$. Suppose $u, v \in V(G)$ are adjacent. Let f be a labeling with $f(u) = 0$ and $f(v) = 1$. Suppose that u is incident with x 1-edges and v is incident with y 1-edges. After swapping the labels of u and v, we let the new labeling be g. Then $e_f(1) - e_g(1) = 2x + 2y - \deg(u) - \deg(v) - 2$.*

Proof: By the assumption, u is incident with $\deg(u) - x$ 0-edges. Similarly, v is incident with $\deg(v) - y$ 0-edges. There are $x + y - 1$ 1-edges incident with either u or v. After swapping the labels of u and v, there are $\deg(u) - x + 1$ 1-edges incident with u and $\deg(v) - y + 1$ 1-edge incident with v. Then there are totally $\deg(u) + \deg(v) - x - y + 1$ 1-edges incident with u or v. Similarly to the proof of Lemma 3.1, we have the lemma. □

Now for $m, n \geq 4$ and $1 \leq i \leq m-3$, $1 \leq j \leq n-3$, the subgraph $G(i, j)$ of $C_m \times P_n$ induced by

$$\begin{array}{llll} u_{i,j+3}, & u_{i+1,j+3}, & u_{i+2,j+3}, & u_{i+3,j+3}, \\ u_{i,j+2}, & u_{i+1,j+2}, & u_{i+2,j+2}, & u_{i+3,j+2}, \\ u_{i,j+1}, & u_{i+1,j+1}, & u_{i+2,j+1}, & u_{i+3,j+1}, \\ u_{i,j}, & u_{i+1,j}, & u_{i+2,j}, & u_{i+3,j}. \end{array}$$

is called the *sub-grid* of $C_m \times P_n$, where the first index i of u's are taking in $\mathbb{Z}_m = \{1, 2, \ldots, m\}$. We shall use the above matrix to represent the subgraph $G(i, j)$.

$G(i, 1)$ is called a *lower sub-grid* of $C_m \times P_n$. $G(i, n-3)$ is called an *upper sub-grid* of $C_m \times P_n$. The boxed part (see below) of the upper sub-grid and the lower sub-grid are called the *box of* $G(i, n-3)$ and the *box of* $G(i, 1)$. That is, the subgraph of $G(i, n-3)$ induced by $u_{i+1,n}$, $u_{i+2,n}$, $u_{i+1,n-1}$, $u_{i+2,n-1}$, $u_{i+1,n-2}$, $u_{i+2,n-2}$. It is similar for the box of $G(i, 1)$.

$$\begin{array}{cccc} u_{i,n}, & u_{i+1,n}, & u_{i+2,n}, & u_{i+3,n}, \\ u_{i,n-1}, & u_{i+1,n-1}, & u_{i+2,n-1}, & u_{i+3,n-1}, \\ u_{i,n-2}, & u_{i+1,n-2}, & u_{i+2,n-2}, & u_{i+3,n-2}, \\ u_{i,n-3}, & u_{i+1,n-3}, & u_{i+2,n-3}, & u_{i+3,n-3}. \end{array} \qquad \begin{array}{cccc} u_{i,4}, & u_{i+1,4}, & u_{i+2,4}, & u_{i+3,4}, \\ u_{i,3}, & u_{i+1,3}, & u_{i+2,3}, & u_{i+3,3}, \\ u_{i,2}, & u_{i+1,2}, & u_{i+2,2}, & u_{i+3,2}, \\ u_{i,1}, & u_{i+1,1}, & u_{i+2,1}, & u_{i+3,1}, \end{array}$$

Suppose f is a labeling of a graph $C_m \times P_n$. We shall use an $m \times n$ matrix whose (i,j)-th entry is $f(u_{i,j})$ to represent the labeling f (note that the numbering of rows is from left to right and that of columns is from bottom to top).

Moreover, suppose f is a labeling of $C_m \times P_n$. We shall use the matrix

$$\begin{bmatrix} f(u_{i,j+3}) & f(u_{i+1,j+3}) & f(u_{i+2,j+3}) & f(u_{i+3,j+3}) \\ f(u_{i,j+2}) & f(u_{i+1,j+2}) & f(u_{i+2,j+2}) & f(u_{i+3,j+2}) \\ f(u_{i,j+1}) & f(u_{i+1,j+1}) & f(u_{i+2,j+1}) & f(u_{i+3,j+1}) \\ f(u_{i,j}) & f(u_{i+1,j}) & f(u_{i+2,j}) & f(u_{i+3,j}) \end{bmatrix}_{i,j}$$

to denote the labeled sub-grid $G(i,j)$.

Let f be a friendly labeling of $C_m \times P_n$. If the labeled sub-grid $G(i,j)$ is of the form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{i,j} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{i,j},$$

then we call it a *full sub-grid*. The subscript of the matrix will be omitted when there is no ambiguity.

Lemma 3.3 *Let f be a friendly labeling of $C_m \times P_n$. Suppose $C_m \times P_n$ contains an upper (or a lower) full sub-grid under f. Then there is a friendly labeling g such that $e_g(1) = e_f(1) - k$, for each k, $4 \le k \le 7$.*

Proof: For a lower sub-grid, it is just a vertical reflection of an upper sub-grid. So we only need to consider an upper sub-grid. It suffices to find a friendly labeling g such that $k = e_f(1) - e_g(1)$ for each k from 4 to 7. Next, by swapping a 0-vertex and a 1-vertex inside the box of the sub-grid suitably, we get the result by Lemma 3.1 or 3.2. Namely, without loss of generality we may assume the upper full sub-grid is labeled as $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. The required labelings are the following ones:

$$\underset{k=4,}{\begin{bmatrix} 1 & \underline{1} & \underline{0} & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}} \quad \underset{k=5,}{\begin{bmatrix} 1 & \underline{1} & 1 & 0 \\ 0 & \underline{0} & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}} \quad \underset{k=6,}{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \underline{1} & 1 \\ 1 & 0 & \underline{0} & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}} \quad \underset{k=7.}{\begin{bmatrix} 1 & \underline{1} & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & \underline{0} & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}} \qquad \square$$

Two boxes are *non-adjacent* if the vertices between the two boxes are not adjacent, in the sense that there is a column and a row between the two boxes.

