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GBRDs with block size 3 over odd order groups and groups of orders divisible by 2 but not 4

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Abstract

Well-known necessary conditions for the existence of a generalized Bhaskar Rao design, GBRD$(v, 3, \lambda; \mathbb{G})$ with $v \geq 4$ are: (i) $\lambda \equiv 0 \pmod{|\mathbb{G}|}$, (ii) $\lambda(v-1) \equiv 0 \pmod 2$, (iii) $\lambda v(v-1) \equiv 0 \pmod 3$, (iv) if $|\mathbb{G}| \equiv 0 \pmod 2$ then $\lambda v(v-1) \equiv 0 \pmod 8$. In this paper we show that these conditions are sufficient whenever (i) the group $\mathbb{G}$ has odd order or (ii) the order is of the form $2q$ for $q = 3^m$ or q an odd number which is not a multiple of 3.

1 Introduction

Throughout this paper $\mathbb{G}$ is a finite (multiplicative) group, $0 \notin \mathbb{G}$ is a zero symbol, t, v, b, r, k, λ are positive integers and K is a finite set of positive integers. The cyclic group of order n is denoted $C(n)$.

1.1 Block designs and generalized Bhaskar Rao designs

Definition 1. Let v and λ be positive integers, K be a set of positive integers and X be a set of v elements. A *pairwise balanced design*, or PBD$(v; K; \lambda)$, is a collection of subsets of X, called *blocks*, for which each pair of distinct elements of X appears together in exactly λ blocks and if a block contains exactly k elements of X then k

belongs to K. A *balanced incomplete block design*, BIBD(v,k,λ), is a PBD$(v;\{k\};\lambda)$ in which $k<v$.

Definition 2. Let $\underline{x}=(x_1,x_2,\ldots,x_b)$ and $\underline{y}=(y_1,y_2,\ldots,y_b)$ be vectors, where each x_i and y_j is either 0 or an element of $\mathbb{G}$. Then we say $\underline{x}$ and $\underline{y}$ are λ-*orthogonal* if the list

$$x_i y_i^{-1} : i=1,2,\ldots,b, \quad x_i\neq 0, \quad y_i\neq 0,$$

contains each group element exactly $\frac{\lambda}{|\mathbb{G}|}$ times.

Clearly, if $\underline{x}$ and $\underline{y}$ are λ-orthogonal, then $\lambda\equiv 0 \pmod{|\mathbb{G}|}$.

Definition 3. A *generalized Bhaskar Rao design* GBRD$(v,K,\lambda;\mathbb{G})$ is a rectangular array with v rows (and b columns), each entry of which is either 0 or an element of $\mathbb{G}$ and such that for each column the number of group entries is an element of K and each pair of distinct rows is λ-orthogonal. When the set K has only one element, we usually use the notation GBRD$(v,k,\lambda;\mathbb{G})$ rather than GBRD$(v,\{k\},\lambda;\mathbb{G})$.

For a GBRD$(v,K,\lambda;\mathbb{G})$ to exist requires $v\geq k$ for each $k\in K$ and that $\lambda\equiv 0 \pmod{|\mathbb{G}|}$. It is often convenient to write GBRD$(v,K,t|\mathbb{G}|;\mathbb{G})$ rather than GBRD$(v,K,\lambda;\mathbb{G})$.

Note that replacing the group entries in a GBRD$(v,k,\lambda;\mathbb{G})$ by 1 and leaving the others 0, results in an incidence matrix for a balanced incomplete block design, (or an all 1's matrix if $v=k$). Hence (i) the number of group entries is constant for each row and equals $r=\lambda(v-1)/(k-1)$, (ii) the number of columns in the GBRD is $b=\lambda v(v-1)/(k(k-1))$. For a GBRD$(v,K,\lambda;\mathbb{G})$, $|K|\neq 1$, the number of group entries in rows is not necessarily constant and the number of columns b is not determined by the parameters of the design. It is often helpful to think of a GBRD as the incidence matrix of a BIBD (or PBD) which has been 'signed' over a group $\mathbb{G}$, or alternatively to think of the incidence matrix of a BIBD (or PBD) as a GBRD over the trivial group.

We are interested in constructing GBRDs, and in determining for which values of $v,K,\lambda,\mathbb{G}$ there does exist a GBRD$(v,K,\lambda;\mathbb{G})$. In this paper we focus on the case $K=\{3\}$, and we begin with some direct constructions of new GBRDs which we use in the proof of our main result.