Corollary 3.1 *Let f be a friendly labeling of $C_m \times P_n$. Suppose $C_m \times P_n$ contains two non-adjacent boxes and each box is contained in an upper (or a lower) full sub-grid under f. Then there is a friendly labeling g such that $e_g(1) = e_f(1) - k$, for each k, $4 \le k \le 14$.*

Proof: For $4 \le k \le 7$, we are done by Lemma 3.3. For $8 \le k \le 14$, we can choose k_1 and k_2 with $4 \le k_1, k_2 \le 7$ such that $k = k_1 + k_2$. Since there are two non-adjacent boxes, by Lemma 3.3, there exists a labeling g such that $e_g(1) = e_f(1) - k_1 - k_2$. □

There are some special labelings for the sub-grid $G(m-3, n-3)$. We use the following matrices to represent them. Also they are named by the matrices, respectively.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ * & 1 & 0 & * \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ * & 1 & 0 & * \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ * & 1 & 0 & * \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ * & 1 & 0 & * \end{bmatrix},$$
$$E = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ * & 1 & 0 & * \end{bmatrix}, F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ * & 1 & 0 & * \end{bmatrix}, G = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ * & 0 & 0 & * \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ * & 1 & 0 & * \end{bmatrix}.$$

The '$*$' means either 0 or 1.

Lemma 3.4 *Let f be a friendly labeling of $C_m \times P_n$. Suppose $C_m \times P_n$ contains a sub-grid C under f. Then there is a friendly labeling under g such that $e_g(1) = e_f(1) + 1$.*

Proof: Change $C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ * & 1 & 0 & * \end{bmatrix}$ to $\begin{bmatrix} 0 & \underline{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & \underline{0} & 1 \\ * & 1 & 0 & * \end{bmatrix}$. Then by Lemma 3.1 we obtain the lemma. □

Lemma 3.5 *Let f be a friendly labeling of $C_m \times P_n$. Suppose $C_m \times P_n$ contains a sub-grid A under f. Then there is a friendly labeling g such that $e_g(1) = e_f(1) - k$, for each k, $2 \le k \le 7$.*

Proof: $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ * & 1 & 0 & * \end{bmatrix}$. By Lemma 3.1 or 3.2, the required labelings are the following ones:

$$\underset{k=2,}{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \underline{1} & 0 \\ 1 & 0 & \underline{0} & 1 \\ * & 1 & 0 & * \end{bmatrix}} \quad \underset{k=3,}{\begin{bmatrix} 1 & 0 & \underline{0} & 0 \\ 0 & 1 & \underline{1} & 0 \\ 1 & 0 & 1 & 1 \\ * & 1 & 0 & * \end{bmatrix}} \quad \underset{k=4,}{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \underline{1} & \underline{0} & 1 \\ * & 1 & 0 & * \end{bmatrix}}$$

$$\underset{k=5}{\begin{bmatrix} 1 & \underline{1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & \underline{0} & 1 \\ * & 1 & 0 & * \end{bmatrix}} \quad \underset{k=6,}{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \underline{0} & 0 & 0 \\ 1 & \underline{1} & 1 & 1 \\ * & 1 & 0 & * \end{bmatrix}} \quad \underset{k=7,}{\begin{bmatrix} 1 & 0 & \underline{0} & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \underline{1} & 1 & 1 \\ * & 1 & 0 & * \end{bmatrix}}$$

where $k = e_f(1) - e_g(1)$. □

Similarly we get the following lemmas.

Lemma 3.6 *Let f be a friendly labeling of $C_m \times P_n$. Suppose $C_m \times P_n$ contains a sub-grid B under f. Then there is a friendly labeling g such that $e_g(1) = e_f(1) - k$, for each k, $2 \leq k \leq 7$, $k \neq 5$.*

Lemma 3.7 *Let f be a friendly labeling of $C_m \times P_n$. Suppose $C_m \times P_n$ contains a sub-grid D under f. Then there is a friendly labeling g such that $e_g(1) = e_f(1) - k$, for each k, $2 \leq k \leq 7$.*

Lemma 3.8 *Let f be a friendly labeling of $C_m \times P_n$. Suppose $C_m \times P_n$ contains a sub-grid E under f. Then there is a friendly labeling g such that $e_g(1) = e_f(1) - k$, for each k, $2 \leq k \leq 5$.*

Lemma 3.9 *Let f be a friendly labeling of $C_m \times P_n$. Suppose $C_m \times P_n$ contains a sub-grid F under f. Then there is a friendly labeling g such that $e_g(1) = e_f(1) - k$, for each $-1 \leq k \leq 3$.*

Lemma 3.10 *Let f be a friendly labeling of $C_m \times P_n$. Suppose $C_m \times P_n$ contains a sub-grid G under f. Then there is a friendly labeling g such that $e_g(1) = e_f(1) - k$, for each k, $-1 \leq k \leq 3$.*

Lemma 3.11 *Let f be a friendly labeling of $C_m \times P_n$. Suppose $C_m \times P_n$ contains a sub-grid H under f. Then there is a friendly labeling g such that $e_g(1) = e_f(1) - k$, for each k, $3 \leq k \leq 7$.*

4 Finding Potential Values

In this section we will realize all potential friendly indices of $C_m \times P_n$ for even m and n with $m \leq 2n$ and $m, n \geq 4$.

Theorem 4.1 *For even m and n with $4 \leq m \leq 2n - 4$ and $n \geq 4$, then*

$$\mathrm{FFI}(C_m \times P_n) = \{-2mn + m + 2i \mid i \in [m + 4, 2mn - m - 4] \cup \{m, 2mn - m\}\}.$$

To prove Theorem 4.1 we have to find a friendly g such that the friendly index of g is $-2mn + m + 2i$ for $i \in [m + 4, 2mn - m - 4] \cup \{m, 2mn - m\}$, here $i = e_g(1)$.

The following algorithm for finding potential friendly indices is to swap some labels from an existing labeling π which contains at least two non-adjacent full sub-grids (or one special sub-grid). Let us call this labeling π as a *pivot labeling*. By using Corollary 3.1 (or some special lemma), for each $i \in [e_\pi(1) - 14, e_\pi(1) - 4]$ (or

$i \in [e_\pi(1) - l_1, e_\pi(1) - l_2]$ for some special values of l_1 and l_2) there is a friendly labeling g such that $e_g(1) = i$.

For even m, the friendly labeling $f_{\max}$ attains the maximum friendly index, which is defined by $f_{\max}(u_{a,b}) = 1$ if and only if $a + b \equiv 1 \pmod 2$ for $1 \le a \le m$ and $1 \le b \le n$. For this labeling, $e_{f_{\max}}(1) = 2mn - m$. This is the first pivot labeling that we use. Thus, by Corollary 3.1 we have the following theorem.

Theorem 4.2 *For even m with $m \ge 4$, and $n \ge 8$, there is a friendly labeling g of $C_m \times P_n$ such that $e_g(1) = i$ for each $i \in [2mn - m - 14, 2mn - m - 4]$.*

Theorem 4.3 *For even n with $m \le 2n$ and $m \ge 4$, $n \ge 6$, there exists a friendly labeling g of $G = C_m \times P_n$ such that $e_g(1) = i$ for each $i \in [m + 4, m + 8]$.*

Proof: Consider the friendly labeling $f_{\min}$ which is defined by $f_{\min}(u_{a,b}) = 1$ if and only if $1 \le a \le m$, $1 \le b \le \frac{n}{2}$. Then $e_{f_{\min}}(1) = m$ which attains the minimum friendly index of G. In $f_{\min}$, there are three categories of 0-vertices. They are
vertices incident with three 0-edges;
vertices incident with three 0-edges and one 1-edge; and
vertices incident with four 0-edges.
There are three categories of 1-vertices. They are
vertices incident with three 1-edges;
vertices incident with three 1-edges and one 0-edge; and
vertices incident with four 1-edges.

For each category of 0-vertices and each category of 1-vertices, there is a pair of non-adjacent vertices consisting of 0-vertex and a 1-vertex in the selected categories, respectively. By Lemma 3.1, all possible combinations of the change of $e_{f_{\min}}(1)$ are +4, +5, +6, +7 and +8. Thus the proof is completed. □

Similar to the proof of Theorem 4.3 we have

Theorem 4.4 *For $4 \le m \le 8$, there exists a friendly labeling g of $G = C_m \times P_4$ such that $e_g(1) = i$ for each $i \in [m + 4, m + 6]$.*

Theorem 4.5 *For even m, n with $m \ge 4$, $n \ge 10$ and $m \le 2n$, there is a friendly labeling g of $C_m \times P_n$ such that $e_g(1) = i$ for each $i \in [mn + 7m - 14, 2mn - m - 12]$ when $n \equiv 0 \pmod 4$ and $i \in [mn + 6m - 14, 2mn - m - 12]$ when $n \equiv 2 \pmod 4$.*

Proof: We introduce a procedure as follows:

Write $m = 2h$ and $n = 2k$. The initial labeling is $\alpha_{h,0} = f_{\max}$.

Procedure A: Let $y = 1$.

Step 1: If $y > \lceil \frac{k}{2} \rceil - 2$ then stop. Based on the labeling $\alpha_{h,y-1}$, for fixed x with $1 \le x \le h$, swap the labels of $u_{2i-1,2k-2-2y}$ and $u_{2i-1,3+2y}$ for $1 \le i \le x$, where both $u_{2i-1,2k-2-2y}$ and $u_{2i-1,3+2y}$ are incident to four 1-edges. Denote this labeling by $\alpha_{x,y}$.

Step 2: Increase y by 1 and repeat Step 1.

For $1 \le x \le h$, $1 \le y \le \lceil \frac{k}{2} \rceil - 2 = N$, by Lemmas 3.1 and 3.2,

$$e_{\alpha_{x,y}}(1) = \begin{cases} 2mn + 3m - 8x - 4my & \text{if } k = 2N+4,\ 1 \le x \le h \text{ and } 1 \le y \le N; \\ 2mn + 3m - 8x - 4my & \text{if } k = 2N+3,\ 1 \le x \le h \text{ and } 1 \le y \le N-1; \\ mn + 9m - 6x & \text{if } k = 2N+3,\ 1 \le x \le h \text{ and } y = N. \end{cases}$$

By using these pivot labelings $\alpha_{x,y}$ and Corollary 3.1, we get the theorem. □

Theorem 4.6 *For even m, n with $m \ge 4$, $n \ge 12$ and $m \le 2n$, there is a friendly labeling g of $C_m \times P_n$ such that $e_g(1) = i$, for each $i \in [16m - 14, mn + 7m - 12]$ when $n \equiv 0 \pmod 4$ and $i \in [16m - 14, mn + 6m - 12]$ when $n \equiv 2 \pmod 4$.*

Proof: We introduce a procedure as follows:

Procedure B: Write $m = 2h$ and $n = 2k$. This procedure is similar to Procedure A. The initial labeling is $\beta_{h,0} = \alpha_{h,\lceil \frac{k}{2} \rceil - 2}$ defined in Procedure A. Let $y = 1$.