Example 4. The following array is a GBRD$(9,8,7;C(7))$ over the cyclic group of order 7. For notational simplicity we use additive notation (the usual convention with cyclic groups). We use $\bullet$ to replace the 'zero symbol'.

$$\begin{bmatrix}
\bullet & 1 & 2 & 0 & 4 & 4 & 0 & 2 & 1\\
1 & \bullet & 1 & 2 & 0 & 4 & 4 & 0 & 2\\
2 & 1 & \bullet & 1 & 2 & 0 & 4 & 4 & 0\\
0 & 2 & 1 & \bullet & 1 & 2 & 0 & 4 & 4\\
4 & 0 & 2 & 1 & \bullet & 1 & 2 & 0 & 4\\
4 & 4 & 0 & 2 & 1 & \bullet & 1 & 2 & 0\\
0 & 4 & 4 & 0 & 2 & 1 & \bullet & 1 & 2\\
2 & 0 & 4 & 4 & 0 & 2 & 1 & \bullet & 1\\
1 & 2 & 0 & 4 & 4 & 0 & 2 & 1 & \bullet
\end{bmatrix}$$

Example 5. A GBRD$(8, \{8, 7\}, 7; C(7))$ can be obtained by taking any eight rows of the GBRD$(9, 8, 7; C(7))$ given in Example 4.

Example 6. The following array is a GBRD$(9, 5, 5; C(5))$ over the cyclic group of order 5. Once again, for simplicity we use additive notation and use $\bullet$ to replace the 'zero symbol'.

$$
\left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0 & 4 & 4 & 1 & \bullet & \bullet & 1 & \bullet & \bullet & 0 & \bullet & \bullet & \bullet & 1 & 4 & \bullet & 4 & 1 \\
4 & 0 & 4 & \bullet & 1 & \bullet & \bullet & 1 & \bullet & \bullet & 0 & \bullet & 4 & \bullet & 1 & 1 & \bullet & 4 \\
4 & 4 & 0 & \bullet & \bullet & 1 & \bullet & \bullet & 1 & \bullet & \bullet & 0 & 1 & 4 & \bullet & 4 & 1 & \bullet \\
\hline
1 & \bullet & \bullet & 0 & 4 & 4 & 1 & \bullet & \bullet & \bullet & 4 & 1 & 0 & \bullet & \bullet & \bullet & 1 & 4 \\
\bullet & 1 & \bullet & 4 & 0 & 4 & \bullet & 1 & \bullet & 1 & \bullet & 4 & \bullet & 0 & \bullet & 4 & \bullet & 1 \\
\bullet & \bullet & 1 & 4 & 4 & 0 & \bullet & \bullet & 1 & 4 & 1 & \bullet & \bullet & \bullet & 0 & 1 & 4 & \bullet \\
\hline
1 & \bullet & \bullet & 1 & \bullet & \bullet & 0 & 4 & 4 & \bullet & 1 & 4 & \bullet & 4 & 1 & 0 & \bullet & \bullet \\
\bullet & 1 & \bullet & \bullet & 1 & \bullet & 4 & 0 & 4 & 4 & \bullet & 1 & 1 & \bullet & 4 & \bullet & 0 & \bullet \\
\bullet & \bullet & 1 & \bullet & \bullet & 1 & 4 & 4 & 0 & 1 & 4 & \bullet & 4 & 1 & \bullet & \bullet & \bullet & 0
\end{array}\right]
$$

If any row of this array is deleted, the remaining 8×18 array is a GBRD$(8, \{5, 4\}, 5; C(5))$.

From Abel et al. [5] we have the following two existence results:

Lemma 7. *For* $v \geq 3$*, there exists a* PBD$(v; \{3, 4, 5, 6, 8\}; 1)$.

Lemma 8. *For* $v \geq 3$*,* $v \equiv 1 \pmod 2$ *there exists a* PBD$(v; \{3, 5\}; 1)$.

We use construction theorems which are based on normal subgroup structure.