Step 1: If $y > \lfloor \frac{k}{2} \rfloor - 2$ then stop. Based on the labeling $\beta_{h,y-1}$, for fixed x with $1 \le x \le h$, swap the labels of $u_{2i,2k-3-2y}$ and $u_{2i,4+2y}$ for $1 \le i \le x$, where both $u_{2i,2k-3-2y}$ and $u_{2i,4+2y}$ are incident to four 1-edges. Denote this labeling by $\beta_{x,y}$.

Step 2: Increase y by 1 and repeat Step 1.

For $1 \le x \le h$ and $1 \le y \le \lfloor \frac{k}{2} \rfloor - 2 = N$, by Lemmas 3.1, 3.2 and Corollary 3.1,

$$e_{\beta_{x,y}}(1) = \begin{cases} mn + 11m - 8x - 4my & \text{if } k = 2N+4,\ 1 \le x \le h \text{ and } 1 \le y \le N-1; \\ 19m - 6x & \text{if } k = 2N+4,\ 1 \le x \le h \text{ and } y = N; \\ mn + 10m - 8x - 4my & \text{if } k = 2N+3,\ 1 \le x \le h \text{ and } 1 \le y \le N-1. \end{cases}$$

By using these pivot labelings $\beta_{x,y}$ we have the theorem. □

Remark 4.1 For even m, n with $m \ge 4$, $n \ge 12$ and $m \le 2n$, combining Theorems 4.2, 4.5 and 4.6 there is a friendly labeling g of $C_m \times P_n$ such that $e_g(1) = i$, for each $i \in [16m - 14, 2mn - m - 4]$. For $n = 10$, combining Theorems 4.2 and 4.5, we have the same result.

Example 4.1 Some of the labelings of $C_6 \times P_{14}$ that appear in Procedures A and B:

Procedure A Procedure B

$$\left[\begin{array}{cccccc} 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ \hline \underline{0}&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ \underline{1}&1&0&1&0&1\\ \hline 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1 \end{array}\right] \quad \left[\begin{array}{cccccc} 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ \hline 0&0&0&0&0&0\\ 0&1&0&1&0&1\\ 0&0&0&0&\underline{0}&0\\ 1&1&1&1&\underline{1}&1\\ 1&0&1&0&1&0\\ 1&1&1&1&1&1\\ \hline 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1 \end{array}\right] \quad \left[\begin{array}{cccccc} 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&\underline{0}\\ 0&0&0&0&0&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&\underline{1}\\ 1&1&1&1&1&1\\ \hline 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1 \end{array}\right]$$

$e_{\alpha_{1,1}}(1) = 154$ $e_{\alpha_{3,2}}(1) = 120$ $e_{\beta_{3,1}}(1) = 96$

Theorem 4.7 *For even m, n with $m \geq 4$, $n \geq 10$ and $m \leq 2n$, there is a friendly labeling g of $C_m \times P_n$ such that $e_g(1) = i$ for each $i \in [m+31, 16m-8]$.*

Proof: We write $m = 2h$ and $n = 2k$. We will introduce some procedures in the following.
The initial labeling is $\beta_{h,\lfloor k/2\rfloor-2}$ when $n \equiv 0 \pmod 4$ ($\beta_{h,\lfloor k/2\rfloor-3}$ when $n \equiv 2 \pmod 4$) defined in Procedure B.

Procedure C: Swap the labels of $u_{2h,2k-1}$ and $u_{2h,2k-2}$, where both $u_{2h,2k-1}$ and $u_{2h,2k-2}$ are incident to four 1-edges. Let this labeling be γ'_0.

Step 1: Starting from the labeling γ'_0, for a fixed x with $1 \leq x \leq h$, swap the labels of $u_{2i-1,3}$ and $u_{2i-1,4}$ for all $1 \leq i \leq x$. Note that $u_{2i-1,3}$ is incident to four 1-edges and $u_{2i-1,4}$ is incident to three 1-edges. Let γ'_x be the new labeling.

Step 2: Based on the labeling γ'_h, for a fixed x with $1 \leq x \leq h$, swap the labels of $u_{2i-1,1}$ and $u_{2i-1,3}$ for all $1 \leq i \leq x$, where $u_{2i-1,1}$ is incident to three 1-edges and $u_{2i-1,3}$ is incident to one 1-edge for $1 \leq i \leq x$. Let γ_x be the new labeling.

By Lemma 3.2 we have $e_{\gamma'_x}(1) = 16m - 6 - 4x$ for $0 \leq x \leq h$. Each of these pivot labelings contains a sub-grid A. By Lemma 3.5, there exists a friendly labeling g such that $e_g(1) = i$, for $i \in [14m-13, 16m-8]$. Similarly, $e_{\gamma_x}(1) = 14m - 6 - x$ for $1 \leq x \leq h$. By Lemma 3.5 again, there exists a friendly labeling g such that $e_g(1) = i$, for $i \in [\frac{27m}{2} - 13, 14m - 9]$.