Theorem 9. *[9] If there exist a* PBD$(v; K; \lambda)$ *and, for each* $h \in K$*, a* GBRD$(h, k, \mu; \mathbb{G})$*, then a* GBRD$(v, k, \lambda\mu; \mathbb{G})$ *exists.*

This generalizes to:

Theorem 10. *[7] Let* $\mathbb{N}$ *be a normal subgroup of a finite group* $\mathbb{G}$*. Then, if there exists a* GBRD$(v, K, \lambda; \mathbb{G}/\mathbb{N})$*, and for each* $h \in K$ *a* GBRD$(h, k, \mu; \mathbb{N})$ *exists, a* GBRD$(v, k, \lambda\mu; \mathbb{G})$ *also exists.*

1.2 GBRDs with $k = 3$

For $v = 3$, the existence of a GBRD$(v, 3, |\mathbb{G}|; \mathbb{G})$ is equivalent to the existence of a complete mapping of the group $\mathbb{G}$, or, equivalently, of an orthomorphism of $\mathbb{G}$. See, for example, Evans [11] or Palmer [17].

The following theorem, for $v = k = 3$ but where λ is not necessarily equal to $|\mathbb{G}|$, follows as a consequence of the Hall-Paige Conjecture [14]. This long standing conjecture has recently been proved (Evans [10], Wilcox [22], and Wilcox, Evans and Bray [6] - New results) so it is now the Hall-Paige Theorem about the existence of complete mappings for all groups without non-trivial cyclic Sylow 2-subgroups.

Theorem 11. *[4] If $\mathbb{G}$ is an even order group with a cyclic Sylow 2-subgroup, then there exists a* $\mathrm{GBRD}(3,3,\lambda;\mathbb{G})$ *if and only if* $\lambda \equiv 0 \pmod{2|\mathbb{G}|}$. *For all other groups* $\mathbb{G}$, *a* $\mathrm{GBRD}(3,3,\lambda;\mathbb{G})$ *exists for all* $\lambda \equiv 0 \pmod{|\mathbb{G}|}$.

For $v \geq 4$, necessary conditions for a $\mathrm{GBRD}(v,3,\lambda;\mathbb{G})$ are given, for example, in Abel et al. [4]:

Lemma 12. *For $v \geq 4$ the following conditions are necessary for the existence of a* $\mathrm{GBRD}(v,3,\lambda;\mathbb{G})$*:*

(*i*) $\lambda \equiv 0 \pmod{|\mathbb{G}|}$;

(*ii*) $\lambda(v-1) \equiv 0 \pmod{2}$;

(*iii*) $\lambda v(v-1) \equiv 0 \pmod{3}$;

(*iv*) if $|\mathbb{G}| \equiv 0 \pmod{2}$ then $\lambda v(v-1) \equiv 0 \pmod{8}$.

Abel et al. [4] conjecture that these necessary conditions are sufficient:

Conjecture 13. *The necessary conditions in Lemma 12 are sufficient for existence of a* $\mathrm{GBRD}(v,3,\lambda;\mathbb{G})$ *for $v \geq 4$.*

We view Conjecture 13 as a generalized Hall-Paige Conjecture (*GHP Conjecture*). There is much evidence for the conjecture.

It is known that the necessary conditions are sufficient when the group is abelian, and the proof of this developed over some time and involved many people. For example, the result for the cyclic group C_2 was obtained by Seberry [20] and for the group $C_2 \times C_4$ by Palmer and Seberry [18]. For C_4 the result was obtained by de Launey, Sarvate and Seberry [8]. For some other abelian groups, including all elementary abelian groups, the result was obtained by Lam and Seberry [15] and Seberry [21]. Recently Ge et al. [13] completed the proof for all abelian groups.

The necessary conditions have been shown to be sufficient for any odd order nilpotent group by Palmer [16], for any dihedral group, for nilpotent groups in general and for 2-groups or semi-dihedral groups and for any sufficiently small group or any dicyclic group by Abel et al. in [1], [2], [3] and [4]. We summarise the evidence for the GHP Conjecture in the following theorem:

Theorem 14. *Let $\mathbb{G}$ be a finite group. Then, for $v \geq 4$, in each of the following cases, the necessary conditions in Lemma 12 are sufficient for the existence of a* $\mathrm{GBRD}(v,3,\lambda;\mathbb{G})$*:*

- *For $\mathbb{G}$ nilpotent, and in particular for $\mathbb{G}$ abelian;*
- *For $\mathbb{G}$ dihedral, dicyclic or semi-dihedral;*
- *For $\mathbb{G}$ with $|\mathbb{G}| = pq$ for p, q primes;*
- $|\mathbb{G}| \in \{2^n p\colon p \geq 5$ *prime, n the smallest positive integer with* $p | (2^n - 1)\}$, (*the only numbers of this form with* $2^n p \leq 10,000$ *are* 56, 80, 992, 4352)*;*

- *For* $\mathbb{G}$ *with* $|\mathbb{G}| \leq 100$ *with the possible exception of* $|\mathbb{G}| \in \{36, 48, 54, 60, 72, 96\}$.