Procedure D: Start from the labeling γ_h. For a fixed x with $1 \leq x \leq h$, swap the labels of $u_{2i,2}$ and $u_{2i,3}$ for all $1 \leq i \leq x$, where both $u_{2i,2}$ and $u_{2i,3}$ are incident to four 1-edges. Let ζ_x be the new labeling.

If $m = 4$, then stop here. We have $e_{\zeta_x}(1) = \frac{27m}{2} - 6 - 6x$ for $1 \leq x \leq h$. By Lemma 3.5, there is a friendly labeling g such that $e_g(1) = i$ for $i \in [\frac{21m}{2} - 13, \frac{27m}{2} - 14]$.

Procedure E: Based on the labeling ζ_h, for a fixed x with $1 \leq x \leq h-2$, swap the labels of $u_{2i-1,2k}$ and $u_{2i-1,2k-3}$ for each $1 \leq i \leq x$, where both $u_{2i-1,2k}$ and $u_{2i-1,2k-3}$ are incident to three 1-edges. Let θ_x be the new labeling.

Procedure F: Based on the labeling θ_{h-2}, for a fixed x with $1 \le x \le h-2$, swap the labels of $u_{2i,2k-1}$ and $u_{2i,2k-2}$ for each $1 \le i \le x$, where both $u_{2i,2k-1}$ and $u_{2i,2k-2}$ are incident to four 1-edges. Let ξ_x be the new labeling.

Procedure G: Starting from the labeling is ξ_{h-2}, for a fixed x with $1 \le x \le 2h-4$, swap the labels of $u_{i,2k-2}$ and $u_{i,2k-3}$ with those of $u_{i,3}$ and $u_{i,4}$, respectively, for each $1 \le i \le x$. Let π_x be the new labeling.

Now we have $e_{\theta_x}(1) = \frac{21m}{2} - 6 - 5x$ for $1 \le x \le h-2$, $e_{\xi_x}(1) = 8m + 4 - 6x$ for $1 \le x \le h-2$ and $e_{\pi_1}(1) = 5m + 20$, $e_{\pi_x}(1) = 5m + 24 - 4x$ for $2 \le x \le m-5$, $e_{\pi_{m-4}}(1) = m + 38$. By Lemma 3.5 for $i \in [m+31, \frac{21m}{2} - 13]$, there is a friendly labeling g such that $e_g(1) = i$. Hence we get the theorem. □

Example 4.2 Following are some labelings of $C_6 \times P_{14}$ in Procedures C, D, E, F and G:

Procedure C

$$\left[\begin{array}{cccccc} 1&0&1&0&1&0\\ 0&1&0&1&0&\underline{0}\\ 1&0&1&0&1&\underline{1}\\ 0&1&0&1&0&1\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ \hline 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1 \end{array}\right] \quad \left[\begin{array}{cccccc} 1&0&1&0&1&0\\ 0&1&0&1&0&0\\ 1&0&1&0&1&1\\ 0&1&0&1&0&1\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ \hline 0&0&0&0&\underline{0}&0\\ 1&1&1&1&\underline{1}&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1 \end{array}\right] \quad \left[\begin{array}{cccccc} 1&0&1&0&1&0\\ 0&1&0&1&0&0\\ 1&0&1&0&1&1\\ 0&1&0&1&0&1\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ \hline 0&0&0&0&0&0\\ 0&1&0&1&\underline{0}&1\\ 1&0&1&0&1&0\\ 1&1&1&1&\underline{1}&1 \end{array}\right]$$

$e_{\gamma'_0}(1) = 90$ $\qquad$ $e_{\gamma'_3}(1) = 78$ $\qquad$ $e_{\gamma_3}(1) = 75$

Procedure D $\qquad$ Procedure E

$$\left[\begin{array}{cccccc} 1&0&1&0&1&0\\ 0&1&0&1&0&0\\ 1&0&1&0&1&1\\ 0&1&0&1&0&1\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ \hline 0&0&0&0&0&0\\ 0&0&0&\underline{0}&0&1\\ 1&1&1&\underline{1}&1&0\\ 1&1&1&1&1&1 \end{array}\right] \quad \left[\begin{array}{cccccc} \underline{0}&0&1&0&1&0\\ 0&1&0&1&0&0\\ 1&0&1&0&1&1\\ \underline{1}&1&0&1&0&1\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1 \end{array}\right]$$

$e_{\zeta_2}(1) = 63$ $\qquad$ $e_{\theta_1}(1) = 52$

Procedure F

$$\left[\begin{array}{cccccc} 0&0&1&0&1&0\\ 0&\underline{0}&0&1&0&0\\ 1&\underline{1}&1&0&1&1\\ 1&1&0&1&0&1\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1 \end{array}\right]$$
$e_{\xi_1}(1) = 46$

Procedure G

$$\left[\begin{array}{cccccc} 0&0&1&0&1&0\\ 0&0&0&1&0&0\\ \underline{0}&1&1&0&1&1\\ \underline{0}&1&0&1&0&1\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ \hline \underline{1}&0&0&0&0&0\\ \underline{1}&0&0&0&0&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1 \end{array}\right] \qquad \left[\begin{array}{cccccc} 0&0&1&0&1&0\\ 0&0&0&1&0&0\\ 0&\underline{0}&1&0&1&1\\ 0&\underline{0}&0&1&0&1\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ \hline 1&\underline{1}&0&0&0&0\\ 1&\underline{1}&0&0&0&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1 \end{array}\right]$$
$e_{\pi_1}(1) = 50$ $\qquad$ $e_{\pi_2}(1) = 44$

Theorem 4.8 *For even m, n with $m \geq 4$, $n \geq 10$ and $m \leq 2n$, there is a friendly labeling g of $C_m \times P_n$ such that $e_g(1) = i$ for each $i \in [m+9, m+30]$.*

Proof: We will introduce a procedure as follows.