For a group $\mathbb{G}$, we say *the GHP Conjecture holds for* $\mathbb{G}$ if the necessary conditions in Lemma 12 are sufficient for the existence of a GBRD$(v, 3, \lambda; \mathbb{G})$ for $v \geq 4$.

We note that the necessary conditions include a dependence on the $\gcd(|\mathbb{G}|, 12)$. In our main result in this paper we show that the GHP Conjecture holds for all groups of odd order, or of order $|\mathbb{G}| = 2q$, where $q = 3^m$ or q is an odd number not divisible by 3.

In our investigations of odd order groups we can sometimes use the following lemma:

Lemma 15. *Let* $\mathbb{G}$ *be an odd order group with a normal subgroup* $\mathbb{N}$ *such that if* $|\mathbb{G}| \equiv 0 \pmod 3$ *then* $|\mathbb{G}/\mathbb{N}| \equiv 0 \pmod 3$. *Then, if the GHP Conjecture holds for the quotient* $\mathbb{G}/\mathbb{N}$, *then the GHP Conjecture also holds for* $\mathbb{G}$.

Since for any group $\mathbb{N}$ of odd order there exists a GBRD$(3, 3, |\mathbb{N}|; \mathbb{N})$, Lemma 15 is a particular case of Lemma 16.

Lemma 16. *[2] Let* $\mathbb{G}$ *be a group with normal subgroup* $\mathbb{N}$ *such that a* GBRD$(3, 3, |\mathbb{N}|; \mathbb{N})$ *exists and* $\gcd(|\mathbb{G}/\mathbb{N}|, 12) = \gcd(|\mathbb{G}|, 12)$. *Then, if the GHP Conjecture holds for the quotient* $\mathbb{G}/\mathbb{N}$, *the GHP Conjecture also holds for* $\mathbb{G}$.

2 New Results

In this section we prove that the GHP Conjecture holds for groups of odd order and for groups with order $2q$ where $q = 3^m$ or q is prime to 6. We use the Feit-Thompson Theorem [12] that groups of odd order are solvable. We also use the theorem that if a group has order $2q$ for odd q then the group has a normal 2-complement (this follows from a theorem of Burnside, see for example Theorem 6.2.11 in Scott [19]). We state this as a theorem:

Theorem 17. *If* $|\mathbb{G}| \equiv 2 \pmod 4$, *then* $\mathbb{G}$ *has a normal subgroup of index* 2.

2.1 Designs with $k = 3$ over groups of odd order

Let $\mathbb{G}$ have odd order, and $v \geq 4$. Then the necessary conditions for a GBRD$(v, 3, \lambda; \mathbb{G})$ can be written as $\lambda \equiv 0 \pmod{|\mathbb{G}|}$, $\lambda(v-1) \equiv 0 \pmod 2$ and $\lambda v(v-1) \equiv 0 \pmod 3$.

Theorem 18. *The GHP Conjecture holds for groups of odd order not divisible by* 3.

Proof. Suppose that $\mathbb{G}$ is a minimal group of odd order, not divisible by 3, such that it has not been determined that Conjecture 13 holds for $\mathbb{G}$. Then, $\mathbb{G}$ cannot be cyclic, and hence $\mathbb{G}$ is not simple, so there is a proper non-trivial normal subgroup $\mathbb{N}$. Clearly $\mathbb{N}$ and $\mathbb{G}$ satisfy the conditions of Lemma 15, and hence Conjecture 13 holds for $\mathbb{G}$. Therefore the Conjecture holds for groups of odd order not divisible by 3. □

Theorem 19. *The GHP Conjecture holds for groups of odd order.*

Proof. Let $\mathbb{G}$ be a group of odd order. Since the GHP Conjecture is known to hold for abelian groups and for groups of odd order not divisible by 3, we assume $\mathbb{G}$ is non-abelian with $|\mathbb{G}| \equiv 0 \pmod 3$. Since the order is odd, $\mathbb{G}$ is solvable, so there is a proper non-trivial normal subgroup $\mathbb{N}$ with (non-trivial) cyclic quotient $\mathbb{G}/\mathbb{N} \cong C(p)$ for some prime p. If we can find such an $\mathbb{N}$ with $p = 3$, then, since the GHP Conjecture holds for $C(3)$, we apply Lemma 15 to determine that the GHP Conjecture holds for $\mathbb{G}$.

So, for the remainder of this proof, assume that $\mathbb{G}$ is a group of odd order $|\mathbb{G}| \equiv 0 \pmod 3$ and that $\mathbb{N}$ is a normal subgroup with odd order $|\mathbb{N}| \equiv 0 \pmod 3$, and $\mathbb{G}/\mathbb{N} \cong C(p)$ for some prime $p \geq 5$. Assume that $\mathbb{G}$ is a minimal group of odd order for which it is not determined whether or not the GHP Conjecture holds for $\mathbb{G}$, so that we know that the GHP Conjecture holds for $\mathbb{N}$.

Since $|\mathbb{G}| \equiv 0 \pmod 3$, the necessary conditions for a $\mathrm{GBRD}(v, 3, \lambda; \mathbb{G})$ reduce to $\lambda \equiv 0 \pmod{|\mathbb{G}|}$ and $\lambda(v-1) \equiv 0 \pmod 2$ so that either $\lambda \equiv 0 \pmod 2$ or $v \equiv 1 \pmod 2$. To show that the GHP Conjecture holds for $\mathbb{G}$ we need to prove that these conditions are sufficient to ensure the existence of a $\mathrm{GBRD}(v, 3, \lambda; \mathbb{G})$. Note that it is sufficient to show that a $\mathrm{GBRD}(v, 3, t|\mathbb{G}|; \mathbb{G})$ exists for (i) $t = 1$ and $v \equiv 1 \pmod 2$ and for (ii) $t = 2$ and any $v \geq 3$. In each case we show the existence of designs with certain small values of v and then use PBD-closure results.

Case (i) $t = 1$ and $v \equiv 1 \pmod 2$. By Lemma 8, we have that for $v \geq 3$, $v \equiv 1 \pmod 2$ there exists a $\mathrm{PBD}(v; \{3, 5\}; 1)$. Hence, by Theorem 9, it is enough to show that there exists a $\mathrm{GBRD}(v, 3, |\mathbb{G}|; \mathbb{G})$ for $v \in \{3, 5\}$. There exists a $\mathrm{GBRD}(3, 3, \lambda; \mathbb{G})$ for any group $\mathbb{G}$ of odd order (see for example Theorem 11). For $v = 5$, we use the normal subgroup $\mathbb{N}$ which has cyclic quotient $\mathbb{G}/\mathbb{N} \cong C(p)$ for some prime $p \geq 5$. It is well known that there exists a $\mathrm{GBRD}(p, p, p; C(p))$ for any odd prime p (for example the multiplication table of the finite field of order p is a $\mathrm{GBRD}(p, p, p; C(p))$ over the additive group of the field.) By removing all but the first 5 rows of this, gives a $\mathrm{GBRD}(5, 5, p; C(p))$ and hence a $\mathrm{GBRD}(5, 5, p; \mathbb{G}/\mathbb{N})$. By minimality of the choice of $\mathbb{G}$, we have that the GHP Conjecture holds for $\mathbb{N}$. Since $|\mathbb{N}| \equiv 0 \pmod 3$ the necessary and sufficient conditions give that there exists an $\mathrm{GBRD}(v, 3, |\mathbb{N}|; \mathbb{N})$ for all $v \geq 3$, $v \equiv 1 \pmod 2$. Hence there exists a $\mathrm{GBRD}(5, 3, |\mathbb{N}|; \mathbb{N})$ and so, by Theorem 10, there exists a $\mathrm{GBRD}(5, 3, |\mathbb{G}|; \mathbb{G})$. Therefore there exists a $\mathrm{GBRD}(v, 3, t|\mathbb{G}|; \mathbb{G})$ when $t = 1$ and $v \equiv 1 \pmod 2$.

Case (ii) $t = 2$ and any $v \geq 3$. By Lemma 7 we have that for $v \geq 3$, there exists a $\mathrm{PBD}(v; \{3, 4, 5, 6, 8\}; 1)$. Hence, by Theorem 9, it is enough to show that there exists a $\mathrm{GBRD}(v, 3, 2|\mathbb{G}|; \mathbb{G})$ for $v \in \{3, 4, 5, 6, 8\}$.