Procedure H: If $m = 4$, then the initial labeling is ζ_2; if $m \geq 6$, then the initial labeling is π_{m-4}.

Step 1: Swap the labels of $u_{m,n-2}$ and $u_{m,n-3}$ with those of $u_{m-3,3}$ and $u_{m-3,4}$, respectively, and let ϕ_1 be the new labeling.

Step 2: Swap the labels of $u_{m-3,n}$ and $u_{m-3,n-3}$, denote the labeling by ϕ_2.

Step 3: Swap the labels of $u_{m-3,n-2}$ and $u_{m-2,3}$, denote the labeling by ϕ_3.

Step 4: Swap the labels of $u_{m-3,n-3}$ and $u_{m-2,4}$, denote the labeling by ϕ_4.

Step 5: Swap the labels of $u_{m-2,n-1}$ and $u_{m-1,3}$, denote the labeling by ϕ_5.

Step 6: Swap the labels of $u_{m-2,n-3}$ and $u_{m-1,4}$, denote the labeling by ϕ_6.

Step 7: Swap the labels of $u_{m-1,n-1}$ and $u_{m-1,4}$, denote the labeling by ϕ_7.

Based on the labeling ϕ_1. Swap the labels of $u_{m-3,n-2}$ and $u_{m,n-2}$, denote the labeling by ϕ_8. Based on the labeling ϕ_6. Swap the labels of $u_{m-1,n-2}$ and $u_{m,n-1}$; swap the labels of $u_{m,k}$ and $u_{m-1,k}$ with those of $u_{m,3}$ and $u_{m,4}$, respectively, denote the labeling by ϕ_9.

By Lemma 3.1 or 3.2 we have $e_{\phi_1}(1) = m+32$, $e_{\phi_2}(1) = m+29$, $e_{\phi_3}(1) = m+27$, $e_{\phi_4}(1) = m+23$, $e_{\phi_5}(1) = m+19$, $e_{\phi_6}(1) = m+13$, $e_{\phi_7}(1) = m+15$, $e_{\phi_8}(1) = m+30$, $e_{\phi_8}(1) = m+9$. Applying Lemmas 3.11, 3.7, 3.8, 3.9 and 3.10 to ϕ_1, ϕ_2, ϕ_4, ϕ_5 and ϕ_6, respectively, and combining with ϕ_7, ϕ_8 and ϕ_9, there is a friendly labeling g such that $e_g(1) = i$ for $i \in [m+9, m+30]$. □

Theorem 4.9 *For even m with $6 \leq m \leq 16$, there is a friendly labeling g of $C_m \times P_8$ such that $e_g(1) = i$ for each $i \in [m+8, 15m-8]$.*

Proof: We apply Procedure C to Procedure G. In this case $\beta_{h,0} = \alpha_{h,0} = f_{\max}$. Applying Lemma 3.2 or Lemma 3.1 we have $e_{\gamma'_x}(1) = 15m - 6 - 6x$ for $0 \le x \le \frac{m}{2}$; $e_{\gamma_x}(1) = 12m - 6 - x$ for $1 \le x \le \frac{m}{2}$; $e_{\zeta_x}(1) = \frac{23m}{2} - 6 - 6x$ for $1 \le x \le \frac{m}{2}$; $e_{\theta_x}(1) = \frac{17m}{2} - 6 - 5x$ for $1 \le x \le \frac{m}{2} - 2$; $e_{\xi_x}(1) = 6m + 4 - 6x$ for $1 \le x \le \frac{m}{2} - 2$; $e_{\pi_1}(1) = 3m + 22$; $e_{\pi_x}(1) = 3m + 24 - 2x$ for $2 \le x \le m - 5$; $e_{\pi_{m-4}}(1) = m + 30$. After that we apply Procedure H. Applying Lemma 3.2 or Lemma 3.1 we have $e_{\phi_1}(1) = m + 26$; $e_{\phi_2}(1) = m + 21$; $e_{\phi_4}(1) = m + 17$ and $e_{\phi_6}(1) = m + 11$. Applying Lemma 4.6 to γ'_x, γ_x, ζ_x, θ_x, ξ_x and Lemmas 3.11, 3.7, 3.8 and 3.10 to ϕ_1, ϕ_2, ϕ_4 and ϕ_6, respectively, we get a friendly labeling g such that $e_g(1) = i$ for $i \in [m+8, 15m-8]$. □

Theorem 4.10 *For even m with $4 \le m \le 12$, there is a friendly labeling g of $C_m \times P_6$ such that $e_g(1) = i$ for each $i \in [m+8, 11m-4]$.*

Proof: Based on the labeling $f_{\max}$. We swap the labels of $u_{m,5}$ and $u_{m,4}$. Let this labeling be γ. Then $e_\gamma(1) = 11m - 6$. Applying Lemma 3.3 and Lemma 3.5 to $f_{\max}$ and γ respectively, there is a friendly labeling g of $C_m \times P_6$ such that $e_g(1) = i$ for each $i \in [11m - 13, 11m - 4]$. We still write $m = 2h$.