- For each of $v = 3, 5$, we have just shown there exists a $\mathrm{GBRD}(v, 3, |\mathbb{G}|; \mathbb{G})$, and two copies of one of these, placed side by side yields a $\mathrm{GBRD}(v, 3, 2|\mathbb{G}|; \mathbb{G})$.

- For $v = 4, 6$, which are each $\equiv 0, 1 \pmod 3$, we again use the normal subgroup $\mathbb{N}$ with quotient $C(p)$, with $p \geq 5$. Since the GHP Conjecture holds for $C(p)$ there exists a $\mathrm{GBRD}(v, 3, 2|C(p)|; C(p))$ for $v \equiv 0, 1 \pmod 3$, and hence there exists a GBRD$(v, 3, 2|\mathbb{G}/\mathbb{N}|; \mathbb{G}/\mathbb{N})$ for $v = 4, 6$. There exists a GBRD$(3, 3, |\mathbb{N}|; \mathbb{N})$ since $\mathbb{N}$ has odd order. By Theorem 10 this gives a GBRD$(v, 3, 2|\mathbb{G}|; \mathbb{G})$, for $v = 4, 6$.

- For $v = 8$, again we make use of $\mathbb{N}$, with $\mathbb{G}/\mathbb{N} \cong C(p)$. Since the GHP Conjecture holds for $\mathbb{N}$, and $|\mathbb{N}| \equiv 0 \pmod 3$, we have that for all $v \geq 3$ there exists a GBRD$(v, 3, 2|\mathbb{N}|; \mathbb{N})$, and in particular a GBRD$(p, 3, 2|\mathbb{N}|; \mathbb{N})$. If the prime $p \geq 8$, we again make use of the existence of a GBRD$(p, p, p; C(p))$ and remove all but the first 8 rows to determine the existence a GBRD$(8, 8, p; C(p))$, that is a GBRD$(8, 8, p; \mathbb{G}/\mathbb{N})$. Since we have a GBRD$(8, 3, 2|\mathbb{N}|; \mathbb{N})$, Theorem 10 ensures the existence of a GBRD$(8, 3, 2|\mathbb{G}|; \mathbb{G})$.

  If the prime $p < 8$, then $p = 5$ or 7. For $p = 7$, we use the GBRD$(8, \{8, 7\}, 7; C(7))$ given in Example 5. Since we have GBRD$(u, 3, 2|\mathbb{N}|; \mathbb{N})$ for each $u \in \{7, 8\}$, Theorem 10 ensures the existence of a GBRD$(8, 3, 2|\mathbb{G}|; \mathbb{G})$. For $p = 5$, we use the GBRD$(8, \{5, 4\}, 5; C(5))$ given in Example 6. Since we have GBRD$(u, 3, 2|\mathbb{N}|; \mathbb{N})$ for each $u \in \{4, 5\}$, Theorem 10 ensures the existence of a GBRD$(8, 3, 2|\mathbb{G}|; \mathbb{G})$.

□

2.2 Designs with $k = 3$ over groups of order 2×3^m

Lemma 20. *If $\mathbb{G}$ is a group with order $|\mathbb{G}| = 2 \times 3^m$, $m \geq 1$, then $\mathbb{G}$ has a normal subgroup of order 3.*

Proof. Let $\mathbb{G}$ be a group with order $|\mathbb{G}| = 2 \times 3^m \geq 6$. Any Sylow 3-subgroup has index 2 and so is normal. Let $\mathbb{P}$ be the unique Sylow 3-subgroup of $\mathbb{G}$, and let τ be any element of order 2 in $\mathbb{G}$. Then the elements of the group can be partitioned into $\mathbb{G} = \mathbb{P} \cup \tau\mathbb{P}$.

We define X to be the set of all elements of order 3 which are in the centre of $\mathbb{P}$. (The set X must be non-empty since $\mathbb{P}$ is a non-trivial group of prime power order.) Furthermore, we can partition the elements of X into pairs of mutually inverse elements, $\{s, s^{-1}\}$. Conjugation by τ permutes these pairs because conjugation is a group homomorphism. Any homomorphism preserves the properties 'belonging to the centre' and 'being of order three' and 'being mutually inverse'. Note that, since τ has order 2, we have $\tau^{-1} = \tau$.

Take any $s \in X$. If $\{s, s^{-1}\}$ is fixed by conjugation by τ, i.e. $\tau s \tau = s$ or $\tau s \tau = s^{-1}$, then the subgroup generated by s is normal in $\mathbb{G}$. If $\{s, s^{-1}\}$ is not fixed by τ, then by setting $t = \tau s \tau$, we have $\tau\{s, s^{-1}\}\tau = \{t, t^{-1}\} \neq \{s, s^{-1}\}$. Since s and t are both elements of order 3 in the centre of $\mathbb{P}$, and $t \notin \{s, s^{-1}\}$, the element st is an element of order 3 in the centre of $\mathbb{P}$. Set $w = st = ts$, then

$$\tau w \tau = \tau s t \tau = (\tau s \tau)(\tau t \tau) = ts = w.$$

So the pair $\{w, w^{-1}\}$ is fixed by τ, and the subgroup generated by w is a normal subgroup of $\mathbb{G}$ of order 3. □

Lemma 21. *If $\mathbb{G}$ is a group with order $|\mathbb{G}| = 2 \times 3^m$, $m \geq 1$, then $\mathbb{G}$ has a normal subgroup of index* 6*.*

Proof. Let $\mathbb{G}$ be a group, $|\mathbb{G}| = 2 \times 3^m \geq 6$. We proceed by induction on m.

If $m = 1$ then $|\mathbb{G}| = 6$ and the result is trivially true.

If $m > 1$ then $|\mathbb{G}| = 2 \times 3^m > 6$. By Lemma 20, there is a normal subgroup of order 3, with quotient group of order $2 \times 3^{m-1} \geq 6$. This quotient group has a normal subgroup of index 6, and this subgroup (of the quotient group) lifts to a normal subgroup of $\mathbb{G}$ which has index 6 in $\mathbb{G}$ as required. □

Theorem 22. *The GHP Conjecture holds for groups of order $|\mathbb{G}| = 2 \times 3^m$, $m \geq 0$.*

Proof. Let $\mathbb{G}$ be a group with order $|\mathbb{G}| = 2q$, with $q = 3^m$. If $|\mathbb{G}| \leq 6$, then from Theorem 14, the GHP Conjecture holds for $\mathbb{G}$. If $|\mathbb{G}| > 6$ then, by Lemma 21, $\mathbb{G}$ has a normal subgroup $\mathbb{N}$ such that the factor group $\mathbb{G}/\mathbb{N}$ has order 6, and hence from Theorem 14, the GHP Conjecture holds for $\mathbb{G}/\mathbb{N}$. The normal subgroup $\mathbb{N}$ has order 3^{m-1} which is odd, and therefore, by Theorem 11, there exists a GBRD$(3, 3, |\mathbb{N}|; \mathbb{N})$. The conditions are satisfied for Lemma 16, and we can conclude that the GHP Conjecture holds for $\mathbb{G}$. □

Groups of order 54 were one case listed in Theorem 14 as possible exceptions for the GHP Conjecture, but, since $54 = 2 \times 3^3$, we have

Corollary 23. *The GHP Conjecture holds for groups of order $|\mathbb{G}| = 54$.*

2.3 Designs with $k = 3$ with $|\mathbb{G}| = 2q$ for q odd not a multiple of 3

Theorem 24. *The GHP Conjecture holds for groups of order $|\mathbb{G}| = 2q$, with q an odd number which is not a multiple of* 3*.*

Proof. From Theorem 17, $\mathbb{G}$ must have a normal subgroup, $\mathbb{N}$, of index 2. Therefore $\gcd(|\mathbb{G}/\mathbb{N}|, 12) = \gcd(|\mathbb{G}|, 12) = 2$. Since $|\mathbb{N}| = q$ is odd, there exists a GBRD$(3, 3, q; \mathbb{N})$. Since $\mathbb{G}/\mathbb{N} \cong C(2)$, the GHP Conjecture holds $\mathbb{G}/\mathbb{N}$ and hence by Lemma 16, the GHP Conjecture holds for $\mathbb{G}$. □

2.4 Summary

We summarise the evidence we now have for the GHP Conjecture. Note that $|\mathbb{G}| = pq$ for primes p, q, is now subsumed within the other cases.