Based on the labeling γ for a fixed x with $1 \le x \le h$, swap the label of $u_{2i-1,1}$ with that of $u_{2i-1,2}$ for all $1 \le i \le x$, where $u_{2i-1,1}$ is adjacent to three 1-vertices, and $u_{2i-1,2}$ is adjacent to four 0-vertices. Let ζ_x be the new labeling. Then $e_{\zeta_x}(1) = 11m-6-5x$. Lemma 3.5, there is a friendly labeling g of $C_m \times P_6$ such that $e_g(1) = i$ for each $i \in [\frac{17m}{2} - 13, 11m - 13]$.

When $m \ge 6$. Then we perform Procedures E and F described in the proof of Theorem 4.7 to get θ_x and ξ_x for $1 \le x \le h - 2$. Now based on ξ_{h-2}, for a fixed x with $1 \le x \le 2h - 4$, swap the labels of $u_{i,4}$ and $u_{i,2}$ for each $1 \le i \le x$. Let π_x be the new labeling. Then $e_{\theta_x}(1) = \frac{17m}{2} - 6 - 5x$, and $e_{\xi_x}(1) = 6m + 4 - 6x$ for $1 \le x \le h - 2$; $e_{\pi_x}(1) = 3m + 20 - 2x$ for $2 \le x \le 2h - 4$. By Lemma 3.5, there is a friendly labeling g of $C_m \times P_6$ such that $e_g(1) = i$ for each $i \in [m + 21, \frac{17m}{2} - 13]$. If $m = 4$, then skip these procedures.

If $m \ge 6$, then the current labeling is π_{2h-4}; if $m = 4$, then the current labeling is ζ_2. Perform the following steps:

Step 1: swap the labels of $u_{m,4}$ and $u_{m,3}$ with those of $u_{m-3,2}$ and $u_{m-2,2}$, respectively, and let ϕ_1 be the new labeling.

Step 2: swap the labels of $u_{m-3,6}$ and $u_{m-1,2}$ and let ϕ_2 be the new labeling.

Step 3: swap the labels of $u_{m-3,4}$ and $u_{m,3}$ and let ϕ_3 be the new labeling.

Step 4: swap the labels of $u_{m,3}$ and $u_{m,2}$ and let ϕ_4 be the new labeling.

Step 5: swap the labels of $u_{m-2,5}$ and $u_{m,3}$ and let ϕ_5 be the new labeling.

When $m \ge 6$. We have $e_{\phi_1}(1) = m + 24$, $e_{\phi_2}(1) = m + 21$, $e_{\phi_3}(1) = m + 19$, $e_{\phi_4}(1) = m + 15$, $e_{\phi_5}(1) = m + 11$. By applying Lemmas 3.11, 3.7, 3.8 and 3.9 to

ϕ_1, ϕ_2, ϕ_4 and ϕ_5 respectively, there is a friendly labeling g of $C_m \times P_6$ such that $e_g(1) = i$ for each $i \in [m+8, m+21]$.

When $m = 4$. We have $e_{\phi_1}(1) = 26$, $e_{\phi_2}(1) = 23$, $e_{\phi_3}(1) = 23$ and $e_{\phi_4}(1) = 17$. By applying Lemmas 3.11, 3.7 and 3.8 to ϕ_1, ϕ_2 and ϕ_4 respectively, there is a friendly labeling g of $C_m \times P_6$ such that $e_g(1) = i$ for each $i \in [12, 23]$.

Hence we have the theorem. □

Theorem 4.11 *For even m, $4 \le m \le 8$, there is a friendly labeling g of $C_m \times P_4$ such that $e_g(1) = i$ for each $i \in [m+5, 7m-4]$.*

Proof: Based on $f_{\max}$, we swap the labels of $u_{m,3}$ and $u_{m,2}$. Let this labeling be γ. Then $e_{f_{\max}}(1) = 7m$ and $e_\gamma(1) = 7m-6$. Applying Lemma 3.5, there is a friendly labeling g of $C_m \times P_4$ such that $e_g(1) = i$ for each $i \in [7m-13, 7m-4]$. We still write $m = 2h$.

For $m \ge 6$, using γ as the initial labeling we perform Procedures E and F described in the proof of Theorem 4.7 to get θ_x and ξ_x for $1 \le x \le h-2$. Then $e_{\theta_x}(1) = 7m-6-6x$, and $e_{\xi_x}(1) = 4m+6-6x$ for $1 \le x \le h-2$. By Lemma 3.5, there is a friendly labeling g of $C_m \times P_4$ such that $e_g(1) = i$ for each $i \in [m+11, 7m-14]$. If $m = 4$, then skip these procedures.

If $m \ge 6$, then the current labeling is ξ_{h-2}; if $m = 4$, then the current labeling is γ. Swap the labels of $u_{m-3,4}$ and $u_{m-3,1}$. Let the new labeling be η_1. Then $e_{\eta_1}(1) = m+12$. By Lemma 3.6, there exists a friendly labeling g such that $e_g(1) = i$ for each $i \in [m+5, m+10] \setminus \{m+7\}$. Swap the labels of $u_{m-2,3}$ and $u_{m-2,2}$ from η_1 to get the new labeling η_2. Then $e_{\eta_2}(1) = m+6$. By Lemma 3.4, there exists a friendly labeling g such that $e_g(1) = m+7$. Hence we obtain the theorem. □

Combining with Theorems 1.2, 1.3, 4.3, 4.8, 4.7 and Remark 4.1, we have Theorem 4.1 for $n \ge 10$. Combining with Theorems 1.2, 1.3, 4.2, 4.3 and 4.9, we have Theorem 4.1 for $n = 8$. Combining with Theorems 1.2, 1.3, 4.3 and 4.10, we have Theorem 4.1 for $n = 6$. Combining with Theorems 1.2, 1.3, 4.4 and 4.11, we have Theorem 4.1 for $n = 4$.