Theorem 25. *Let $\mathbb{G}$ be a finite group. Then, for $v \geq 4$, in each of the following cases, the necessary conditions in Lemma 12 are sufficient for the existence of a* GBRD$(v, 3, \lambda; \mathbb{G})$*:*

- *For $\mathbb{G}$ nilpotent, and in particular for $\mathbb{G}$ abelian;*
- *For $\mathbb{G}$ dihedral, dicyclic or semi-dihedral;*

- *For* $\mathbb{G}$ *with*

  $|\mathbb{G}| \equiv 1 \pmod 2)$*, or*

  $|\mathbb{G}| = 2q$ *where* $q = 3^m$ *or* q *odd not divisible by* 3*, or*

  $|\mathbb{G}| \in \{2^n p\text{: } p \geq 5$ *prime,* n *the smallest positive integer with* $p|(2^n - 1)\}$*,*

  or

  $|\mathbb{G}| \leq 100$ *with the possible exception of* $|\mathbb{G}| \in \{36, 48, 60, 72, 96\}$*.*

References

[1] R.J.R. Abel, D. Combe, A.M. Nelson and W.D. Palmer, GBRDs over groups of order $\leq$ 100 or of order pq with p, q primes, *Discrete Math.* **310** (2010), 1080–1088.

[2] R.J.R. Abel, D. Combe, A.M. Nelson and W.D. Palmer, GBRDs with block size three over 2-groups, semi-dihedral groups and nilpotent groups, *Electron. J. Combin.* **18(1)** (2011), #P32.

[3] R.J.R. Abel, D. Combe and W.D. Palmer, Bhaskar Rao designs and the dihedral groups, *J. Combin. Theory Ser. A* **106** (2004), 145–157.

[4] R.J.R. Abel, D. Combe, G. Price and W.D. Palmer, Existence of Generalised Bhaskar Rao Designs with block size 3, *Discrete Math.* **309** (2009), 4069–4078.

[5] R.J.R. Abel, F.E. Bennett and M. Greig, PBD-closure, 247–254, in: *The CRC Handbook of Combinatorial Designs* (C.J. Colbourn and J.H. Dinitz, Eds), CRC Press, Boca Raton, FL, Second Edition, 2007.

[6] *The CRC Handbook of Combinatorial Designs* (C.J. Colbourn and J.H. Dinitz, Eds.), CRC Press, Boca Raton, FL, Second Edition, 2007. New results can be found at http://www.emba.uvm.edu/dinitz/part6.newresults.html.

[7] D. Combe, W.D. Palmer and W.R. Unger, Bhaskar Rao designs and the alternating group A_4, *Australas. J. Combin.* **24** (2001), 275–283.

[8] W. de Launey, D.G. Sarvate and J. Seberry, Generalized Bhaskar Rao designs of block size 3 over Z_4, *Ars Combin.* **19A** (1985), 273–285.

[9] W. de Launey and J. Seberry, Generalized Bhaskar Rao designs of block size four, *Congr. Numer.* **41** (1984), 229–294.

[10] A.B. Evans, The admissibility of sporadic simple groups, *J. Algebra* **321** (2009), 105–116.

[11] A.B. Evans, *Orthomorphism Graphs of Groups*, Springer-Verlag, 1992.

[12] W. Feit and J.G. Thompson, Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 775–1029.

[13] G. Ge, M. Greig, J. Seberry and R. Seberry, Generalized Bhaskar Rao designs with block size 3 over finite abelian groups, *Graphs Combin.* **23** (2007), no. 3, 271–290.

[14] M. Hall and L.J. Paige, Complete mappings of finite groups, *Pacific J. Math.* **5** (1955), 541–549.

[15] C. Lam and J. Seberry, Generalized Bhaskar Rao designs, *J. Statist. Plann. Inference* **10** (1984), 83–95.

[16] W.D. Palmer, A composition theorem for generalized Bhaskar Rao designs, *Australas. J. Combin.* **6** (1992), 221–228.

[17] W.D. Palmer, *Partial generalized Bhaskar Rao designs*, Ph.D. Thesis, University of Sydney, 1993.

[18] W.D. Palmer and J. Seberry, Bhaskar Rao designs over small groups, *Ars Combin.* **26A** (1988), 125–148.

[19] W.R. Scott, *Group Theory*, Prentice-Hall Inc., 1964.

[20] J. Seberry, Regular group divisible designs and Bhaskar Rao designs with block size three, *J. Statist. Plann. Inference* **10** (1984), 69–82.

[21] J. Seberry, Generalized Bhaskar Rao designs of block size three, *J. Statist. Plann. Inference* **11** (1985), 373–379.

[22] S. Wilcox, Reduction of the Hall-Paige conjecture to sporadic simple groups, *J. Algebra* **321** (2009), 1407–1428.

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