Lemma 4.1 *For even n, there is a friendly labelings f and g of $C_{2n} \times P_n$ such that $e_f(1) = 2n+2$ and $e_g(1) = 2n+3$.*

Proof: Following are the required labelings f and g:

	column		
1 to $n-1$	n	$n+1$	$n+2$ to $2n$
	1	1	
	0	1	
0	0	1	1
	⋮	⋮	
	0	1	
	0	0	

	column		
1 to $n-1$	n	$n+1$	$n+2$ to $2n$
	0	1	
	1	1	
0	0	1	1
	⋮	⋮	
	0	1	
	0	0	

□

Lemma 4.2 *For even n, there is a friendly labeling f of $C_{2n-2} \times P_n$ such that $e_f(1) = 2n$.*

Proof: Let

$$f(u_{i,j}) = \begin{cases} 0 & \text{if } 1 \leq i \leq n-1,\ 1 \leq j \leq n; \\ 1 & \text{if } n \leq i \leq 2n-2,\ 1 \leq j \leq n. \end{cases}$$

It is easy to see that $e_f(1) = 2n$. □

By Lemmas 4.1 and 4.2 we have the following two theorems.

Theorem 4.12 *For even $n \geq 4$ and $m = 2n$,*

$$\text{FFI}(C_{2n} \times P_n) = \{-4n^2 + 2n + 2i \mid i \in [2n+2, 4n^2 - 2n - 4] \cup \{2n, 4n^2 - 2n\}\}.$$

Theorem 4.13 *For even $n \geq 4$ and $m = 2n - 2$,*

$$\text{FFI}(C_{2n-2} \times P_n) =$$
$$\{-4n^2 + 6n - 2 + 2i \mid i \in [2n+2, 4n^2 - 6n - 2] \cup \{2n-2, 2n, 4n^2 - 6n + 2\}\}.$$

We can determine full friendly index sets of $C_m \times P_n$ for other cases by similar algorithms. The reader is referred to [13] for details. The results are as follows.
The full friendly index set of $C_m \times P_n$ is

$$\big\{ -2mn + m + 2i \mid i \in [2n+2, 2mn - m - 4] \cup \{2n, 2mn - m\} \big\}$$
for $m \geq 2n + 2$ and m, n are even;

$$\big\{ -2mn + m + 2i \mid i \in [m+4, 2mn - m - 4] \cup \{m+2, 2mn - m\} \big\}$$
for $m \leq 2n - 2$, m is even and n is odd;

$$\big\{ -2mn + m + 2i \mid i \in [2n+2, 2mn - m - 4] \cup \{2n, 2mn - m\} \big\}$$
for $m \geq 2n$ and m is even and n is odd;

$$\big\{ -2mn + m + 2i \mid i \in [m+4, 2mn - m - n] \cup \{m\} \big\}$$
for $m \leq 2n - 3$, m is odd and n is even;

$$\big\{ -2mn + m + 2i \mid i \in [m+2, 2mn - m - n] \cup \{m\} \big\}$$
for $m = 2n - 1$ and n is even;

$$\big\{ -2mn + m + 2i \mid i \in [2n, 2mn - m - n] \big\}$$
for $m \geq 2n + 1$ and m is odd and n is even;

$$\big\{ -2mn + m + 2i \mid i \in [m+4, 2mn - m - n] \cup \{m+2\} \big\}$$
for $m \leq 2n - 3$ and m, n are odd;

$$\big\{ -2mn + m + 2i \mid i \in [2n+1, 2mn - m - n] \big\}$$
for $m \geq 2n - 1$ and m, n are odd.

References

[1] L.W. Beineke, R.J. Wilson and P.J. Cameron, *Topics in Algebraic Graph Theory*, Cambridge, 2004.

[2] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, 1976.

[3] M. Gao, The edge difference sets of the direct product of two paths (in Chinese), *MSc. Thesis*, Fuzhou Unviersity, 2010.

[4] H. Kwong, S-M. Lee and H.K. Ng, On friendly index sets of 2-regular graphs, *Discrete Math.* **308** (2008), 5522–5532.

[5] H. Kwong and S-M. Lee, On friendly index sets of generalized books, *J. Combin. Math. Combin. Comput.* **66** (2008), 43–58.

[6] S-M. Lee and H.K. Ng, On friendly index sets of bipartite graphs, *Ars Combin.* **86** (2008), 257–271.

[7] E. Salehi and S-M. Lee, On friendly index set of trees, *Congr. Numer.* **178** (2006), 173–183.

[8] E. Salehi and D. Bayot, On friendly index set of $P_m \times P_n$, *Util. Math.* **81** (2010), 121–130.

[9] W.C. Shiu and M.H. Ling, Extreme friendly indices of $C_m \times C_n$, *Congr. Numer.* **188** (2007), 175–182.

[10] W.C. Shiu and H. Kwong, Full friendly index sets of $P_2 \times P_n$, *Discrete Math.* **308** (2008), 3688–3693.

[11] W.C. Shiu and F.S. Wong, Extreme friendly indices of $C_m \times P_n$, *Congr. Numer.* **197** (2009), 65–75.

[12] W.C. Shiu and M.H. Ling, Full friendly index sets of Cartesian products of two cycles, *Acta Math. Sinica* **26** (2010), 1233–1244.

[13] F.S. Wong, Full friendly index sets of the Cartesian product of cycles and paths, *M. Phil. Thesis*, Hong Kong Baptist University, 2010.

(Received 31 Mar 2011; revised 3 Oct 2011, 3 Nov 2011)